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Stability of Integral Manifold and Orbital Attraction of Quasi-periodic Motion

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1. INTRODUCTION

Consider

$$\begin{aligned} z' &= A(\theta, t, \lambda) z + F(z, \theta, t, \lambda) \\ \theta' &= Q(\theta, t, \lambda) + G(z, \theta, t, \lambda), \end{aligned} \qquad (1.1)_\lambda$$

where $(z, \theta, \lambda) \in R^n \times T^k \times R^m$, $F = O(|z|^2)$, $G = O(|z|)$, as $\lambda = 0$.

Assume that

$$z' = A(\hat{\theta}(t), t, 0) z, \qquad (1.2)$$

where $\hat{\theta}(t)$, the solution of $\theta' = Q(\theta, t, 0)$ with $\hat{\theta}(0) = \theta_0$, has an Exponential Dichotomy on R uniformly for $\theta_0 \in T^k$. Then it is shown in Yi [35] that $(1.1)_\lambda$ has for each "small" λ a unique integral manifold of type

$$S_\lambda = \{(f_\lambda(\theta, t), \theta, t) | \theta \in T^k, t \in R\} \qquad (1.3)_\lambda$$

and S_λ enjoys the same kind of "smoothness" as the original system $(1.1)_\lambda$.

In the classical integral manifold theories (see Hale [9, 10], Chow and Hale [4] for the cases that A is a constant matrix), it is known that the integral manifold S_λ looks locally like a "saddle node," that is, the stable and unstable manifolds of S_λ, W_λ^+, and W_λ^- exist and can be characterized by

$$W_\lambda^\pm = \{(z_0, \theta_0, \tau) | \text{ the solution } (\hat{z}, \hat{\theta}) \text{ of } (1.1)_\lambda \text{ through } (z_0, \theta_0) \text{ at time } \tau \text{ satisfies } |\hat{z} - f_\lambda(\hat{\theta}, t)| \to 0 \text{ "exponentially" as } t \to \pm\infty\}. \qquad (1.4)_\lambda$$

From this definition, we see that any "trajectory" $(\hat{z}(t), \hat{\theta}(t), t) \in W_\lambda^+$ (W_λ^-) (by trajectory, we mean that $(\hat{z}(t), \hat{\theta}(t))$ solves $(1.1)_\lambda$) is attracted (repelled)

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by a "curve" $(f_\lambda(\hat{\theta}(t), t), \hat{\theta}(t), t)$ on S_λ $((f_\lambda(\hat{\theta}(t), t), \hat{\theta}(t))$ is not necessarily a solution of $(1.1)_\lambda$), and hence is attracted (repelled) by S_λ.

We will show in Section 5 that the classical stability results hold true in our generalized case $(1.1)_\lambda$ and (1.2). However, compared to the classical theories, our stable (unstable) manifold W_λ^+ (W_λ^-) of S_λ is constructed in a different but equivalent way so that any "trajectory" on W_λ^+ (W_λ^-) is attracted (repelled) by a true "*trajectory*" on S_λ. That is, S_λ actually enjoys "orbital" stability (see Henry [13] for a similar result). To be more precise, we have the following theorem.

THEOREM. *Consider* $(1.1)_\lambda$, (1.2). *Assume that* A, Q *are* C^r $(r \geqslant 1)$ *in* θ, λ, *and* F, G *are* C^r in Z, θ, *and* λ *so that* A, Q, F, G *and all their partial derivatives are uniformly bounded and uniformly continuous on* $T^k \times R \times I$ *or* $E \times T^k \times R \times I$, *where* $I = \{\lambda \in R^n \mid |\lambda| \leqslant 1\}$ *and* $E \subset R^n$ *is an arbitrary compact subset. Let* δ_0 *be the smallest Lyapunov exponent of* (1.2) *in absolute value,* $L_0 =: \sup_{\theta \in T^k, t \in R} |\partial_\theta Q(\theta, t, 0)|$. *Suppose that* $L_0 < \delta_0/(r+1)$. *Then for any* ε_0, $0 < \varepsilon_0 < (\delta_0 - (r+1) L_0)/2$, *there is a* $\lambda_0 = \lambda_0(\varepsilon_0)$, $0 < \lambda_0 \leqslant 1$, *such that for each* $\lambda \in I_{\lambda_0} =: \{\lambda \in R^m \mid |\lambda| \leqslant \lambda_0\}$, *there are integral manifolds* $W_\lambda^+(S_\lambda)$, $W_\lambda^-(S_\lambda)$ *to* $(1.1)_\lambda$, *referred to as the stable and unstable manifolds of* S_λ. *Moreover, the following are satisfied*:

