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Note

Estimates of weighted Hardy–Littlewood averages on the p-adic vector space

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Abstract

In the p-adic vector space $\mathbb{Q}_p^n$, we characterize those non-negative functions ψ defined on $\mathbb{Z}_p^* = \{w \in \mathbb{Q}_p\colon\ 0 < |w|_p \leqslant 1\}$ for which the weighted Hardy–Littlewood average $U_\psi : f \to \int_{\mathbb{Z}_p^*} f(t\,\cdot)\psi(t)\,dt$ is bounded on $L^r(\mathbb{Q}_p^n)$ $(1 \leqslant r \leqslant \infty)$, and on $BMO(\mathbb{Q}_p^n)$. Also, in each case, we find the corresponding operator norm $\|U_\psi\|$.

Keywords: p-Adic field; p-Adic vector space; Weighted Hardy–Littlewood averages; *BMO*

1. Introduction

For a prime number p, a non-zero element w of the p-adic field $\mathbb{Q}_p$ is uniquely represented as a canonical form $w = \sum_{j=\gamma}^{\infty} a_j p^j$, where $a_j \in \mathbb{Z}/p\mathbb{Z}$, $\gamma \in \mathbb{Z}$ and $a_\gamma \neq 0$. We call $\gamma = \mathrm{ord}_p(w)$ the order of w. And then $|w|_p = p^{-\mathrm{ord}_p(w)}$ becomes a non-Archimedean norm in $\mathbb{Q}_p$ and $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with $|\cdot|_p$ [7]. We let $\mathbb{Z}_p = \{w \in \mathbb{Q}_p\colon\ |w|_p \leqslant 1\}$ be the class of all p-adic integers of $\mathbb{Q}_p$ and denote $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus \{0\}$.

For a given $n \in \mathbb{N}$, the space $\mathbb{Q}_p^n$ denotes a vector space over $\mathbb{Q}_p$, which contains all n-tuples of $\mathbb{Q}_p$. If we define $\|x\| = \max_{1 \leqslant k \leqslant n} |x_k|_p$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{Q}_p^n$, then $\mathbb{Q}_p^n$ becomes a Banach space with the norm $\|\cdot\|$. For $\gamma \in \mathbb{Z}$, we denote B_γ as a γ-ball of $\mathbb{Q}_p^n$ with center at 0,

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containing all x with $\|x\| \leqslant p^\gamma$, and $S_\gamma = B_\gamma \setminus B_{\gamma-1} = \{x \in \mathbb{Q}_p^n\colon \|x\| = p^\gamma\}$ its boundary. Also, for $a \in \mathbb{Q}_p^n$, $B_\gamma(a)$ consists of all x with $x - a \in B_\gamma$. Similarly, $x \in S_\gamma(a)$ means that $x - a \in S_\gamma$.

Since $\mathbb{Q}_p^n$ is a locally compact Hausdorff space, there is the Haar measure dx on the additive group $\mathbb{Q}_p^n$, normalized by $\int_{B_0} dx := |B_0| = 1$, where $|E|$ denotes the Haar measure of a measurable set $E \subset \mathbb{Q}_p^n$. By a simple calculation the Haar measures of any balls and spheres can be obtained. Especially, we frequently use $|B_\gamma| = p^{n\gamma}$ and $|S_\gamma| = p^{n\gamma}(1 - p^{-n})$.

