On the sharpness of two-sided bounds for the Perron root and of the related eigenvalue inclusion sets

L. Yu. Kolotilina

St. Petersburg Branch of the V.A., Steklov Mathematical Institute of the Russian Academy of Sciences, Fontanka 27, St. Petersburg 191023, Russian Federation

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Dedicated to Richard S. Varga at the occasion of his 80th birthday

Abstract

The paper considers the sharpness problem for certain two-sided bounds for the Perron root of an irreducible nonnegative matrix. The results obtained are applied to prove the sharpness of the related eigenvalue inclusion sets in classes of matrices with fixed diagonal entries, bounded above deleted absolute row sums, and a partly specified irreducible sparsity pattern.

Keywords: Nonnegative matrices; Perron root; Two-sided bounds; Directed graph; Simple circuits; Eigenvalue inclusion sets

1. Introduction

In the papers [13,11] and the remarkable monograph [12], Richard Varga considered the sharpness problem for the Gerschgorin, Ostrowski–Brauer, and Brualdi eigenvalue inclusion sets. Recall that for a given matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$, all these sets are defined in terms of the same $2n$ values.
\[ a_{ii}, \quad i = 1, \ldots, n; \]
\[ r'_i(A) = \sum_{j=1, j\neq i}^{n} |a_{ij}|, \quad i = 1, \ldots, n \]
and the Brualdi set also depends on the set \( \mathcal{C}(A) \) of all simple circuits in the directed graph of the matrix \( A \). Consequently, if
\[
A' \in \omega(A) := \{ B = (b_{ij}) \in \mathbb{C}^{n \times n} : b_{ii} = a_{ii}, r'_i(B) = r'_i(A), i = 1, \ldots, n \},
\]
then the Gerschgorin and Ostrowski–Brauer sets for \( A \) and \( A' \) are the same. Furthermore, the Gerschgorin and Ostrowski–Brauer sets for \( A \) contain those for any matrix
\[
A' \in \hat{\omega}(A) := \{ B = (b_{ij}) \in \mathbb{C}^{n \times n} : b_{ii} = a_{ii}, r'_i(B) \leq r'_i(A), i = 1, \ldots, n \}.
\]
Similarly, if
\[
A' \in \omega_B(A) := \{ B = (b_{ij}) \in \mathbb{C}^{n \times n} : b_{ii} = a_{ii}, r'_i(B) = r'_i(A), i = 1, \ldots, n ; \mathcal{C}(B) = \mathcal{C}(A) \},
\]
then the Brualdi sets for \( A \) and \( A' \) coincide, and for any
\[
A' \in \hat{\omega}_B(A) := \{ B = (b_{ij}) \in \mathbb{C}^{n \times n} : b_{ii} = a_{ii}, r'_i(B) \leq r'_i(A), i = 1, \ldots, n; \mathcal{C}(B) = \mathcal{C}(A) \},
\]
the Brualdi eigenvalue inclusion set for \( A' \) is contained in the Brualdi set for \( A \).

The sharpness problem for the above eigenvalue inclusion sets can be stated as follows: Is it true that each point of the set in question is an eigenvalue of a matrix from the corresponding matrix class \( \omega(A), \hat{\omega}(A), \omega_B(A), \) or \( \hat{\omega}_B(A) \), or of a matrix from the closures of these classes? For the solutions of these problems we refer the reader to [12].

In this paper, we consider the sharpness problem for some two-sided bounds for the Perron root of a nonnegative matrix. These bounds, recalled in Section 2, involve the row sums of a given nonnegative matrix and also either its zero/nonzero pattern or the set of simple circuits in the associated directed graph. Since the interval determined by a two-sided bound can be regarded as an inclusion set for the Perron root, the congeniality of our sharpness problems and the sharpness problems for eigenvalue inclusion sets is obvious. Furthermore, as is known (see e.g., [9,10]), the Gerschgorin, Ostrowski–Brauer, and Brualdi inclusion theorems can be derived from the corresponding upper bounds for the Perron root. This makes the interrelation of the sharpness problems for two-sided bounds for the Perron root and for eigenvalue inclusion sets even closer.

This paper is organized as follows. In Section 2, we recall the relevant two-sided bounds for the Perron root of a nonnegative matrix. The corresponding sharpness results are presented in Section 3. Finally, in Section 4, the sharpness results for the Perron root are used to derive sharpness theorems for the Brualdi and an Ostrowski–Brauer-type eigenvalue inclusion sets. The latter theorem is new and extends a result of Varga on the classical Ostrowski–Brauer inclusion set, which is the union of all the Cassini ovals, to the case of an arbitrary symmetric off-diagonal sparsity pattern.

We conclude this introduction by specifying the notation used throughout the paper.

- For a positive integer \( n \geq 1 \), we denote \( \langle n \rangle = \{1, \ldots, n\} \).
- \( I_n \) is the identity matrix of order \( n \).
- For a matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \).
\[ r_i(A) = \sum_{j=1}^{n} |a_{ij}|, \quad i = 1, \ldots, n \]

are the absolute row sums of \( A \), and
\[ r'_i(A) = r_i(A) - |a_{ii}|, \quad i = 1, \ldots, n, \]
are the deleted absolute row sums of \( A \).

- For a matrix \( A = (a_{ij}) \in \mathbb{C}^{n \times n} \) and \( K \in \langle n \rangle \), \( A[K] = (a_{ij})_{i,j \in K} \) is the principal submatrix specified by \( K \).
- For real matrices \( A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}^{n \times n}, n \geq 1 \), the matrix inequality \( A \geq B \)

is understood componentwise, i.e., it means that \( a_{ij} \geq b_{ij}, i, j \in \langle n \rangle \). In particular, \( A \geq 0 \) means that the matrix \( A \) is nonnegative.
- For \( A \geq 0 \), by \( \rho(A) \) we denote the Perron root of \( A \), i.e., the nonnegative eigenvalue of \( A \) equal to its spectral radius, i.e., to the value \( \max_{\lambda \in \text{Spec } A} |\lambda| \).
- For \( A = (a_{ij}) \in \mathbb{C}^{n \times n}, n \geq 1 \), \( G_A \) is the directed graph of the matrix \( A \), and \( \mathcal{C}(A) \) is the set of simple circuits in \( G_A \), i.e., \( \gamma \in \mathcal{C}(A) \) if and only if \( \gamma = (i_1, \ldots, i_p, i_{p+1} = i_1) \), where \( p \geq 1 \) and \( i_1, \ldots, i_p \in \langle n \rangle \) are pairwise distinct. The set \( \bar{\gamma} := \{i_1, \ldots, i_p\} \) is called the support of \( \gamma \), and \( |\gamma| := p \) is called the length of \( \gamma \).

2. Two-sided bounds for the Perron root

In this section, we recall the two-sided bounds for the Perron root of a nonnegative matrix whose sharpness will be considered in the next section.

**Theorem 2.1** (Frobenius [5]). Let \( A = (a_{ij}) \) be a nonnegative matrix of order \( n \geq 1 \). Then
\[ \min_{i \in \langle n \rangle} r_i(A) \leq \rho(A) \leq \max_{i \in \langle n \rangle} r_i(A). \tag{2.1} \]
Furthermore, if the matrix \( A \) is irreducible, then either both inequalities in (2.1) hold with equality or both are strict.

**Theorem 2.2** [6]. Let \( A = (a_{ij}) \) be a nonnegative matrix of order \( n \geq 1 \). Then for any \( \alpha, 0 \leq \alpha \leq 1 \)
\[ \min_{i,j : a_{ij} \neq 0} r_i(A)^\alpha r_j(A)^{1-\alpha} \leq \rho(A) \leq \max_{i,j : a_{ij} \neq 0} r_i(A)^\alpha r_j(A)^{1-\alpha}. \tag{2.2} \]
Furthermore, if the matrix \( A \) is irreducible, then either both inequalities in (2.2) hold with equality or both are strict.