(1) $W_\lambda^\pm(S_\lambda) = \{(z, \theta, \tau) \in R^n \times T^k \times R \mid$ *solution* $(\hat{z}(t), \hat{\theta}(t))$ *of* $(1.1)_\lambda$ *through* (z, θ) *at time* τ *satisfies* $\sup_{t \in R^\pm} |\hat{z}(t) - f_\lambda(\hat{\theta}(t), t)|\, e^{\pm \eta t} < \infty$ *for some* $\eta \in [L_0 + \varepsilon_0, \delta_0 - \varepsilon_0]\} = \{(z, \theta, \tau) \in R^n \times T^k \times R \mid$ *there is a unique* $(f_\lambda(\theta_0, \tau), \theta_0, \tau) \in S_\lambda$ *such that* $\sup_{t \in R^\pm} |N_t^\lambda(z, \theta, \tau) - N_t^\lambda(f_\lambda(\theta_0, \tau), \theta_0, \tau)|\, e^{\pm \eta t} < \infty$ *for some* $\eta \in [L_0 + \varepsilon_0, \delta_0 - \varepsilon_0]\}$, *where* $N_t^\lambda(z, \theta, \tau)$ *is the solution of* $(1.1)_\lambda$ *with* $N_\tau^\lambda(z, \theta, \tau) = (z, \theta)$.

(2) $W_\lambda^\pm(S_\lambda)$ *are foliated by disjoint immersed and invriant* C^r *leaves* $W_\lambda^\pm(\theta_0, \tau) = \{(z, \theta, \tau) \in R^n \times T^k \times \{\tau\} \mid \sup_{t \in R^\pm} |N_t(z, \theta, \tau) = N_t^\lambda(f_\lambda(\theta_0, \tau), \theta_0, \tau)|\, e^{\pm \eta t} < \infty$ *for some* $\eta \in [L_0 + \varepsilon_0, \delta_0 - \varepsilon_0]\}$, *that is,* $W_\lambda^\pm(S_\lambda) = \bigcup_{\theta_0 \in T^k, \tau \in R} W_\lambda^\pm(\theta_0, \tau)$. *Moreover,* $W_\lambda^+(\theta_0, \tau)$, $W_\lambda^-(\theta_0, \tau)$ *are locally* C^r *diffeomorphic to the stable and unstable subspaces of*

$$z' = A(\theta, t, \lambda)\, e, \tag{1.4$_\lambda$}$$

respectively, where θ *satisfies*

$$\theta' = Q(\theta, t, \lambda) + G(f_\lambda(\theta, t), \theta, t, \lambda) \tag{1.5$_\lambda$}$$

and $\theta(\tau) = \theta_0$. *They also interest transversally at* $(f_\lambda(\theta_0, \tau), \theta_0, \tau) \in S_\lambda$ (*the "invariance" of leaves* $W_\lambda^\pm(\theta_0, z)$ *means if* $n_t(\theta_0, \tau)$ *is denoted as the solution of* $(1.5)_\lambda$ *through* θ_0 *at time* τ, *then* $N_t((W_\lambda^\pm(\theta_0, \tau)), t + \tau) = W_\lambda^\pm(n_t(\theta_0, \tau))$).

(3) $W_\lambda^\pm(\theta_0, \tau)$ *varies* C^{r-1} *smoothly in* λ *and* θ_0, C^1 *smoothly in* τ.

(4) *If* $(1.1)_\lambda$ *is autonomous,* S_λ, $W_\lambda^\pm(S_\lambda)=\bigcup_{\theta_0\in T^k} W_\lambda^\pm(\theta_0)$ *then reduce to invariant manifolds to the flow generated by* $(1.1)_\lambda$. *In this case,* $W_\lambda^\pm(\theta_0)$ *are* C^r *manifolds which vary* C^{r-1} *smoothly in* θ_0, λ; *hence* $W_\lambda^\pm(S_\lambda)$ *are* C^{r-1} *manifolds which vary* C^{r-1} *smoothly in* λ *(see* [5] *for a reason that the leaves* $W_\lambda^\pm(\theta_0,\tau)$ *vary only* C^{r-1} *continuously).*

As an application of our stability results, we discuss in Section 6 the so-called orbital stability of quasi-periodic motions. We consider

$$x'=f(x), \qquad x\in R^n, \tag{1.5}$$

where f is a C^2 function. Suppose that (1.5) has a quasi-periodic solution $\phi(t)\equiv q(\omega_1 t, ..., \omega_k t)$ with k frequencies $\omega_1, ..., \omega_k$ ($q\colon T^k\to R^n$), and that the Sacker–Sell spectrum (see Sect. 2) of the variational equation

$$y'=f_x(\phi(t))\,y \tag{1.6}$$

satisfies $\Sigma_N\subset(-\infty,0)$. Here Σ_N is referred to as the normal spectrum (see Sect. 2) of (1.6). Then $\{\phi(t)\}$ is of asymptotic orbital stability in the sense that there is an $\varepsilon>0$ such that if a solution $x(t)$ of (1.5) satisfies $|x(t_2)-q(\omega_1 t_1, ..., \omega_k t_1)|<\varepsilon$ for some $t_1, t_2\in R$, then there exist constants $h_1, ..., h_k$ (asymptotic phases) so that $\lim_{t\to+\infty}|x(t)-q(h_1+\omega_1 t, ..., h_k+\omega_k t)|=0$ exponentially. We then generalize the classical asymptotic orbital stability of periodic motions (see, for example, [2]).