A measurable real-valued function f on $\mathbb{Q}_p^n$ is said to be in $L^r(\mathbb{Q}_p^n)$ $(1 \leqslant r < \infty)$ provided

$$\|f\|_{L^r(\mathbb{Q}_p^n)} := \left(\int_{\mathbb{Q}_p^n} \left|f(x)\right|^r dx\right)^{1/r} < \infty. \tag{1.1}$$

Also, $L^\infty(\mathbb{Q}_p^n)$ consists of the set of all measurable real-valued functions f on $\mathbb{Q}_p^n$ satisfying

$$\|f\|_{L^\infty(\mathbb{Q}_p^n)} := \operatorname*{ess\,sup}_{x \in \mathbb{Q}_p^n} \left|f(x)\right| < \infty. \tag{1.2}$$

The space $BMO(\mathbb{Q}_p^n)$ which consists of all measurable functions $f \in L^1_{\mathrm{loc}}(\mathbb{Q}_p^n)$ with bounded mean oscillation

$$\|f\|_{BMO} := \sup_B \frac{1}{|B|} \int_B \left|f(x) - f_B\right| dx < \infty, \tag{1.3}$$

where the supremum is taken over all balls and $f_B = |B|^{-1} \int_B f(x)\,dx$ stands for the average of f over B. To mention some of the previous results on harmonic analysis on the p-adic field, Haran [2,3] obtained the explicit formula of Riesz potentials and developed an analytic potential theory in the p-adic field. In [1], authors proved conditions of boundedness of maximal operators over the p-adic field $\mathbb{Q}_p$.

For a function f on $\mathbb{Q}_p^n$ and a function $\psi : \mathbb{Z}_p^* \to [0, \infty)$, we define the weighted p-adic Hardy–Littlewood average $U_\psi f$ on $\mathbb{Q}_p^n$ as

$$U_\psi f(x) = \int_{\mathbb{Z}_p^*} f(tx)\psi(t)\,dt.$$

In case f is defined on the real field $\mathbb{R}$ and $\psi \equiv 1$, U_ψ is just reduced to the classical Hardy–Littlewood average Uf such as

$$Uf(x) = \frac{1}{x} \int_0^x f(y)\,dy \quad (x \neq 0). \tag{1.4}$$

In Theorem 327 of [4], Hardy proved the following inequalities: For $1 < r < \infty$,

$$\left(\int_0^\infty \left|Uf(x)\right|^r dx\right)^{1/r} \leqslant \frac{r}{r-1} \left(\int_0^\infty \left|f(x)\right|^r dx\right)^{1/r}, \tag{1.5}$$

where the constant $r/(r-1)$ is the best possible. For $r = \infty$,

$$\operatorname*{ess\,sup}_{x \neq 0} \left|Uf(x)\right| \leqslant \operatorname*{ess\,sup}_{x \neq 0} \left|f(x)\right|, \tag{1.6}$$

which is also a sharp inequality. Hardy's result still remains to be an important one as it is closely related to the Hardy–Littlewood maximal functions in harmonic analysis [5].

In 2001, Xiao generalized inequalities (1.5) and (1.6) to the n-dimensional Euclidean space [8]. However, as far as we understand, theories of functions from $\mathbb{Q}_p^n$ into $\mathbb{R}$ play an important role in the p-adic quantum mechanics and the theory of p-adic probability in which real-valued random variables have to be considered to solve the covariance problems [6]. For this reason, the purpose of this paper is to prove the continuities of weighted p-adic Hardy–Littlewood averages on $L^r(\mathbb{Q}_p^n)$, and on $BMO(\mathbb{Q}_p^n)$. Indeed, we classify those functions ψ for which the operator U_ψ is bounded on $L^r(\mathbb{Q}_p^n)$ $(1 \leqslant r \leqslant \infty)$, and on $BMO(\mathbb{Q}_p^n)$. And then we determine the corresponding operator norms. Here are our main results.

Theorem 1.1. *For $r \in [1, \infty]$, U_ψ is bounded on $L^r(\mathbb{Q}_p^n)$ if and only if*

$$\int_{\mathbb{Z}_p^*} |t|_p^{-n/r} \psi(t)\,dt < \infty.$$

Moreover,

$$\|U_\psi\| = \int_{\mathbb{Z}_p^*} |t|_p^{-n/r} \psi(t)\,dt.$$

Theorem 1.2. *U_ψ is bounded on $BMO(\mathbb{Q}_p^n)$ if and only if*

$$\int_{\mathbb{Z}_p^*} \psi(t)\,dt < \infty.$$

Moreover,

$$\|U_\psi\| = \int_{\mathbb{Z}_p^*} \psi(t)\,dt.$$

The proofs of Theorems 1.1 and 1.2 are given in Section 2.