**Theorem 2.3** ([1], see also [8]). Let \( A = (a_{ij}) \) be a nonnegative matrix of order \( n \geq 1 \) free of zero rows. Then
\[ \min_{\gamma \in \mathcal{C}(A)} \left[ \prod_{i \in \bar{\gamma}} r_i(A) \right]^{1/|\gamma|} \leq \rho(A) \leq \max_{\gamma \in \mathcal{C}(A)} \left[ \prod_{i \in \bar{\gamma}} r_i(A) \right]^{1/|\gamma|}. \tag{2.3} \]
Furthermore, if the matrix \( A \) is irreducible, then either both inequalities in (2.3) hold with equality or both are strict.
3. Sharpness results

In this section, we analyze the two-sided bounds of Theorems 2.1–2.3 and show that they cannot be improved in the associated classes of irreducible nonnegative matrices with prescribed row sums, which will be specified below.

3.1. Sharpness of the Frobenius bounds

The sharpness of the Frobenius two-sided bounds recalled in Theorem 2.1 immediately follows from the theorem below. In the sequel, by \( P = P(r_1, \ldots, r_n) \) we denote the set of all matrices \( A \geq 0 \) of order \( n \), \( n \geq 1 \), with prescribed row sums \( r_i(A) = r_i, i = 1, \ldots, n \).

**Theorem 3.1.** Given \( n \geq 1 \) and arbitrary \( r_i > 0, i = 1, \ldots, n \), let \( \Psi_{Ir} = \Psi_{Ir}(r_1, \ldots, r_n) \) be the set of nonnegative irreducible matrices \( A \) of order \( n \) with row sums \( r_i(A) = r_i, i = 1, \ldots, n \). If \( \min_{i \in \langle n \rangle} r_i < \max_{i \in \langle n \rangle} r_i \), then

\[
\{ \rho(A) : A \in \Psi_{Ir} \} = \left( \min_{i \in \langle n \rangle} r_i, \max_{i \in \langle n \rangle} r_i \right);
\]

(3.1)

otherwise

\[
\{ \rho(A) : A \in \Psi_{Ir} \} = \min_{i \in \langle n \rangle} r_i = \max_{i \in \langle n \rangle} r_i.
\]

In order to prove this theorem, we will use the following three lemmas.

**Lemma 3.1.** Given \( r_1 > r_2 > 0 \), for any \( \xi \)

\[ r_2 < \xi < r_1, \]

there is an irreducible nonnegative \( 2 \times 2 \) matrix \( A \) with \( r_i(A) = r_i, i = 1, 2 \), such that

\[ \rho(A) = \xi. \]

**Proof.** First assume that

\[ \sqrt{r_1 r_2} \leq \xi < r_1 \]

and observe that from the latter assumption and the condition \( r_1 > r_2 \) it follows that \( \xi > r_2 \). Consider the matrix

\[
B = \begin{bmatrix}
\frac{\xi^2 - r_1 r_2}{\xi - r_2} & \frac{\xi (r_1 - \xi)}{\xi - r_2} \\
\frac{\xi (r_1 - \xi)}{\xi - r_2} & \frac{r_2 (r_1 - \xi)}{\xi - r_2}
\end{bmatrix},
\]

(3.2)

which obviously is nonnegative and irreducible. The two eigenvalues \( \lambda_{1,2}(B) \) of \( B \) are the roots of the characteristic equation

\[
\lambda \left( \lambda - \frac{\xi^2 - r_1 r_2}{\xi - r_2} \right) - \xi r_2 \frac{r_1 - \xi}{\xi - r_2} = 0.
\]

It is immediately seen that \( \lambda_1(B) = \xi \). Since

\[
\lambda_2(B) = \det B / \lambda_1(B) = -\frac{r_2 (r_1 - \xi)}{\xi - r_2} < 0,
\]

we conclude that \( \rho(B) = \xi \).
If
\[ r_2 < \xi < \sqrt{r_1 r_2}, \]
then \( \xi < r_1 \), and for the irreducible nonnegative matrix
\[ B = \begin{bmatrix} 0 & r_1 \\ \frac{\xi (\xi - r_2)}{r_1 - \xi} & \frac{r_1 r_2 - \xi^2}{r_1 - \xi} \end{bmatrix}, \]
the equality \( \rho(B) = \xi \) is established in the same way as for the matrix \((3.2)\). □

**Lemma 3.2.** Given \( n \geq 3 \) and arbitrary quantities \( r_1 \geq r_2 \geq \cdots \geq r_n \) such that \( r_1 > r_n > 0 \), for any
\[ \xi \in (\sqrt{r_1 r_n}, r_1), \]
there is an \( n \times n \) nonnegative matrix of the form
\[ A = \begin{bmatrix} \alpha & \varepsilon & \cdots & \varepsilon & [r_1 - \varepsilon (n - 2) - \alpha] \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \]
with
\[ \alpha > 0, \quad \varepsilon > 0, \quad \alpha + \varepsilon (n - 2) < r_1 \]
such that
\[ \rho(A) = \xi. \]

**Proof.** As is readily seen, the characteristic equation of the matrix \((3.5)\) is as follows:
\[ \lambda^{n-2} \left[ \lambda^2 - \alpha (\lambda - r_n) - r_1 r_n - \varepsilon \sum_{i=2}^{n-1} r_i + \varepsilon r_n (n - 2) \right] = 0. \]
Therefore, \( \lambda = 0 \) is an eigenvalue of \( A \) of multiplicity \( n - 2 \).
Furthermore, one can readily ascertain that for
\[ \alpha = \frac{\xi^2 - r_1 r_n - \varepsilon \sum_{i=2}^{n-1} (r_i - r_n)}{\xi - r_n}, \]
\( \lambda = \xi \) is a root of the characteristic equation, i.e., \( \xi \) is a positive eigenvalue of \( A \). Therefore, the remaining nonzero eigenvalue of \( A \) equals
\[ \text{tr } A - \xi = \alpha - \xi = \frac{-(r_1 - \xi) r_n - \varepsilon \sum_{i=2}^{n-1} (r_i - r_n)}{\xi - r_n} \]
and is negative, whence \( \rho(A) = \xi \).

Now it only remains to ascertain that conditions \((3.6)\) are fulfilled for a positive \( \varepsilon \) and the corresponding \( \alpha \) specified by \((3.7)\). Indeed, by \((3.4)\), from \((3.7)\) it follows that \( \alpha > 0 \) whenever \( \varepsilon \) is sufficiently small. The last inequality in \((3.6)\) also holds for sufficiently small \( \varepsilon \) because
\[ r_1 - \alpha = \frac{\xi (r_1 - \xi) + \varepsilon \sum_{i=2}^{n-1} (r_i - r_n)}{\xi - r_n} \geq \frac{\xi (r_1 - \xi)}{\xi - r_n} > 0. \]
This completes the proof of the lemma. □
Lemma 3.3. Given \( n \geq 3 \) and arbitrary quantities \( r_1 \geq r_2 \geq \cdots \geq r_n \) such that \( r_1 > r_n > 0 \), for any
\[
\xi \in (r_n, \sqrt{r_1 r_n}),
\]
there is an \( n \times n \) nonnegative matrix of the form
\[
A = \begin{bmatrix}
    r_1 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    r_n - \varepsilon(n - 2) - \alpha & \varepsilon & \cdots & \varepsilon \\
    \varepsilon & \cdots & \varepsilon & \alpha
\end{bmatrix} \tag{3.8}
\]
with
\[
\alpha > 0, \quad \varepsilon > 0, \quad \alpha + \varepsilon(n - 2) < r_n
\]
such that
\[
\rho(A) = \xi.
\]
This lemma is proved similarly to Lemma 3.2.

Remark 3.1. From the proof of Theorem 3.1 it follows that equality (3.1) also holds for the proper subclass of the class \( \Psi_{Ir} \) consisting of structurally symmetric matrices.

Proof of Theorem 3.1. Since in the case \( \max_{i \in \{n\}} r_i = \min_{i \in \{n\}} r_i \) the desired result immediately follows from Theorem 2.1, we may assume that \( n \geq 2 \) and that
\[
\min_{i \in \{n\}} r_i < \max_{i \in \{n\}} r_i.
\]
Furthermore, without loss of generality, we may assume that
\[
r_1 \geq r_2 \geq \cdots \geq r_n.
\]
In view of Theorem 2.1 and the latter assumption, we have
\[
\{ \rho(A) : A \in \Psi_{Ir} \} \subseteq (r_n, r_1)
\]
and only the opposite inclusion
\[
\{ \rho(A) : A \in \Psi_{Ir} \} \supseteq (r_n, r_1)
\]
must be established. To this end, for an arbitrary fixed \( \xi, r_n < \xi < r_1 \), we must show that \( \xi = \rho(A) \) for a matrix \( A \in \Psi_{Ir} \).