Many difficulties arise when we deal with the present case rather than the classical one. First of all, like the usual idea of working with non-autonomous systems, we would like to work with the "hull" of $(1.1)_\lambda$ rather than $(1.1)_\lambda$ itself because of the needs of "compactness" and "uniformity" in our problems. These ideas were first introduced by G. Sell (see [32]). However, as far as the smoothness is concerned, it is convenient to use a modified idea as developed in Johnson [16] and Yi [35]. Following this idea, $(1.1)_\lambda$ then gives rise to a (nonlinear) skew-product flow Λ_λ on $R^n\times T^k\times\Omega_0$, where Ω_0 is a topological manifold with compact closure. S_λ therefore "becomes" an invariant topological manifold $\mathscr{S}_\lambda$ to Λ_λ. Once we determine the stable and unstable manifolds to $\mathscr{S}_\lambda$, $W_\lambda^\pm(S_\lambda)$ follows immediately as a special case.

Second, since the matrix A in $(1.1)_\lambda$ is time dependent, we need more detailed understanding about exponential dichotomy (ED) (see Coppel [3], Sacker and Sell [27], Johnson *et al.* [17]). For examples, the following play an important role in our proofs: smoothness of projections associated to ED (see Palmer [20], Yi [35]), control of ED constants (Yi [35]), Sacker–Sell spectrum [27, 28] and other related concepts to stability of linear systems.

Finally, the classical uniform contraction mapping principal (see

Hale [12]) does not apply when we prove our smoothness results. The smoothness problems are nontrivial even in autonomous cases (see Chow *et al.* [5], Lu [19] for nice arguments). To overcome it, we derive a generalized version of the uniform contraction mapping principal in Section 4 similar to Rybakowski [24] and Vanderbauwhede and Van Gils [34]. We remark here that our techniques also work for foliations of general invariant manifolds (for example, center stable manifolds) to differentiable systems.

The stability problems of invariant manifolds (sets) in dynamical systems have been studied by many authors. J. Hale [9] seems to be the first one to study the stability of integral manifolds. N. Fenichel [8] and Hirsch *et al.* [14] derived stable–unstable manifolds for general hyperbolic invariant sets of flows (continuous and discrete flows, respectively). G. Sell [30] and R. Johnson [16] proved existence of smooth spectrum subbundles (stable and unstable) which are more or less special cases of $(1.1)_\lambda$ without the θ variable. The stability of center manifolds has been discussed by J. Carr [1], Palmer [23], Chow and Hale [4], Sakamoto [29] (a singular perturbed system), etc. The work of Henry [13] in systems of FDE and the work of Chow *et al.* [5] in smooth invariant foliations is also of importance in this context. For more literature, see Chow and Hale [4], Hale [9], and Sell [33].

2. Preliminaries and Notations

(1) *Exponential Dichotomy* (ED). We say that the equation

$$x' = A(t)\,x, \qquad x \in R^n \tag{2.1}$$

has an ED on R, if there exists a projection $P\colon R^n \to R^n$ and positive constants K, δ such that

$$\begin{aligned} |\Phi(t)\,P\Phi^{-1}(s)| &\leqslant Ke^{-\delta(t-s)}, \qquad t \geqslant s,\\ |\Phi(t)(I-P)\,\Phi^{-1}(s)| &\leqslant Ke^{\delta(t-s)}, \qquad t \leqslant s, \end{aligned} \tag{2.2}$$

where $\Phi(t)$ is the fundamental matrix of (2.1) with $\Phi(0) = I$.

If (2.1) has ED on R, we define so-called *stable* and *unstable* subspaces (V^+, V^- resp.) by

$$V^{\pm} = \{\xi \in R^n \mid |\Phi(t)\,\xi| \to 0 \text{ as } t \to \pm\infty\};$$

then $V^+ = \text{Range } P$, $V^- = \text{Null } P$, and $R^n = V^+ \oplus V^-$.

(2) (*Linear*) *Skew-Product Flow*. Let X be a finite-dimensional vector space, and let Y be a Hausdorff space. Consider the trivial bundle space

$(X\times Y, Y, p)$ where p is the natural projection onto Y, $p^{-1}(y)=X_y=X\times\{y\}$.

A flow π on $X\times Y$ is said to be a *(linear) skew-product flow* if

$$\pi(x, y, t)=(\Phi(y, t)\, x, y\cdot t), \tag{2.3}$$

where $y\cdot t$ is a flow on Y, $\Phi(y, t)\colon X_y\to X_{y\cdot t}$ is linear.

It is easy to verify that (a) $\Phi(y, t)$ is jointly continuous in y and t, (b) $\Phi(y, 0)=I$, (c) $\Phi(y\cdot t, s)\,\Phi(y, t)=\Phi(y, t+s)$, (d) $\Phi(y, t)$ is nonsingular, $\Phi^{-1}(y, t)=\Phi(y\cdot t, -t)$, (e) $\dim X_y=\dim X_{y\cdot t}$, $\forall t\in R$.