In case $\psi \equiv 1$, we have

$$\int_{\mathbb{Z}_p^*} |t|_p^{-n/r}\,dt = \sum_{-\infty<\gamma\leqslant 0} \int_{S_\gamma} |t|_p^{-n/r}\,dt \tag{1.7}$$

$$= \sum_{-\infty<\gamma\leqslant 0} p^{-\gamma n(1/r-1)}\big(1-p^{-n}\big) \tag{1.8}$$

$$= \big(1-p^{-n}\big)\sum_{\gamma=0}^{\infty} p^{\gamma n(1/r-1)}, \tag{1.9}$$

which has a finite value of $\mathbb{R}$ if and only of $1 < r \leqslant \infty$. Hence by Theorem 1.1, U_ψ is bounded on $L^r(\mathbb{Q}_p^n)$ for $1 < r \leqslant \infty$, but *not* bounded on $L^1(\mathbb{Q}_p^n)$, which is an analogue of the classical Hardy–Littlewood average case on the real field. Also form Theorems 1.1 and 1.2, we can see that on $BMO(\mathbb{Q}_p^n)$, the boundedness condition of U_ψ and its operator norm are the same as those on $L^\infty(\mathbb{Q}_p^n)$.

2. Proofs of main theorems

First we provide proofs of Theorem 1.1 and then by using a lemma on *BMO* norm of p-adic logarithmic function, we prove Theorem 1.2.

Proof of Theorem 1.1. Since the case of $r = \infty$ is trivial, it suffices to consider $r \in [1, \infty)$. Suppose $\int_{\mathbb{Z}_p^*} |t|_p^{-n/r} \psi(t)\, dt < \infty$. Using the Minkowski's inequality for integrals and the change of variables $tx = y$ [7, (4.2)], we have

$$\|U_\psi f\|_{L^r(\mathbb{Q}_p^n)} \leqslant \int_{\mathbb{Z}_p^*} \left(\int_{\mathbb{Q}_p^n} \left| f(tx) \right|^r dx \right)^{1/r} \psi(t)\, dt = \|f\|_{L^r(\mathbb{Q}_p^n)} \int_{\mathbb{Z}_p^*} |t|_p^{-n/r} \psi(t)\, dt. \tag{2.1}$$

Thus U_ψ maps boundedly $L^r(\mathbb{Q}_p^n)$ into itself.

Conversely, suppose $r \in [1, \infty)$ and U_ψ is bounded operator on $L^r(\mathbb{Q}_p^n)$. Then there exists a constant $C = C(r) > 0$ such that

$$\|U_\psi f\|_{L^r(\mathbb{Q}_p^n)} \leqslant C \|f\|_{L^r(\mathbb{Q}_p^n)}, \quad f \in L^r\left(\mathbb{Q}_p^n\right). \tag{2.2}$$

Now, for any *rational* number $\epsilon > 0$, we let

$$f_\epsilon(x) = \begin{cases} 0, & \|x\| < 1, \\ \|x\|^{-n/r-\epsilon}, & \|x\| \geqslant 1. \end{cases} \tag{2.3}$$