In the case \( n = 2 \), this immediately follows from Lemma 3.1.

Now let \( n \geq 3 \). For \( \xi \neq \sqrt{r_1 r_n} \), a matrix \( A \in \Psi_{Ir} \) with \( \rho(A) = \xi \) exists by virtue of Lemmas 3.2 and 3.3. So it remains to consider the case \( \xi = \sqrt{r_1 r_n} \).

For \( 0 < \varepsilon < r_n \), we have \( r'_i := r_i - \varepsilon > 0, i = 1, \ldots, n \), and \( \xi' := \sqrt{r_1 r_n} - \varepsilon > 0 \). Observe that from the assumption \( r_1 > r_n \) it readily follows that \( \xi' \geq \sqrt{r'_1 r'_n} \). Thus, as we have already demonstrated, there is a matrix \( A' \in \Psi_{Ir}(r'_1, \ldots, r'_n) \) such that \( \rho(A') = \xi' = \sqrt{r_1 r_n} - \varepsilon \). But then for the shifted matrix \( A = \varepsilon I_n + A' \) we obviously have \( \rho(A) = \sqrt{r_1 r_n} \) and \( A \in \Psi_{Ir}(r_1, \ldots, r_n) \).

Theorem 3.1 is proved completely. \( \square \)
Remark 3.2. Theorem 2.2 implies that, in general (unless \(\max_{i \neq j} (r_i r_j)^{1/2} = \max_{i \in \langle n \rangle} r_i\) and \(\min_{i \neq j} (r_i r_j)^{1/2} = \min_{i \in \langle n \rangle} r_i\)), the set
\[
\{\rho(A): A \in \Psi^0(r_1, \ldots, r_n)\},
\]
where \(\Psi^0(r_1, \ldots, r_n)\) is the subclass of the class \(\Psi(r_1, \ldots, r_n)\) consisting of nonnegative matrices with zero diagonal entries and prescribed row sums, is not dense in the interval \([\min_{i \in \langle n \rangle} r_i, \max_{i \in \langle n \rangle} r_i]\).

By Theorem 2.1, the equality
\[
\{\rho(A): A \in \Psi_{Ir}\} = \left[\min_{i \in \langle n \rangle} r_i, \max_{i \in \langle n \rangle} r_i\right]
\]
does not hold for the class \(\Psi_{Ir} = \Psi_{Ir}(r_1, \ldots, r_n)\) of irreducible nonnegative matrices with prescribed row sums, unless \(\min_{i \in \langle n \rangle} r_i = \max_{i \in \langle n \rangle} r_i\). However, considering reducible triangular matrices of the form
\[
\begin{pmatrix}
\alpha & 0 & \cdots & 0 & r_1 - \alpha \\
0 & \cdots & 0 & r_2 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & r_{n-1} & r_n
\end{pmatrix},
\]
where \(r_n \leq \alpha \leq r_1 \leq \cdots \geq r_n\), we immediately arrive at the following result on the sharpness of the Frobenius two-sided bounds in the class \(\Psi(r_1, \ldots, r_n)\) of all nonnegative matrices with prescribed row sums.

**Theorem 3.2.** Given \(n \geq 1\) and arbitrary quantities \(r_i \geq 0, i = 1, \ldots, n\), we have
\[
\{\rho(A): A \in \Psi(r_1, \ldots, r_n)\} = \left[\min_{i \in \langle n \rangle} r_i, \max_{i \in \langle n \rangle} r_i\right].
\]

### 3.2. Sharpness of the circuit bounds

We begin this section by introducing some notions related to matrix sparsity patterns.

In what follows, an arbitrary square matrix \(S = (s_{ij})\) of order \(n \geq 1\) whose entries equal either 0 or 1 will be referred to as a sparsity pattern of order \(n\).

Since all matrices with a prescribed sparsity pattern \(S\) are irreducible (or reducible) simultaneously with \(S\), it is correct to speak of irreducible and reducible sparsity patterns.

The inequality \(S_1 \leq S_2\) between sparsity patterns is regarded as an inequality between nonnegative matrices, i.e., componentwise. The sum \(S_1 + S_2\) of two sparsity patterns of the same order is defined as the result of Boolean addition of their entries.

For a given matrix \(A = (a_{ij})\) of order \(n \geq 1\), its sparsity pattern \(S_A = (s_{ij})\) is defined by the standard relations
\[
s_{ij} = \begin{cases} 
1 & \text{if } a_{ij} \neq 0, \\
0 & \text{if } a_{ij} = 0, 
\end{cases} \quad i, j \in \langle n \rangle.
\]

For a simple circuit \(\gamma = (i_1, \ldots, i_p, i_{p+1} = i_1), i_j \in \langle n \rangle, j = 1, \ldots, p, p \geq 1\), the associated sparsity pattern \(S_\gamma = (s_{ij})\) is defined as follows:
\[
s_{ij} = \begin{cases} 
1 & \text{if } i = i_k, j = i_{k+1}, 1 \leq k \leq p, \\
0 & \text{otherwise,} 
\end{cases} \quad i, j \in \langle n \rangle.
\]
Given \( n \geq 2, r_i > 0, i = 1, \ldots, n \), and a sparsity pattern \( S \) of order \( n \), by \( \Psi_S = \Psi(S; r_1, \ldots, r_n) \) we denote the class of nonnegative matrices of order \( n \) with sparsity pattern \( S \) and prescribed row sums \( r_i(A) = r_i \), \( i = 1, \ldots, n \).

For \( \gamma \in \mathcal{C}(S) \) and given \( r_i > 0, i = 1, \ldots, n \), the weight, \( w(\gamma) \), of \( \gamma \) is defined by the relation

\[
 w(\gamma) = \left( \prod_{i \in \tilde{\gamma}} r_i \right)^{1/|\gamma|},
\]

where \( \tilde{\gamma} \) is the support of \( \gamma \), and \( |\gamma| \) is its length.

The main result of this subsection is the following Theorem 3.3, showing that for an irreducible sparsity pattern \( S \), the circuit bounds of Theorem 2.3 cannot be improved in the subclass \( \Psi_S = \Psi(S; r_1, \ldots, r_n) \) of the class \( \Psi = \Psi(r_1, \ldots, r_n) \).

Note that if \( S \) is an irreducible sparsity pattern, then it is uniquely determined by the set \( \mathcal{C}(S) \). Indeed, since \( S \) is irreducible, every arc in \( G_S \) necessarily belongs to a circuit \( \gamma \in \mathcal{C}(S) \). Consequently, the set of arcs in \( G_S \), which uniquely determines \( S \), coincides with the set of all arcs belonging to the circuits in \( \mathcal{C}(S) \). Thus, for an irreducible sparsity pattern \( S \), we have

\[
 \Psi(S; r_1, \ldots, r_n) = \{A: A \in \Psi(r_1, \ldots, r_n), \mathcal{C}(A) = \mathcal{C}(S)\}.
\]

**Theorem 3.3.** Let \( S \) be an irreducible sparsity pattern of order \( n \geq 1 \) and let arbitrary positive values \( r_i, i = 1, \ldots, n \), be given. If

\[
 \min_{\gamma \in \mathcal{C}(S)} w(\gamma) < \max_{\gamma \in \mathcal{C}(S)} w(\gamma), \tag{3.9}
\]

then the set \( \{\rho(A): A \in \Psi_S\} \) is a dense subset of the interval

\[
 \left( \min_{\gamma \in \mathcal{C}(S)} w(\gamma), \max_{\gamma \in \mathcal{C}(S)} w(\gamma) \right); \tag{3.10}
\]

otherwise

\[
 \{\rho(A): A \in \Psi_S\} = \min_{\gamma \in \mathcal{C}(S)} w(\gamma) = \max_{\gamma \in \mathcal{C}(S)} w(\gamma).
\]

The proof of this theorem will be based on several lemmas below.