We say that the skew-product flow (2.3) has an ED on Y, if there is a continuous family of projections $P(y)\colon X\to X$ and positive constants K, δ which are independent of $y\in Y$ such that

$$\begin{aligned} |\Phi(y, t)\, P(y)\, \Phi^{-1}(y, s)| &\leqslant Ke^{-\delta(t-s)}, & t&\geqslant s,\\ |\Phi(y, t)(I-P(y))\, \Phi^{-1}(y, s)| &\leqslant Ke^{\delta(t-s)}, & t&\leqslant s. \end{aligned} \tag{2.4}$$

Note that $P(y\cdot t)\,\Phi(y, t)=\Phi(y, t)\, P(y)$ holds true for any $y\in Y$, $t\in R$. In this case, the bundles

$$V^{\pm}=\{(x, y)\in X\times Y \mid |\Phi(y, t)\, x|\to 0, \text{ as } t\to\pm\infty\}$$

are well defined and invariant (under flow π). They are referred to as *stable* (with "+") and *unstable* (with "−") subbundles. Their corresponding fibers

$$V_y^{\pm}=\{x\in R^n \mid (x, y)\in V^{\pm}\} \tag{2.5}$$

are called *stable* and *unstable* subspaces.

Similar to (1), we have $V_y^+=\text{Range } P(y)$, $V_y^-=\text{Null } P(y)$, $X=V_y^+\oplus V_y^-$. $V_y^{\pm}$ are *invariant* in the sense that

$$\Phi(y, t)(V_y^{\pm})=V_{y\cdot t}^{\pm}.$$

(3) *Lyapounov Exponents.* Let $\pi(x, y, t)=(\Phi(y, t)\, x, y\cdot t)$ be a linear skew-product flow on $R^n\times Y$, where Y is a compact Hausdorff space. Let μ be an invariant probability measure on Y (it always exists by the Krylov–Bogoliubov Theorem). Then it has been proved in [17] that there is an invariant set $Y_\mu\subseteq Y$ with $\mu(Y_\mu)=1$ such that for any $y\in Y_\mu$ there exist $\lambda_1(y)>\lambda_2(y)>\cdots>\lambda_k(y)$ for some k $(1\leqslant k\leqslant n)$ and a decomposition $R^n=W_1(y)+\cdots+W_k(y)$, where $\{W_i\}$ are linearly independent and measurable subspaces, such that

$$\lim_{|t|\to\infty}\frac{1}{t}\ln|\Phi(y, t)\, x|=\lambda_j(y)$$

for every nonzero $x \in W_j(y)$, $j = 1, 2, ..., k$. If μ is ergodic, then λ_j's are independent of y. The $\lambda_j(y)$'s so defined are called *Lyapounov exponents* of π.

For the linear system

$$x' = A(t)\,x, \qquad x \in R^n \tag{2.6}$$

we denote $\Omega = \text{cl}(A_\tau)$ in compact open topology, where $A_\tau(t) \equiv A(t+\tau)$; then Ω is compact [32]. For each $\omega \in \Omega$, let $\Phi(\omega, t)$ be the fundamental matrix of

$$x' = \omega(t)\,x, \qquad x \in R^n, \tag{2.7}$$

such that $\Phi(\omega, 0) \equiv I$ (identity). Then

$$\pi(x, \omega, t) = (\Phi(\omega, t)\,x, \omega \cdot t) \tag{2.8}$$

defines a linear skew-product flow on $R^n \times \Omega$, where $(\omega \cdot t)(s) = \omega(t+s)$. We define the *Lyapounov exponents* of π in (2.8) as the *Lyapounov exponents* of Eq. (2.6). We shall agree to fix an invariant measure on Ω.

(4) *Sacker–Sell Spectrum.* Let

$$\pi_\lambda(x, y, t) = (e^{-\lambda t}\Phi(y, t)\,x, y \cdot t) \tag*{(2.9)$_\lambda$}$$

be the linear skew-product flow generated by

$$x' = (A(y \cdot t) - \lambda I)\,x, \qquad x \in R^n, \tag*{(2.10)$_\lambda$}$$

where $\Phi(y, t)$ is the fundamental matrix of

$$x' = A(y \cdot t)\,x \tag{2.11}$$

such that $\Phi(y, 0) \equiv I$.

For each $y \in Y$, define the *resolvent* $\rho(y)$ of (2.11) by

$$\rho(y) = \{\lambda \in R \mid \pi_\lambda \text{ has an ED on } R\},$$

and the *spectrum* of (2.11) by

$$\Sigma(y) = R \backslash \rho(y).$$

For any subset $M \subseteq Y$, define

$$\Sigma(M) = \bigcup_{y \in M} \Sigma(y)$$

$$\rho(M) = \bigcap_{y \in M} \rho(y) = \Sigma(M)^c.$$

The following results can be found in [25].