Then

$$\|f_\epsilon\|_{L^r(\mathbb{Q}_p^n)}^r = \int_{\|x\| \geqslant 1} \|x\|^{-n-\epsilon r}\, dx \tag{2.4}$$

$$= \sum_{k=0}^{\infty} \int_{\|x\| = p^k} p^{-(n+\epsilon r)k}\, dx \tag{2.5}$$

$$= \sum_{k=0}^{\infty} p^{-(n+\epsilon r)k} p^{nk} \left(1 - \frac{1}{p^n}\right) \tag{2.6}$$

$$= \frac{1}{1 - p^{-\epsilon r}} \left(1 - \frac{1}{p^n}\right). \tag{2.7}$$

Thus $f_\epsilon \in L^r(\mathbb{Q}_p^n)$, for each ϵ. Since $0 < |t|_p \leqslant 1$ for $t \in \mathbb{Z}_p^*$, we have

$$U_\psi f_\epsilon(x) = \begin{cases} 0, & \|x\| < 1, \\ \|x\|^{-n/r-\epsilon} \int_{1/\|x\| \leqslant |t|_p \leqslant 1} |t|_p^{-n/r-\epsilon} \psi(t)\, dt, & \|x\| \geqslant 1. \end{cases} \tag{2.8}$$

Evaluating $\|U_\psi f_\epsilon\|_{L^r(\mathbb{Q}_p^n)}$ for ϵ such that $|\epsilon|_p > 1$, we have

$$C^r \|f_\epsilon\|_{L^r(\mathbb{Q}_p^n)}^r \tag{2.9}$$

$$\geqslant \|U_\psi f_\epsilon\|_{L^r(\mathbb{Q}_p^n)}^r \tag{2.10}$$

$$= \int_{\|x\| \geqslant 1} \left(\|x\|^{-n/r-\epsilon} \int_{1/\|x\| \leqslant |t|_p \leqslant 1} |t|_p^{-n/r-\epsilon} \psi(t)\, dt \right)^r dx \tag{2.11}$$

$$\geqslant \int_{\|x\| \geqslant |\epsilon|_p} \left(\|x\|^{-n/r-\epsilon} \int_{1/|\epsilon|_p \leqslant |t|_p \leqslant 1} |t|_p^{-n/r-\epsilon} \psi(t)\, dt \right)^r dx \tag{2.12}$$

(put $x = \epsilon y$)

$$= \int_{\|y\|\geqslant 1} \|y\|^{-n-\epsilon r}\, dy\, |\epsilon|_p^{-\epsilon r} \left(\int_{1/|\epsilon|_p \leqslant |t|_p \leqslant 1} |t|_p^{-n/r-\epsilon} \psi(t)\, dt \right)^r$$

$$= \|f_\epsilon\|^r_{L^r(\mathbb{Q}_p^n)} \left(|\epsilon|_p^{-\epsilon} \int_{1/|\epsilon|_p \leqslant |t|_p \leqslant 1} |t|_p^{-n/r-\epsilon} \psi(t)\, dt \right)^r, \tag{2.13}$$

which implies

$$\int_{1/|\epsilon|_p \leqslant |t|_p \leqslant 1} |t|_p^{-n/r-\epsilon} \psi(t)\, dt \leqslant C|\epsilon|_p^{\epsilon}. \tag{2.14}$$

Now we take $\epsilon = 1/p^k$ $(k = 1, 2, 3, \ldots)$. Then $|\epsilon|_p = p^k > 1$. Letting $k \to \infty$, then $\epsilon \to 0$ and $|\epsilon|_p^{\epsilon} = p^{k/p^k} \to 1$. Thus from (2.14) and Fatou's lemma, we have

$$\int_{0<|t|_p \leqslant 1} |t|_p^{-n/r} \psi(t)\, dt \leqslant C. \tag{2.15}$$

Moreover, if we assume that U_ψ is bounded on $L^r(\mathbb{Q}_p^n)$, then from (2.1) we have

$$\|U_\psi\| \leqslant \int_{\mathbb{Z}_p^*} |t|_p^{-n/r} \psi(t)\, dt. \tag{2.16}$$

On the other hand, by using the above $f_\epsilon \in L^r(\mathbb{Q}_p^n)$ to obtain

$$\|U_\psi\|^r \|f_\epsilon\|^r_{L^r(\mathbb{Q}_p^n)} \geqslant \|U_\psi f_\epsilon\|^r_{L^r(\mathbb{Q}_p^n)}$$
$$\geqslant \|f_\epsilon\|^r_{L^r(\mathbb{Q}_p^n)} \left(|\epsilon|_p^{-\epsilon} \int_{1/|\epsilon|_p \leqslant |t|_p \leqslant 1} |t|_p^{-n/r-\epsilon} \psi(t)\, dt \right)^r. \tag{2.17}$$