**Lemma 3.4** [2]. Let \( A \) be an irreducible matrix of order \( n \geq 2 \). Then, for an arbitrary proper principal submatrix \( A_{11} \) of \( A \), the matrix \( A \) is permutationally similar to a block-partitioned matrix of the form

\[
 \begin{bmatrix}
 A_{11} & A_{12} & A_{13} & \cdots & A_{1m-1} & A_{1m} \\
 A_{21} & A_{22} & A_{23} & \cdots & A_{2m-1} & A_{2m} \\
 0 & A_{32} & A_{33} & \cdots & A_{3m-1} & A_{3m} \\
 \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 0 & 0 & 0 & \cdots & A_{mm-1} & A_{mm}
 \end{bmatrix},
\]

where \( m \geq 2 \) and all the subdiagonal blocks \( A_{ii-1}, i = 2, \ldots, m \), are free of zero rows.

**Proof.** First the matrix \( A \) is symmetrically permuted to the form

\[
 A' = \begin{bmatrix}
 A_{11} & A'_{12} \\
 A'_{21} & A_{22}
 \end{bmatrix}.
\]
By the irreducibility of \( A \), the block \( A'_{21} \) is nonzero. If it is free of zero rows, then the desired form is obtained. Otherwise, by a symmetric permutation of rows and columns, one can bring \( A' \) to the form

\[
\begin{bmatrix}
A_{11} & A_{12} & A'_{13} \\
A_{21} & A_{22} & A'_{23} \\
0 & A'_{32} & A'_{33}
\end{bmatrix},
\]

where \( A_{21} \) is free of zero rows. If \( A'_{32} \) has no zero rows, then the result is established. Otherwise, since \( A'_{32} \) is nonzero by the irreducibility of \( A \), one can permute the rows in \([0 \ A'_{32} \ A'_{33}]\) and the corresponding columns of \( A \) to single out the next subdiagonal block free of zero rows, and so on. \( \square \)

The next lemma considers the case of an irreducible sparsity pattern of the form \( S_{\gamma_1} + S_{\gamma_2} \) subject to certain additional conditions. Note that for such sparsity patterns, the related inverse eigenvalue problem of finding a matrix \( A \in \Psi(S; r_1, \ldots, r_m) \) with prescribed \( \rho(A) \) is solved constructively.

Let \((i, j)\) denote the arc from vertex \( i \) to vertex \( j \) and, for \( \gamma_1, \gamma_2 \in \mathcal{C}(S) \), set

\[
K(\gamma_1, \gamma_2) := \{i \in \langle m\rangle : i \in \tilde{\gamma}_1 \cap \tilde{\gamma}_2, (i, j) \notin \gamma_1 \cap \gamma_2 \text{ for all } j \in \langle m\rangle\},
\]

i.e., \( i \in K(\gamma_1, \gamma_2) \) if and only if the \( i \)th row of the sparsity pattern \( S_{\gamma_1} + S_{\gamma_2} \) contains two unit entries.

**Lemma 3.5.** Let positive values \( r_i, i = 1, \ldots, m, m \geq 2 \), be given and let \( S = S_{\gamma_1} + S_{\gamma_2} \), where \( \gamma_1 \neq \gamma_2 \) are simple circuits. Assume that \( \tilde{\gamma}_i \in \langle m\rangle, i = 1, 2; \tilde{\gamma}_1 \cup \tilde{\gamma}_2 = \langle m\rangle \), and \( K(\gamma_1, \gamma_2) = \{i_0\} \). Denote

\[
w_1 = w(\gamma_1), \quad w_2 = w(\gamma_2).
\]

If \( w_1 \neq w_2 \), then for every

\[
\xi \in (\min\{w_1, w_2\}, \max\{w_1, w_2\}), \quad \text{(3.11)}
\]

there is a matrix \( A \in \Psi(S; r_1, \ldots, r_m) \) such that

\[
\mathcal{C}(A) = \{\gamma_1, \gamma_2\} \quad \text{and} \quad \rho(A) = \xi.
\]

**Proof.** Given \( \alpha \in \mathbb{R} \), define the matrix \( A = A(\alpha) = (a_{ij}) \in \mathbb{R}^{m \times m} \) by the relations

\[
a_{ij} = \begin{cases} 
    r_i & \text{if } i \neq i_0 \text{ and } (i, j) \in \gamma_1 \cup \gamma_2; \\
    \alpha r_i & \text{if } i = i_0, (i, j) \in \gamma_1, \text{ and } (i, j) \notin \gamma_2; \\
    (1 - \alpha)r_i & \text{if } i = i_0, (i, j) \in \gamma_2, \text{ and } (i, j) \notin \gamma_1; \\
    0 & \text{if } (i, j) \notin \gamma_1 \cup \gamma_2.
\end{cases} \quad \text{(3.12)}
\]

Obviously, the rows of \( A \) are linear combinations of the respective rows of the two matrices in \( \Psi_S \) whose directed graphs coincide with the circuits \( \gamma_1 \) and \( \gamma_2 \). Note that if \( 0 < \alpha < 1 \), then \( A \in \Psi_S \).

Let a value \( \xi \) satisfying condition (3.11) be fixed.

Since the condition \( |K(\gamma_1, \gamma_2)| = 1 \) readily implies that \( \mathcal{C}(S) = \{\gamma_1, \gamma_2\} \), by the definition of the determinant, we have

\[
\lambda^m - \alpha w_1^{|\gamma_1|} \lambda^{m - |\gamma_1|} - (1 - \alpha) w_2^{|\gamma_2|} \lambda^{m - |\gamma_2|} = 0. \quad \text{(3.13)}
\]

Obviously, for \( \xi \) to be a root of Eq. (3.13), it is sufficient to take
\begin{equation}
\alpha = \frac{\xi^m - w_2|\gamma_2|\xi^{m-|\gamma_2|}}{w_1|\xi|^{m-|\gamma_1|} - w_2|\gamma_2|} = \frac{1 - (w_2/\xi)|\gamma_2|}{(w_1/\xi)|\gamma_1| - (w_2/\xi)|\gamma_2|}.
\end{equation}

Note that in view of (3.11), \(\alpha\) is correctly defined and satisfies the conditions \(0 < \alpha < 1\). Therefore, for \(\alpha\) specified by (3.14), the matrix (3.12) belongs to \(\Psi_3\), and \(\xi\) is a positive eigenvalue of \(A\). Thus, it only remains to ascertain that \(\xi = \rho(A)\). To this end, first assume that \(|\gamma_1| \leq |\gamma_2|\). Since \(\xi\) is a root of Eq. (3.13), we have
\begin{equation}
\xi|\gamma_2| - \alpha w_1|\gamma_2|\xi^{1-|\gamma_1|} = \xi|\gamma_2| - \alpha w_1|\gamma_1| (\xi|\gamma_1| - \alpha w_1|\gamma_1|) = (1 - \alpha)w_2|\gamma_2|.
\end{equation}

Since the right-hand side of (3.15) is positive, we conclude that
\begin{equation}
\xi|\gamma_1| > \alpha w_1|\gamma_1|.
\end{equation}

Now suppose \(\rho(A) \neq \xi\). Then
\begin{equation}
\rho(A) > \xi
\end{equation}
and, by (3.17), (3.16), and (3.15), we have
\begin{equation}
\rho(A)|\gamma_2| - \alpha w_1|\gamma_1| (\rho(A)|\gamma_1| - \alpha w_1|\gamma_1|) > \xi|\gamma_2| - \alpha w_1|\gamma_1| (\xi|\gamma_1| - \alpha w_1|\gamma_1|) = (1 - \alpha)w_2|\gamma_2|.
\end{equation}

Using (3.18), we derive
\begin{equation}
\rho(A)^m - \alpha w_1|\gamma_1| \rho(A)^{m-|\gamma_1|} = \rho(A)^{m-|\gamma_2|} \left[ \rho(A)^{|\gamma_2|} - \alpha w_1|\gamma_1| \right] > \rho(A)^{m-|\gamma_2|} (1 - \alpha)w_2|\gamma_2|,
\end{equation}
which shows that \(\rho(A)\) is not a root of (3.13). The contradiction obtained proves the equality \(\rho(A) = \xi\). The case \(|\gamma_1| > |\gamma_2|\) is considered similarly.