THEOREM 2.1 (Sacker and Sell). *Let $M \subseteq Y$ be an invariant connected compact subset. Then the spectrum $\Sigma(M)$ is a union of compact intervals, that is,*

$$\Sigma(M) = [a_1, b_1] \cup \cdots \cup [a_k, b_k] \qquad \textit{for some} \quad 1 \leqslant k \leqslant n.$$

THEOREM 2.2. (S-S Perturbation Theorem). *Denote $K(Y)$ the collection of all compact invariant subsets of Y. Let $M_0 \in K(Y)$.*

(1) *If $\lambda \in \rho(M_0)$, then there is an $\alpha > 0$ and a neighborhood V of M_0 in Y such that $M \in K(Y)$, $M \subseteq V$ implies $(\lambda - \alpha, \lambda + \alpha) \subseteq \rho(M)$.*

(2) *For every neighborhood U of $\Sigma(M_0)$, there is a neighborhood V of M_0 such that $M \in K(Y)$, $M \subseteq V$ implies $\Sigma(M) \subseteq U$.*

Note that we can define the S-S spectrum for general linear skew-product flow (2.3) in the same fashion.

Let V_N, V_T be the normal and tangential bundles of $\{0\} \times Y$ if they exist. Then, the restriction of π to V_N and V_T defines two linear skew-product flows π_N and π_T on V_N and V_T, respectively (see Sell [31], Sacker and Sell [28]); the S-S spectra Σ_N, Σ_T of π_N and π_T are hereby referred to as the *normal* and *tangential* spectra of (2.11). It is proved (Sell [31]) that $\Sigma \subset \Sigma_N \cup \Sigma_T$, and, if $\Sigma_N \cap \Sigma_T = \emptyset$ then $\Sigma = \Sigma_N \cup \Sigma_T$.

(5) *Notations.* (a) Let X be a Banach space. Define for any $\beta \in R$

$$X_{\beta+} = \{f: R^+ \to X \mid |f|_{\beta+} := \sup_{t \geqslant 0} |f(t)| \, e^{\beta t} < \infty\},$$

$$X_{\beta-} = \{f: R^- \to X \mid |f|_{\beta-} := \sup_{t \leqslant 0} |f(t)| \, e^{-\beta t} < \infty\}.$$

Then $X_{\beta\pm}$ are Banach spaces and there are continuous embeddings $X_{\beta\pm} \to X_{\alpha\pm}$ if $\beta \geqslant \alpha$.

(b) For Banach spaces $Y_1, ..., Y_p, Y, X$, we denote by $L(Y_1 \times Y_2 \times \cdots \times Y_p, Y)$ the Banach space of continuous p-linear mappings A: $Y_1 \times \cdots \times Y_p \to Y$. We also define $L^k(X, Y)$ inductively by $L^0(X, Y) = Y$, $L^{k+1}(X, Y) = L(X, L^k(X, Y))$ $(k = 0, 1, ...)$. We will identify $L^k(X, Y)$ with $L(X \underbrace{\times \cdots \times}_{k} X, Y)$ because of the isomorphism between them.

(c) As in [24], we denote by N^j the set of j-tuples of elements of $N = \{1, 2, 3, ...\}$. For any $v = (v(1), ..., v(j)) \in N^j$, let $|v| = \sum_{s=1}^{j} v(s)$. We define $N^0 = \emptyset$ and $|v| = 0$ for $v \in N^0$.

(d) All norm or absolute value symbols will have their obvious meaning unless specified otherwise.

3. Mappings on Weigted Norm Space

Consider

$$x' = A(y, t)\, x, \tag{3.1}$$

where $x \in R^n$, $y \in Y$, Y is a C^r Finsler manifold.

Assume that (3.1) has ED on R for each $y \in Y$ uniformly; that is, the ED constants δ, K are independent of $y \in Y$. We denote the family of ED projections by $P(y): R^n \to R^n$.

Let $\Phi(y, t)$ be the fundamental matrix of (3.1) satisfying $\Phi(y, 0) \equiv I$. We define mappings $S_\pm : Y \to L(R^n, X_{\rho\pm})$, $K_\pm : Y \to L(X_{\rho\pm}, X_{\rho\pm})$ for $0 \leqslant \rho < \delta$ as the following:

$$(S_+(y)\,\xi)(t) = \Phi(y, t)\, P(y)\, \xi \qquad (t \geqslant 0), \tag{3.2}$$

$$(S_-(y)\,\xi)(t) = \Phi(y, t)(I - P(y))\, \xi \qquad (t \leqslant 0), \tag{3.3}$$

for any $\xi \in R^n$;

$$(K_+(y)\,\phi)(t) = \int_0^t \Phi(y, t)\, P(y)\, \Phi^{-1}(y, s)\, \phi(s)\, ds$$
$$- \int_t^\infty \Phi(y, t)(I - P(y))\, \Phi^{-1}(y, s)\, \phi(s)\, ds \qquad (t \geqslant 0), \tag{3.4}$$

$$(K_-(y)\,\phi)(t) = -\int_0^t \Phi(y, t)(I - P(y))\, \Phi^{-1}(y, s)\, \phi(s)\, ds$$
$$+ \int_{-\infty}^t \Phi(y, t)\, P(y)\, \Phi^{-1}(y, s)\, \phi(s)\, ds \qquad (t \leqslant 0), \tag{3.5}$$

for any $\phi \in X_{\pm\rho}$.