Once again, by taking $\epsilon = 1/p^k$ $(k = 1, 2, 3, \ldots)$ in (2.17) and letting $k \to \infty$, we have

$$\|U_\psi\| \geqslant \int_{\mathbb{Z}_p^*} |t|_p^{-n/r} \psi(t)\, dt. \tag{2.18}$$

Hence, from (2.16) and (2.18), we obtain the operator norm $\|U_\psi\|$ and this completes the proof of Theorem 1.1. □

For $t \in \mathbb{Q}_p$, we define a dilation tB_γ of B_γ by $tB_\gamma = B_{\gamma - \mathrm{ord}_p(t)}$. Then we get $|tB_\gamma| = |t|_p^n |B_\gamma|$.

Similarly we define $tB_\gamma(a) = tB_\gamma + a$. Since the Haar measure is translation-invariant, for any $t \in \mathbb{Q}_p$ and for any ball $B \subset \mathbb{Q}_p^n$, we get $|tB| = |t|_p^n |B|$.

We need the following lemma to prove Theorem 1.2.

Lemma 2.1. *The p-adic logarithmic function has a non-trivial BMO norm, i.e., if* $g(x) = \log \|x\|$, *then* $g \in BMO(\mathbb{Q}_p^n)$ *and* $\|g\|_{BMO} > 0$.

Proof. Let $\delta \in \mathbb{Q}_p$. For $f \in BMO(\mathbb{Q}_p^n)$, let $f_\delta(x) = f(\delta x)$ be a scaling function of f by δ. By change of variables, we have

$$\frac{1}{|B|}\int\limits_B f_\delta(x)\,dx = \frac{1}{|\delta|_p^n|B|}\int\limits_{\delta B} f(x)\,dx = \frac{1}{|\delta B|}\int\limits_{\delta B} f(x)\,dx, \tag{2.19}$$

where B is any ball. Thus scaling transformations map $BMO(\mathbb{Q}_p^n)$ into itself. Under these scalings, $g(x) = \log\|x\|$ is changed by at most an additive constant. To show that $g \in BMO(\mathbb{Q}_p^n)$ thus it suffices to check the alternative assertions that there is a constant C such that

$$\int\limits_{B_0(a)} \big|\log\|x\|\big|\,dx \leqslant C \quad \text{for } \|a\| \leqslant 1, \tag{2.20}$$

$$\int\limits_{B_0(a)} \big|\log\|x\| - \log\|a\|\big|\,dx \leqslant C \quad \text{for } \|a\| > 1. \tag{2.21}$$

To see this, if $\|a\| < 1$, then $B_0(a) = B_0$. Thus we have

$$\begin{aligned}\int\limits_{B_0(a)} \big|\log\|x\|\big|\,dx &= \int\limits_{B_0}\big|\log\|x\|\big|\,dx \\ &= \big(1 - 1/p^n\big)\log p \sum_{k=-\infty}^{0} |k|p^{kn} \equiv C < \infty.\end{aligned} \tag{2.22}$$

Also if $\|a\| = 1$, then $B_0(a) \subset B_0$. Thus we have

$$\int\limits_{B_0(a)} \big|\log\|x\|\big|\,dx \leqslant \int\limits_{B_0}\big|\log\|x\|\big|\,dx = C, \tag{2.23}$$

where the equality follows from (2.22).

Finally if $\|a\| > 1$, then for $x \in B_0(a)$, we get $\|x\| = \|x - a + a\| = \|a\|$.