This completes the proof of Lemma 3.5. □

From the proof of Lemma 3.5 it readily follows that if we take \(\xi = w_2\) (\(\xi = w_1\)), then (3.14) yields \(\alpha = 0\) (\(\alpha = 1\)). For \(\alpha = 0\) and \(\alpha = 1\), the matrix (3.12), whose Perron root equals \(w_2\) or \(w_1\), respectively, has the required row sums but is sparser than \(S\), i.e., \(C(A) \subseteq \{\gamma_1, \gamma_2\}\), and may prove to be reducible.

Thus, we also have the following result, supplementing Lemma 3.5.

**Corollary 3.1.** Under the hypotheses of Lemma 3.5, for any
\begin{equation}
\xi \in [\min\{w(\gamma_1), w(\gamma_2)\}, \max\{w(\gamma_1), w(\gamma_2)\}].
\end{equation}
there is a nonnegative matrix \(A = (a_{ij})\) such that \(C(A) \subseteq \{\gamma_1, \gamma_2\}\), \(r_i(A) = r_i, i = 1, \ldots, m,\) and \(\rho(A) = \xi\).

**Lemma 3.6.** Let \(S\) be an irreducible sparsity pattern of order \(n \geq 2\). If \(\gamma, \xi \in C(S), \gamma \neq \xi,\) then there is a finite sequence \(\gamma_1, \ldots, \gamma_p,\) where \(p \geq 2\) and \(\gamma_i \in C(S), i = 1, \ldots, p,\) such that \(\gamma_1 = \gamma, \gamma_p = \xi,\) and
\begin{equation}
|K(\gamma_i, \gamma_{i+1})| = 1, \quad i = 1, \ldots, p - 1.
\end{equation}

**Proof.** First assume that \(\gamma \cap \xi \neq \emptyset\).

If \(|K(\gamma, \xi)| = 1\), then set \(p = 2, \gamma_1 = \gamma,\) and \(\gamma_2 = \xi\).
If \( |K(\gamma, \zeta)| \geq 2 \), then we can find two distinct vertices \( i, j \in K(\gamma, \zeta) \). In this case, in \( G_S \), there are distinct simple paths \( \pi_{ij}(\gamma) \) and \( \pi_{ji}(\zeta) \) from \( i \) to \( j \) and distinct simple paths \( \pi_{ji}(\gamma) \) and \( \pi_{ij}(\zeta) \) from \( j \) to \( i \), which are portions of the circuits \( \gamma \) and \( \zeta \). The paths \( \pi_{ij}(\gamma) \) and \( \pi_{ji}(\zeta) \) (as well as the paths \( \pi_{ji}(\gamma) \) and \( \pi_{ij}(\zeta) \)) form a circuit, say, \( \gamma' \). If \( \gamma' \) is simple, i.e., \( \gamma' \in C(S) \), then, obviously, \( |K(\gamma, \gamma')| = |K(\zeta, \gamma')| = 1 \), and we may set \( p = 3, \gamma_1 = \gamma, \gamma_2 = \gamma', \gamma_3 = \zeta \). Otherwise, \( \gamma' \) contains a simple circuit \( \gamma'' \in C(S) \) such that \( i \in \gamma'' \), and we change \( \gamma_2 = \gamma' \) for \( \gamma_2 = \gamma'' \).

Now let \( \tilde{\gamma} \cap \tilde{\zeta} = \emptyset \). Since \( S \) is irreducible, for arbitrary \( i' \in \tilde{\gamma} \) and \( j' \in \tilde{\zeta} \), in \( G_S \) there is a simple path \( \pi' \) from \( i' \) to \( j' \). Let \( \tilde{\pi} \) denote the support of \( \pi' \), i.e., \( \tilde{\pi} \) is the set of vertices belonging to \( \pi' \). If \( \tilde{\pi} \cap \tilde{\gamma} \supset \{i\}, i \neq i' \) and/or \( \tilde{\pi} \cap \tilde{\zeta} \supset \{j\}, j \neq j' \), we can find a subpath \( \pi \) of \( \pi' \) going from \( i \) to \( j \) such that \( \tilde{\pi} \cap \tilde{\gamma} = \{i\} \) and \( \tilde{\pi} \cap \tilde{\zeta} = \{j\} \). Similarly, we can find a simple path \( \pi'' \) from certain \( j' \in \tilde{\zeta} \) to a certain \( i' \in \tilde{\gamma} \) such that \( \tilde{\pi} \cap \tilde{\gamma} = \{i'\} \) and \( \tilde{\pi} \cap \tilde{\zeta} = \{j\} \). Consider the circuit \( \gamma' \) that consists of the paths \( \pi \) and \( \pi'' \) and also of the portions of \( \gamma \) and \( \zeta \) going from \( i' \) to \( i \) and from \( j \) to \( j' \), respectively. Note that the vertices \( i \) and \( i' \) (and \( j \) and \( j' \)) may coincide, in which case the circuit \( \gamma(\zeta) \) is not used. If \( \gamma' \in C(S) \), then we set \( p = 3, \gamma_1 = \gamma \), \( \gamma_2 = \gamma' \), \( \gamma_3 = \zeta \). Otherwise \( \gamma' \) decomposes in a chain of simple circuits \( \gamma_2, \ldots, \gamma_{q-p+1}, \ldots, \gamma_{p-2}, \gamma_{p} \), \( p > 4 \), such that \( |\tilde{\gamma}_i \cap \tilde{\gamma}_{i+1}| = 1, i = 2, \ldots, p \), and we have \( |K(\gamma_i, \gamma_{i+1})| = 1 \), \( i = 2, \ldots, p-2 \), and also \( |K(\gamma_2, \gamma_{p})| = |K(\gamma_{p-1}, \zeta)| = 1 \). This completes the proof of Lemma 3.6. □

**Proof of Theorem 3.3.** First note that in the case where \( \min_{\gamma \in C(S)} w(\gamma) = \max_{\gamma \in C(S)} w(\gamma) \), the required assertion immediately follows from Theorem 2.3.

Now let condition (3.9) be fulfilled and let \( \xi \in (\min_{\gamma \in C(S)} w(\gamma), \max_{\gamma \in C(S)} w(\gamma)) \). We will show that there is a matrix \( A \in \Psi_S \) such that \( \rho(A) \) is arbitrarily close to \( \xi \).

Let
\[
  w(\gamma) = \min_{\gamma' \in C(S)} w(\gamma'), \quad w(\xi) = \max_{\gamma' \in C(S)} w(\gamma').
\]

By Lemma 3.6, there is a sequence \( \{\gamma_i\}_{i=1}^p \), \( p \geq 2 \), such that \( \gamma_1 = \gamma \), \( \gamma_p = \xi \), and
\[
  |K(\gamma_i, \gamma_{i+1})| = 1, \quad i = 1, \ldots, p-1.
\]

First assume that
\[
  \xi \in (\min\{w(\gamma_i), w(\gamma_{i+1})\}, \max\{w(\gamma_i), w(\gamma_{i+1})\}), \quad 1 \leq i \leq p-1. \tag{3.20}
\]

In this case, by Lemma 3.5, there is a nonnegative matrix \( B = (b_{ij}) \) such that \( \Psi(B) = \{\gamma_i, \gamma_{i+1}\} \subseteq \Psi(S), r_k(B) = r_k \), for \( k \in J \equiv \gamma_i \cup \gamma_{i+1}, \) and \( \rho(B) = \xi \). Let \( J = \{j_1, \ldots, j_q\} \).