It is easy to verify that

$$|S_\pm(y)\,\xi|_{\rho\pm} \leqslant K\,|\xi|, \tag{3.6}$$

and

$$|K_\pm(y)\,\phi|_{\rho\pm} \leqslant 2K(\delta - \rho)^{-1}\, |\phi|_{\rho\pm}; \tag{3.7}$$

hence, $S_\pm$, $K_\pm$ are well defined.

Our Theorem 3.4 will give (Lipschitz) continuity and smoothness results of the above mappings.

THEOREM 3.1. *Consider* (3.1).

(1) *If A is uniformly bounded on $Y \times R$ and uniformly continuous in y on $Y \times I$, where $I \subset R$ is an arbitrary compact subset, then P is uniformly continuous.*

(2) *If $A(y, t)$ is Lipschitz in y and there are constants $C > 0$, $N \in [0, \delta)$ which are independent of y and t such that* $\mathrm{Lip}_y A \leqslant Ce^{N|t|}$, *then P is uniform Lipschitz.*

(3) *If $A(y, t)$ is C^r $(r \geqslant 1)$ in y, and for any compact subset $I \subset R$, $\partial_y^p A(y, t)$ $(p = 1, 2, ..., r)$ are uniformly continuous in y on $Y \times I$, and moreover $|\partial_y^p A(y, t)| \leqslant Ce^{pN|t|}$ $(1 \leqslant p \leqslant r)$ for any $y \in Y$, $t \in R$, where $C > 0$, $N \in [0, \delta/(r+1))$ are constants which are independent of y, t, and p, then $P(y)$ is C^r and bounded (that is, its derivatives up to order p are continuous and bounded).*

To prove the theorem, we need the following lemma.

LEMMA 3.2 (Palmer). *Consider*

$$x' = A(t)\, x + f(t), \tag{3.8}$$

where $x \in R^n$ or $x \in M_n$ ($n \times n$ matrix space). Suppose that

$$z' = A(t)\, z, \qquad z \in R^n, \tag{3.9}$$

has ED on R with ED constants δ, K and projection P. Let $\beta \in (-\delta, \delta)$. If $|f|_{\beta+} < \infty$ $(|f|_{\beta-} < \infty)$, then (3.8) has for each $\xi \in R^n$ a unique solution $x_+(t)$ $(x_-(t))$ satisfying $|x_+|_{\beta+} < \infty$ $(|x_-|_{\beta-} < \infty)$, $Px_+(0) = P\xi$ $((I-P)\, x_-(0) = (I-P)\,\xi)$. Moreover,

$$|x_\pm|_{\beta\pm} \leqslant K\,|\xi| + 2K(\delta - |\beta|)^{-1}\,|f|_{\beta\pm}. \tag{3.10}$$

This lemma can be found in Palmer [20], *or Yi* [35].

Proof of Theorem 3.1. Parts (2), (3) are proved in Yi [35]. We now prove (1). For any $y_0, y \in Y$, consider the matrix equation

$$x' = A(y_0, t)\, x + f(t), \tag{3.11}$$

where $f(t) = [A(y, t) - A(y_0, t)]\, \Phi(y, t)\, P(y)$. By ED properties, then $f(t) \to 0$, as $t \to +\infty$ uniformly. It follows from our conditions upon A that for any $\varepsilon > 0$, there is $\delta := \delta(\varepsilon) > 0$ such that $d(y, y_0) < \delta$ implies $|f|_{0+} < \varepsilon$, where d is a metric on Y. Now $x^+(y, t) \equiv \Phi(y, t)\, P(y)$ is clearly a solution of (3.11) satisfying $P(y_0)\, x^+(y, 0) = P(y_0)\, P(y)$, $|x^+|_{0+} \leqslant K$. By Lemma 3.2, such a solution is unique. It is easy to verify also that $x^+(y, t)$ satisfies

$$x^+(y, t) = \Phi(y_0, t)\, P(y_0)\, P(y) + \int_0^t \Phi(y_0, t)\, P(y_0)\, \Phi^{-1}(y_0, s) f(s)\, ds$$

$$- \int_t^\infty \Phi(y_0, t)(I - P(y_0))\, \Phi^{-1}(y_0, s) f(s)\, ds.$$

Let $t=0$ in above. Then

$$P(y)=P(y_0)\,P(y)-\int_0^{\infty}(I-P(y_0))\,\Phi^{-1}(y_0,s)f(s)\,ds;$$

therefore,

$$|(I-P(y_0))\,P(y)|\leqslant\frac{K}{\delta}|f|_{0+}<\frac{K}{\delta}\varepsilon. \tag{3.12}$$

If we replace f in (3.11) by $f=[A(y,t)-A(y_0,t)]\,\Phi(y,t)(I-P(y))$ and apply Lemma 3.2 with "$-$" (that is, $t\leqslant 0$), then, a similar argument as above yields

$$|P(y_0)(I-P(y))|\leqslant\frac{K}{\delta}|f|_{0-}<\frac{K}{\delta}\varepsilon. \tag{3.13}$$