Put $\|a\| = p^\gamma$. Then

$$\int\limits_{B_0(a)} \big|\log\|x\| - \log\|a\|\big|\,dx = \int\limits_{S_\gamma}\big|\log\|x\| - \log p^\gamma\big|\,dx = 0. \tag{2.24}$$

Moreover, from (2.22) we have $\|g\|_{BMO} > 0$, which completes the proof of the lemma. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2. First, suppose U_ψ is bounded on $BMO(\mathbb{Q}_p^n)$. Then we consider a constant function $f \equiv 1$ to get $U_\psi 1 \in BMO(\mathbb{Q}_p^n)$. This means that $U_\psi 1(x)$ has a finite value at almost every $x \in \mathbb{Q}_p^n$. Hence

$$\int\limits_{\mathbb{Z}_p^*} \psi(t)\,dt = U_\psi 1(x) < \infty. \tag{2.25}$$

Conversely, suppose $\int_{\mathbb{Z}_p^*}\psi(t)\,dt < \infty$. Let $f \in BMO(\mathbb{Q}_p^n)$ and let B be a ball. Then by Fubini's theorem and change of variables, we have

$$(U_\psi f)_B = \int_{\mathbb{Z}_p^*} \left(\frac{1}{|B|} \int_B f(tx)\,dx \right) \psi(t)\,dt \tag{2.26}$$

$$= \int_{\mathbb{Z}_p^*} f_{tB} \psi(t)\,dt \tag{2.27}$$

and

$$\frac{1}{|B|} \int_B \left| U_\psi f(x) - (U_\psi f)_B \right| dx$$

$$\leqslant \frac{1}{|B|} \int_B \left(\int_{\mathbb{Z}_p^*} \left| f(tx) - f_{tB} \right| \psi(t)\,dt \right) dx$$

$$= \int_{\mathbb{Z}_p^*} \left(\frac{1}{|B|} \int_B \left| f(tx) - f_{tB} \right| dx \right) \psi(t)\,dt = \int_{\mathbb{Z}_p^*} \left(\frac{1}{|tB|} \int_{tB} \left| f(x) - f_{tB} \right| dx \right) \psi(t)\,dt$$

$$\leqslant \|f\|_* \int_{\mathbb{Z}_p^*} \psi(t)\,dt, \tag{2.28}$$

which implies that U_ψ is bounded on $BMO(\mathbb{Q}_p^n)$ and $\|U_\psi\| \leqslant \int_{\mathbb{Z}_p^*} \psi(t)\,dt$.

Therefore, to complete the proof, it remains to show that $\int_{\mathbb{Z}_p^*} \psi(t)\,dt \leqslant \|U_\psi\|$. Now we use $g(x) = \log \|x\|$. We know that, from Lemma 2.1, $g \in BMO(\mathbb{Q}_p^n)$ and $\|g\|_{BMO} \neq 0$. Also, we get

$$U_\psi g(x) = g(x) \int_{\mathbb{Z}_p^*} \psi(t)\,dt + \int_{\mathbb{Z}_p^*} \log |t|_p \psi(t)\,dt. \tag{2.29}$$

Since $\int_{\mathbb{Z}_p^*} \psi(t)\,dt = U_\psi 1(x) < \infty$, the second integral of (2.29) has to be finite. Taking the BMO-norm on both sides of (2.29), we have

$$\|g\|_{BMO} \int_{\mathbb{Z}_p^*} \psi(t)\,dt = \left\| g \int_{\mathbb{Z}_p^*} \psi(t)\,dt \right\|_{BMO}$$

$$= \left\| g \int_{\mathbb{Z}_p^*} \psi(t)\,dt + \int_{\mathbb{Z}_p^*} \log |t|_p \psi(t)\,dt \right\|_{BMO}$$

$$= \|U_\psi g\|_{BMO} \leqslant \|U_\psi\| \|g\|_{BMO}. \tag{2.30}$$

From this we get $\int_{\mathbb{Z}_p^*} \psi(t)\,dt \leqslant \|U_\psi\|$, which completes the proof of Theorem 1.2. □

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