By the continuity of the Perron root, for any \( \varepsilon > 0 \), there is a matrix
\[
  C = (c_{ij}) \in \Psi(S[J]; r_{j_1}, \ldots, r_{j_q})
\]
(with sufficiently small entries \( c_{ij} \) in positions \( (i, j) \) such that \( b_{ij} = 0 \)) whose Perron root satisfies the condition
\[
  |\rho(C) - \xi| = |\rho(C) - \rho(B)| < \varepsilon. \tag{3.21}
\]

If \( S[J] = \langle n \rangle \), then the existence of the desired matrix is established. Otherwise, using Lemma 3.4, permute the matrix \( S \) to an upper block Hessenberg matrix
\[
  S' \equiv P^TSP = \begin{bmatrix}
  S[J] & S_{12} & S_{13} & \cdots & S_{1m-1} & S_{1m} \\
  S_{21} & S_{22} & S_{23} & \cdots & S_{2m-1} & S_{2m} \\
  0 & S_{32} & S_{33} & \cdots & S_{3m-1} & S_{3m} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & S_{mm-1} & S_{mm}
\end{bmatrix},
\]
whose subdiagonal blocks $S_{i+1,i}, i = 1, \ldots, m - 1$, are free of zero rows. Owing to the latter property, one can find a matrix $A' \in \Psi_{S'}$, close to a matrix of the block form:

$$
\begin{bmatrix}
C & 0 & 0 & \cdots & 0 & 0 \\
\ast & 0 & 0 & \cdots & 0 & 0 \\
0 & \ast & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \ast & 0
\end{bmatrix},
$$

(3.22)

whose Perron root coincides with that of $C$. Then, in view of (3.21), $\rho(A')$ also is arbitrarily close to $\rho(B) = \xi$. Therefore, the Perron root of the matrix $A = PA'P^T \in \Psi_S$ is arbitrarily close to $\xi$ as well.

Finally, if $\xi = w(\gamma_i)$, where $1 \leq i \leq p$, then the proof is essentially the same, the only difference being that the matrix $B$ (of order $|\gamma_i|$) is determined by the conditions $S_B = S_{\gamma_i}$ and $r_k(B) = r_k, k \in \tilde{\gamma}_i$, implying, in view of Theorem 2.3, that $\rho(B) = \xi$. □

**Remark 3.3.** If, under the hypotheses of Theorem 3.3, the sparsity pattern $S$ is not irreducible and inequality (3.9) holds true, then, in general, the result of the theorem is not valid, as is the case, for instance, for the diagonal sparsity pattern

$$
S = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
$$

and arbitrary row sums $r_1 > r_2 \geq 0$. On the other hand, we may conjecture that if $S$ is irreducible and $\min_{\gamma \in \varepsilon(S)} w(\gamma) < \max_{\gamma \in \varepsilon(S)} w(\gamma)$, then the set $\{\rho(A): A \in \Psi_S\}$ coincides with the open interval $\left(\min_{\gamma \in \varepsilon(S)} w(\gamma), \max_{\gamma \in \varepsilon(S)} w(\gamma)\right)$.

The following result is the counterpart of Theorem 3.3 for the case where the matrices are allowed to be sparser than prescribed and, consequently, to be reducible.

**Theorem 3.4.** Under the hypotheses of Theorem 3.3, we have

$$
\{\rho(A): A \in \Psi(S'; r_1, \ldots, r_n), S' \subseteq S\} = \left[\min_{\gamma \in \varepsilon(S)} w(\gamma), \max_{\gamma \in \varepsilon(S)} w(\gamma)\right].
$$

(3.23)

**Proof.** The fact that the left-hand side is contained in the right-hand one stems from Theorem 2.3. In order to prove the converse, the proof of Theorem 3.3 is modified as follows. Using Corollary 3.1, for $\xi \in [\min\{w(\gamma_1), w(\gamma_1+1)\}, \max\{w(\gamma_1), w(\gamma_1+1)\}]$, we choose a matrix $B$ such that $\rho(B) = \xi$, $\mathcal{C}(B) \subseteq \{\gamma_1, \gamma_1+1\}$, and $r_k(B) = r_k, k \in \tilde{\gamma}_1 \cup \tilde{\gamma}_1+1$. Next we set $C = B$ and choose $A'$ of the form (3.22). Then $A \in \Psi_{S'}$, where $S' \subseteq S$, and $\rho(A) = \rho(A') = \rho(C) = \rho(B) = \xi$, which completes the proof. □

**Remark 3.4.** Since from $S' \subseteq S$ it trivially follows that $\mathcal{C}(S') \subseteq \mathcal{C}(S)$, we have $\{A \in \Psi(S'; r_1, \ldots, r_n), S' \subseteq S\} \subseteq \{A \in \Psi(r_1, \ldots, r_n): \mathcal{C}(A) \subseteq \mathcal{C}(S)\}$. This inclusion and Theorem 2.3 imply that from (3.23) it follows that

$$
\{\rho(A): A \in \Psi(r_1, \ldots, r_n), \mathcal{C}(A) \subseteq \mathcal{C}(S)\} = \left[\min_{\gamma \in \varepsilon(S)} w(\gamma), \max_{\gamma \in \varepsilon(S)} w(\gamma)\right].
$$

(3.24)
3.3. Sharpness of the arc bounds

Let \( r_i > 0, \ i = 1, \ldots, n, \) and an irreducible sparsity pattern \( S \) of order \( n \geq 2 \) be fixed.

Theorem 2.2 with \( \alpha = 1/2 \) asserts that for any \( A \in \Psi(S; \ r_1, \ldots, r_n) \), the following two-sided bounds are valid:

\[
\min_{i,j: s_{ij} \neq 0} (r_i r_j)^{1/2} \leq \rho(A) \leq \max_{i,j: s_{ij} \neq 0} (r_i r_j)^{1/2} .
\]  

(3.25)

Furthermore, if the left-hand side of (3.25) does not equal its right-hand side, then both inequalities are strict. Comparing these bounds with the result of Theorem 3.3, we immediately arrive at the following criteria on the sharpness of the bounds in (3.25).

**Theorem 3.5.** Let \( r_i > 0, \ i = 1, \ldots, n, n \geq 1, \) be given and let \( S \) be an irreducible sparsity pattern of order \( n \). Then the upper bound in (3.25) is sharp in \( \Psi(S; \ r_1, \ldots, r_n) \) if and only if

\[
\max_{i,j: s_{ij} \neq 0} (r_i r_j)^{1/2} = \max_{\gamma \in \mathcal{E}(S)} w(\gamma) .
\]  

(3.26)

and, similarly, the lower bound in (3.25) is sharp in \( \Psi(S; \ r_1, \ldots, r_n) \) if and only if

\[
\min_{i,j: s_{ij} \neq 0} (r_i r_j)^{1/2} = \min_{\gamma \in \mathcal{E}(S)} w(\gamma).  
\]  

(3.27)

Furthermore, if both conditions (3.26) and (3.27) are fulfilled and \( \max_{i,j: s_{ij} \neq 0}(r_i r_j) > \min_{i,j: s_{ij} \neq 0}(r_i r_j) \), then the set \( \{ \rho(A): A \in \Psi(S; \ r_1, \ldots, r_n) \} \) is a dense subset of the interval \( (\max_{i,j: s_{ij} \neq 0}(r_i r_j), \ \min_{i,j: s_{ij} \neq 0}(r_i r_j)) \), and if \( \max_{i,j: s_{ij} \neq 0}(r_i r_j) = \min_{i,j: s_{ij} \neq 0}(r_i r_j) \), then

\[
\{ \rho(A): A \in \Psi(S; \ r_1, \ldots, r_n) \} = \max_{i,j: s_{ij} \neq 0} (r_i r_j)^{1/2} = \min_{i,j: s_{ij} \neq 0} (r_i r_j)^{1/2} .
\]

Note that in order to judge upon the sharpness of the arc bounds (3.25) by applying the criteria provided by Theorem 3.5, it is necessary to compute the maximal and minimal weights of the circuits in \( \mathcal{E}(S) \), which is quite expensive. For this reason, the simple sufficient sharpness conditions of the theorem below, which may be regarded as conditions of partial structural symmetry, seem very attractive.

**Theorem 3.6.** Let \( r_i > 0, \ i = 1, \ldots, n, n \geq 2, \) be given, and let \( S \) be an irreducible sparsity pattern of order \( n \). If

\[
\max_{i,j: s_{ij} \neq 0} r_i r_j = \max_{i,j: s_{ij} = 0} r_i r_j ,
\]  

(3.28)

then the upper bound in (3.25) is sharp in \( \Psi(S; \ r_1, \ldots, r_n) \).

Similarly, if

\[
\min_{i,j: s_{ij} \neq 0} r_i r_j = \min_{i,j: s_{ij} = 0} r_i r_j ,
\]  

(3.29)

then the lower bound in (3.25) is sharp in \( \Psi(S; r_1, \ldots, r_n) \).