Combine (3.12), (3.13). Then

$$\begin{aligned}|P(y)-P(y_0)|&\leqslant|P(y_0)(I-P(y))|+|(I-P(y_0))\,P(y)|\\&<\frac{2K}{\delta}\varepsilon.\end{aligned}$$

Hence, P is uniformly continuous. ∎

LEMMA 3.3. *Consider* (3.1) *and define*

$$(J_+(y))(t)=-(I-P(y))\,\Phi^{-1}(y,t)\qquad(t\geqslant 0),$$

$$(J_-(y))(t)=P(y)\,\Phi^{-1}(y,t)\qquad(t\leqslant 0).$$

(1) *If the conditions in Theorem* 3.1(1) *hold, then* $J_\pm: Y\to X_{\rho\pm}$ $(0\leqslant\rho<\delta)$ *are continuous.*

(2) *If the conditions in Theorem* 3.1(2) *hold, then* $J_\pm: Y\to X_{\rho\pm}$ $(0\leqslant\rho\leqslant\delta-N)$ *are Lipschitz, and*

$$|J_\pm(y_1)-J_\pm(y_2)|_{\rho\pm}\leqslant M_1\,d(y_1,y_2) \tag{3.14}$$

$(0\leqslant\rho\leqslant\delta-N)$, *for any* $y_1, y_2\in Y$, *where* $M_1=KM_0+2CK^2/N$, $M_0\equiv \text{Lip}\,P$.

(3) *If the conditions in Theorem* 3.1(3) *hold, then* $J_\pm: Y\to X_{\rho\pm}$ *is* p*-times differentiable if* $0\leqslant\rho\leqslant\delta-(p+1)N$, *and is* C^p $(1\leqslant p\leqslant r)$ *and bounded if* $0\leqslant\rho<\delta-(p+1)N$. *Moreover, if* $J_\pm^p$ $(1\leqslant p\leqslant r)$ *are defined by taking* p*th derivatives on the right-hand side of* $J_\pm(y)$, *then* $J_\pm^p: Y\to L((TY)^p, X_{\rho\pm})$ *are well defined if* $0\leqslant\rho\leqslant\delta-pN$, *and are continuous if* $0\leqslant\rho<\delta-pN$.

Proof. The proofs of (2) and the first part of (3) can also be found in Yi [35]. Proofs of (1) and the second part of (3) are quite similar to the following proofs for the results of $S_\pm$. ∎

THEOREM 3.4. *Consider* (3.1) *with* $S_\pm, K_\pm$ *being defined as before.*

(1) *If the conditions in Theorem* 3.1(1) *hold, then* $S_\pm : Y \to L(R^n, X_{\rho\pm})$ *is continuous provided* $0 < \rho < \delta$, *and* $K_\pm : Y \to L(X_{\eta\pm}, X_{\rho\pm})$ *is continuous provided* $0 < \rho < \eta < \delta$.

(2) *If the conditions in Theorem* 3.1(2) *hold, then*

$$S_\pm : Y \to L(R^n, X_{\rho\pm}) \qquad (0 \leqslant \rho \leqslant \delta - N)$$
$$K_\pm : Y \to L(X_{\eta\pm}, X_{\rho\pm}) \qquad (N \leqslant \eta < \delta, 0 \leqslant \rho \leqslant \eta - N)$$

are uniformly Lipschitz.

(3) *If the conditions in Theorem* 3.1(3) *hold, then*

$$S_\pm : Y \to L(R^n, X_{\rho\pm}) \qquad (0 \leqslant \rho < \delta - (p+1) N)$$
$$K_\pm : Y \to L(X_{\eta\pm}, X_{\rho\pm}) \qquad ((p+1) N < \eta < \delta, 0 \leqslant \rho < \eta - (p+1) N)$$

are C^p $(1 \leqslant p \leqslant r)$ *and bounded.*

Moreover, for $p = 1, 2, ..., r$, *if* $S^p_\pm(y)$, $K^p_\pm(y)$ *are defined by taking pth derivatives with respect to* y *on the right-hand sides of* (3.2)–(3.5), *respectively, then*: $S^p_\pm : Y \to L(R^n \times (TY)^p, X_{\rho\pm})$ *are well defined if* $0 \leqslant \rho \leqslant \delta - pN$, *continuous if* $0 \leqslant \rho < \delta - pN$; *and,* $K^p_\pm : Y \to L(X_{\eta\pm} \times (TY)^p, X_{p\pm})$ *are well defined if* $(p+1) N \leqslant \eta < \delta$, $0 \leqslant \rho \leqslant \delta - pN$, *continuous if* $(p+1) N < \eta < \delta$, $0 \leqslant \rho < \delta - pN$.

Remark. We shall see from the proofs of the theorem that $S^p_\pm$, $K^p_\pm$ are just formal derivatives $D^pS_\pm$, $D^pK_\pm$ of $S_\pm, K_\pm$, respectively.