Furthermore, if both conditions (3.28) and (3.29) are fulfilled and \( \max_{i,j: s_{ij} \neq 0}(r_i r_j) > \min_{i,j: s_{ij} \neq 0}(r_i r_j) \), then the set \( \{ \rho(A): A \in \Psi(S; r_1, \ldots, r_n) \} \) is a dense subset of the interval \( (\max_{i,j: s_{ij} \neq 0}(r_i r_j), \ \min_{i,j: s_{ij} \neq 0}(r_i r_j)) \), and if \( \max_{i,j: s_{ij} \neq 0}(r_i r_j) = \min_{i,j: s_{ij} \neq 0}(r_i r_j) \), then

\[
\{ \rho(A): A \in \Psi(S; r_1, \ldots, r_n) \} = \max_{i,j: s_{ij} \neq 0} (r_i r_j) = \min_{i,j: s_{ij} \neq 0} (r_i r_j) .
\]  

(3.30)
In particular, for an arbitrary symmetric irreducible sparsity pattern $S$, the set \( \{ \rho(A) : A \in \mathfrak{P}(S; r_1, \ldots, r_n) \} \) is either a dense subset of the interval \( (\max_{i,j:s_{ij} \neq 0} (r_i r_j), \min_{i,j:s_{ij} \neq 0} (r_i r_j)) \), or relations (3.30) hold true.

**Proof.** In view of Theorem 3.5, suffice it to demonstrate that (3.28) implies (3.26) and that (3.29) implies (3.27). Since, for any \( \gamma = (i_1, i_2, \ldots, i_k, i_{k+1} = i_1) \in \mathcal{C}(S) \)

\[
w(\gamma) = (r_1 r_2 \cdots r_k)^{1/k} = \left[ \prod_{j=1}^{k} (r_{ij} r_{i+1})^{1/2} \right]^{1/k} \leq \max_{i,j:s_{ij} \neq 0} (r_i r_j)^{1/2},
\]

we have

\[
\max_{\gamma \in \mathcal{C}(S)} w(\gamma) \leq \max_{i,j:s_{ij} \neq 0} (r_i r_j)^{1/2}.
\]

On the other hand, if condition (3.28) is fulfilled, then \( \mathcal{C}(S) \) contains a circuit \( \gamma' = (k, l, k) \) such that

\[
\max_{i,j:s_{ij} \neq 0} (r_i r_j)^{1/2} = (r_k r_l)^{1/2} = w(\gamma') \leq \max_{\gamma \in \mathcal{C}(S)} w(\gamma).
\]

Together with (3.31), this proves (3.26). The fact that (3.29) implies (3.27) is established similarly. \( \square \)

Consider an example illustrating the results obtained. Let \( n = 5 \) and let \( r_1 \geq r_2 \geq r_3 \geq r_4 \geq r_5 > 0 \) be given. Consider the irreducible sparsity pattern

\[
S = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

and assume that

\[
\max_{i,j:s_{ij} \neq 0} (r_i r_j) = r_1 r_3 > \min_{i,j:s_{ij} \neq 0} (r_i r_j) = r_3 r_5.
\]

(3.32)

We have

\[
\mathcal{C}(S) = \{(1, 3, 1), (1, 3, 5, 2, 4, 1)\}.
\]

Since condition (3.28) of Theorem 3.6 is fulfilled, the upper bound \( \rho(A) < (r_1 r_3)^{1/2} \) is sharp in \( (S; r_1, \ldots, r_5) \). However, since \( s_3 s_5 s_3 = 0 \), condition (3.29) is violated, but condition (3.27) of Theorem 3.5 can be applied. In this way, we conclude that the lower bound \( \rho(A) > (r_3 r_5)^{1/2} \) is sharp if and only if \( (r_3 r_5)^3 = (r_1 r_2 r_4)^2 \), which is impossible in view of (3.32). Thus, the lower bound \( \rho(A) > (r_3 r_5)^{1/2} \) is not sharp.

**Remark 3.5.** For \( \alpha \neq 1/2 \), the arc bounds of Theorem 2.2 are in general not sharp even for an irreducible symmetric sparsity pattern \( S \). Indeed, let \( n = 2, r_1 > r_2 \), and let \( S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). Then, for \( \alpha > 1/2 \), the interval given by Theorem 2.2 is \( (r_1^{1-\alpha} r_2^{\alpha}, r_1^\alpha r_2^{1-\alpha}) \), whereas for any \( A \in \mathfrak{P}(S; r_1, r_2) \) we actually have \( \rho(A) = (r_1 r_2)^{1/2} \), which stems from Theorem 2.3.
Theorem 3.7. Let \( r_i > 0, i = 1, \ldots, n, n \geq 2 \), be given and let for an irreducible sparsity pattern \( S \) conditions (3.28) and (3.29) be fulfilled. Then

\[
\rho(A) : A \in (S'; r_1, \ldots, r_n), S' \subseteq S = \left[ \min_{i,j:s_{ij} \neq 0} (r_i r_j)^{1/2}, \max_{i,j:s_{ij} \neq 0} (r_i r_j)^{1/2} \right].
\]

In particular, relation (3.33) holds for any asymmetric irreducible sparsity pattern \( S \).

Remark 4.1. If \( S \) is an irreducible sparsity pattern, then, as was explained in Section 3.2, the relations \( S_A = S \) and \( \mathbb{C}(A) = \mathbb{C}(S) \) are equivalent. Thus, for an irreducible \( S \), the set (4.1) is completely analogous to the set (2.60) in [12, p. 59], whereas the set \( \tilde{\omega}_S \) coincides with the closure of the set (2.61) in [12, p. 59].

Theorems 4.1 and 4.2 below state known results on eigenvalue inclusion sets in a slightly different form.
Theorem 4.1 ([4], see also [12]). Let \( S = (s_{ij}) \) be an irreducible off-diagonal sparsity pattern of order \( n \geq 2 \), and let \( \alpha_i \in \mathbb{C}, r'_i > 0, i = 1, \ldots, n \), be given values. For any matrix \( A \in \tilde{\omega}_S(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) \), all its eigenvalues belong to the set

\[
\mathcal{B}_S = \mathcal{B}_S(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) := \bigcup_{\gamma \in \mathcal{C}(S)} \left\{ z \in \mathbb{C}: \prod_{i \in \bar{\gamma}} |\alpha_i - z| \leq \prod_{i \in \bar{\gamma}} r'_i \right\}.
\]  

(4.4)

Theorem 4.2 [7]. Let \( S = (s_{ij}) \) be an irreducible off-diagonal sparsity pattern of order \( n \geq 2 \), and let \( \alpha_i \in \mathbb{C}, r'_i > 0, i = 1, \ldots, n \), be given values. For any matrix \( A \in \tilde{\omega}_S(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) \), all its eigenvalues belong to the set

\[
\mathcal{K}_S = \mathcal{K}_S(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) := \bigcup_{i \neq j, s_{ij} \neq 0} \{ z \in \mathbb{C}: |\alpha_i - z||\alpha_j - z| \leq r'_i r'_j \}.
\]  

(4.5)

In view of Theorems 4.1 and 4.2 and the known inclusion \( \mathcal{B}_S \subseteq \mathcal{K}_S \), we have

\[
\bigcup_{A \in \tilde{\omega}_S} \text{Spec } A \subseteq \mathcal{B}_S \subseteq \mathcal{K}_S.
\]

We will prove that the Brualdi set \( \mathcal{B}_S \) and, for a symmetric sparsity pattern, also the Ostrowski–Brauer-type set \( \mathcal{K}_S \) both are completely filled out with the eigenvalues of matrices belonging to \( \tilde{\omega}_S \), i.e., \( \bigcup_{A \in \tilde{\omega}_S} \text{Spec } A = \mathcal{B}_S \) and, for \( S = S^T \), also \( \bigcup_{A \in \tilde{\omega}_S} \text{Spec } A = \mathcal{K}_S \).

To this end, we will need the following simple result, relating an eigenvalue of a certain complex matrix with prescribed diagonal entries to the unit Perron root of an associated nonnegative matrix with zero diagonal.

Lemma 4.1. Let \( \alpha_i \in \mathbb{C}, i = 1, \ldots, n \), and \( \xi \in \mathbb{C}, \xi \neq \alpha_i, i = 1, \ldots, n, n \geq 2 \), be given values. Let \( A = D - E P \), where

\[
D = \text{diag}(\alpha_1, \ldots, \alpha_n);
\]

\[
E = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n), \quad \varepsilon_i = \frac{\alpha_i - \xi}{|\alpha_i - \xi|}, \quad i = 1, \ldots, n
\]  

(4.6)

and \( P \) is a nonnegative matrix of order \( n \) with zero diagonal entries. If

\[
\rho((D - \xi I_n)^{-1} P) = 1,
\]  

(4.7)

then \( \xi \in \text{Spec } A \).

Proof. By (4.7), for a nonnegative Perron vector \( v \neq 0 \) we have

\[
|D - \xi I_n|^{-1} P v = v.
\]

Using (4.6), we derive

\[
P v = |D - \xi I_n| v = E^{-1}(D - \xi I_n) v,
\]

which amounts to the equality

\[
Av = (D - E P) v = \xi v. \quad \square
\]

Theorem 4.3. Let \( S = (s_{ij}) \) be an irreducible off-diagonal sparsity pattern of order \( n \geq 2 \), and let \( \alpha_i \in \mathbb{C}, r'_i > 0, i = 1, \ldots, n \), be given values. Then
\[ \bigcup_{A \in \tilde{\omega}_S} \text{Spec } A = \mathcal{B}_S, \]

where the sets \( \tilde{\omega}_S = \tilde{\omega}_S(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) \) and \( \mathcal{B}_S = \mathcal{B}_S(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) \) are defined in (4.3) and (4.4).

**Proof.** In view of Theorem 4.1, suffice it to prove that each point of the set \( \mathcal{B}_S \) is an eigenvalue of a matrix \( A \in \tilde{\omega}_S \). Solve \( \xi \in \mathcal{B}_S \). First assume that \( \xi \neq \alpha_i, i = 1, \ldots, n \), and denote

\[ r_i = \frac{r'_i}{|\alpha_i - \xi|}, \quad i = 1, \ldots, n \]  

(4.9)

From (4.4) it follows that for a circuit \( \gamma \in \mathcal{C}(S) \) we have

\[ w(\gamma) = \left( \prod_{i \in \gamma} r_i \right)^{1/|\gamma|} \geq 1, \]

whence

\[ \max_{\gamma \in \mathcal{C}(S)} w(\gamma) \geq 1. \]

If \( \xi \) is not an interior point of at least one of the circuit sets occurring in (4.4), then, similarly

\[ \min_{\gamma \in \mathcal{C}(S)} w(\gamma) \leq 1. \]

In this case, by Theorem 3.4, there is a nonnegative matrix \( R \in \mathcal{P}(r_1, \ldots, r_n) \) such that \( S_R \leq S \) and \( \rho(R) = 1 \). Set \( P = \text{diag}(|\alpha_1 - \xi|, \ldots, |\alpha_n - \xi|)R \) and \( A = D - EP \), where the diagonal matrices \( D \) and \( E \) are defined in Lemma 4.1. Then, by (4.9), we have

\[ A = \text{diag}(\alpha_1, \ldots, \alpha_n) - \text{diag}(\alpha_1 - \xi, \ldots, \alpha_n - \xi) \tilde{R} \in \tilde{\omega}_S \]

and, by Lemma 4.1, \( \xi \in \text{Spec } A \).

If \( \xi \in \mathcal{B}_S \) is a common interior point of all the circuit sets occurring in (4.4), then

\[ 1 < \min_{\gamma \in \mathcal{C}(S)} w(\gamma) \leq \max_{\gamma \in \mathcal{C}(S)} w(\gamma). \]

Set

\[ w = \min_{\gamma \in \mathcal{C}(S)} w(\gamma) \]

and

\[ \bar{r}_i := r_i / w, \quad i = 1, \ldots, n \]

In this case, arguing as above, we find a matrix \( \tilde{R} \in \mathcal{P}(\tilde{r}_1, \ldots, \tilde{r}_n) \), with \( S_{\tilde{R}} \leq S \) and \( \rho(\tilde{R}) = 1 \). Then

\[ A := \text{diag}(\alpha_1, \ldots, \alpha_n) - \text{diag}(\alpha_1 - \xi, \ldots, \alpha_n - \xi) \tilde{R} \in \tilde{\omega}_S \]

and, once again by Lemma 4.1, \( \xi \in \text{Spec } A \).

Finally, if \( \xi = \alpha_i \) for some \( i \in \langle n \rangle \), then, trivially, \( \xi \) is an eigenvalue of a matrix \( A \in \tilde{\omega}_S \) with \( r'_i(A) = 0 \). \( \square \)

In view of Remark 4.1, for an irreducible sparsity pattern \( S \), the assertion of Theorem 4.3 essentially coincides with the last equality in the string of relations (2.71) in Theorem 2.11 of...
Theorem 4.4. Let $S = (s_{ij})$ be a symmetric irreducible off-diagonal sparsity pattern of order $n \geq 2$, and let $\alpha_i \in \mathbb{C}, r'_i > 0, i = 1, \ldots, n$, be given values. Then

$$
\bigcup_{A \in \tilde{\omega}_S} \text{Spec } A = \mathcal{K}_S,
$$

where the sets $\tilde{\omega}_S = \tilde{\omega}_S(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n)$ and $\mathcal{K}_S = \mathcal{K}_S(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n)$ are defined in (4.3) and (4.5).

Theorem 4.4 can be proved in the same way as Theorem 4.3 but using Theorem 3.7 rather than Theorem 3.4. Another possibility to prove Theorem 4.4 is to use equality (4.8) and to observe (cf. the proof of Theorem 3.6) that

$$
B_S = K_S \quad \text{if} \quad S = S^T.
$$

Relation (4.11) is important in itself because it enables one to construct an optimal eigenvalue inclusion set as a union of the associated Cassini ovals $K_{ij}(\alpha_i, \alpha_j; r'_i, r'_j) := \{z \in \mathbb{C}^{n \times n} : |\alpha_i - z||\alpha_j - z| \leq r'_i r'_j\}, i \neq j, s_{ij} = s_{ji} = 1$, which is much simpler than to construct it as a union of Brualdi circuit sets.

In particular, choosing in Theorem 4.4 the sparsity pattern $S = (s_{ij}), s_{ii} = 0, i = 1, \ldots, n, s_{ij} = 1, i \neq j, i, j \in \langle n \rangle$, which is obviously symmetric and irreducible (for $n \geq 2$), we immediately arrive at the following result on the sharpness of the classical Ostrowski–Brauer theorem ([12], see also [12, Theorem 2.2]).

Corollary 4.1. For $n \geq 2$ and arbitrary given values $\alpha_i \in \mathbb{C}, r'_i > 0, i = 1, \ldots, n$

$$
\bigcup_{A \in \tilde{\omega}(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n)} \text{Spec } A = \mathcal{K}(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n),
$$

where

$$
\tilde{\omega}(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) := \{A = (a_{ij}) \in \mathbb{C}^{n \times n} : a_{ii} = \alpha_i, r'_i(A) \leq r'_i, i = 1, \ldots, n\},
$$

$$
\mathcal{K}(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) = \bigcup_{i \neq j} \{z \in \mathbb{C} : |\alpha_i - z||\alpha_j - z| \leq r'_i r'_j\}.
$$

Note that equality (4.12) coincides with equality (2.20) in [12, p. 40].

In conclusion, it is worth recalling that the most well-known Gerschgorin eigenvalue inclusion set

$$
\Gamma(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n) = \bigcup_{i=1}^{n} \{z \in \mathbb{C}^{n \times n} : |\alpha_i - z| \leq r'_i\}
$$

is not always sharp, i.e., for some values $\alpha_1, \ldots, \alpha_n$ and $r'_1, \ldots, r'_n > 0$

$$
\bigcup_{A \in \tilde{\omega}(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n)} \text{Spec } A \not\subset \Gamma(\alpha_1, \ldots, \alpha_n; r'_1, \ldots, r'_n),
$$

(see e.g., [12, p. 42]). This readily follows from the fact that, in general
Note also that from the standpoint adopted in the present paper, the redundancy of the Gerschgorin sets is related to the fact that, in general, the Frobenius bounds for the Perron root are not sharp in the subclass $\Psi^0(r_1, \ldots, r_n)$, see Remark 3.2.

References