Proof. (1) For any $y_0, y \in Y$, $\xi \in R^n$, consider

$$x' = A(y_0, t) x + F(t), \tag{3.15}$$

where $F(t) = [A(y_0, t) - A(y, t)](S_+(y) \xi)(t)$. It is clear that $x^+(t) =: (S_+(y_0) \xi)(t) - (S_+(y) \xi)(t)$ is a solution of (3.15) such that $P(y_0) x^+(0) = P(y_0)(P(y_0) - P(y)) \xi$, $|x^+|_{\rho+} < \infty$ for $\rho \in [0, \delta)$. By Lemma 3.2, we have

$$|x^+|_{\rho+} \leqslant K |P(y_0) - P(y)| \cdot |\xi| + 2K(\delta - \rho)^{-1} |F|_{\rho+}. \tag{3.16}$$

For any $\varepsilon > 0$, let $T = T(\varepsilon)$ be chosen so that $2C_0K \cdot e^{-(\delta - \rho)T} < \varepsilon$, where $C_0 = \sup_{y \in Y, t \in R} |A(y, t)|$. For such T, since $A(y, t)$ is uniformly continuous in y on $Y\times[0, T]$, and $P(y)$ is continuous by Theorem 3.1, there is $\delta=\delta(\varepsilon)>0$ such that $d(y_0, y)<\delta$ implies $|P(y_0)-P(y)|<\varepsilon$, $|A(y_0, t)-A(y, t)|<\varepsilon/K$ for any $t\in[0, T]$. Therefore,

$$\begin{aligned}|F|_{\rho+}&=\sup_{t\geqslant 0}|F(t)|\,e^{\rho t}\\&\leqslant\max\{\sup_{0\leqslant t\leqslant T}|A(y_0, t)-A(y, t)|\cdot K\,|\xi|, 2C_0Ke^{-(\delta-\rho)T}\,|\xi|\}\\&<\varepsilon\,|\xi|.\end{aligned}$$

By (3.12),

$$|x^+|_{\rho+}\leqslant[K+2K(\delta-\rho)^{-1}]\,\varepsilon\,|\xi|,$$

that is, S_+ is continuous. Similarly, S_- is continuous.

To prove the continuity of K_+ in y, we notice that for any $y_0, y\in Y$, and any $\phi\in X_{\eta+}$ $(0\leqslant\eta<\delta)$, $w^+(t):=(K_+(y_0)\,\phi)(t)-(K_+(y)\,\phi)(t)$ satisfies

$$w'=A(y_0, t)\,w+F(t), \tag{3.17}$$

where $F=[(A(y_0, t)-A(y, t))](K_+(y)\,\phi)(t)$. Moreover,

$$P(y_0)\,w^+(0)=P(y_0)\cdot\left(\int_0^\infty[J_+(y_0)(s)-J_+(y)(s)]\,\phi(s)\,ds\right),$$

and $|w^+|_{\rho+}<\infty$. Since $K_+(y)\,\phi e^{\rho t}\to 0$ uniformly as $t\to+\infty$, where $0\leqslant\rho<\eta<\delta$, we can just apply Lemma 3.2, Lemma 3.3(1) and use the exact same arguments as before to prove the continuity of K_+. For K_-, the proofs are similar.

(2) For any $y_0, y\in Y$, $\xi\in R^n$, define $x^+(t)=(S_+(y_0)\,\xi)(t)-(S_+(y)\,\xi)(t)$. Then $x^+(t)$ satisfies (3.15) and (3.16). By the Lipschitz conditions on A, we have $|F|_{\rho+}\leqslant CK\,d(y_0, y)\,|\xi|$ $(0\leqslant\rho\leqslant\delta-N)$. Since P is Lipschitz by Theorem 3.1(2), (3.16) implies

$$|x^+|_{\rho+}\leqslant\left(KM_0+\frac{2CK^2}{N}\right)d(y_1, y_2)\,|\xi|,$$

where $M_0\equiv\operatorname{Lip}P$. Let $M_1\equiv KM_0+2CK^2/N$. Then

$$\|S_+(y_1)-S_+(y_2)\|_{L(R^n, X_{\rho+})}\leqslant M_1\,d(y_1, y_2). \tag{3.18}$$

Similarly,

$$\|S_-(y_1)-S_-(y_2)\|_{L(R^n, X_{\rho-})}\leqslant M_1\,d(y_1, y_2). \tag{3.19}$$

To prove the results for $K_+(y)$, we consider Eq. (3.17) for any $y_0, y \in Y$, and any $\phi \in X_{\eta+}$ $(N \leqslant \eta < \delta)$. Because of (3.7), we have

$$|F|_{\rho+} \leqslant 2CK(\delta-\eta)^{-1} |\phi|_{\eta+} d(y_0, y), \tag{3.20}$$

where $0 \leqslant \rho \leqslant \eta - N$. By (3.14)

$$|J_\pm(y_0) - J_\pm(y)|_{(\phi-N)\mp} \leqslant M_1 d(y_0, y). \tag{3.21}$$

It follows that

$$\left|\int_0^\infty [J_+(y_0)(s) - J_+(y)(s)]\, \phi(s)\, ds\right| \leqslant M_1(\delta - N + \eta)^{-1} |\phi|_{\eta+} d(y_0, y). \tag{3.22}$$

Let $w^+(t) = (K_+(y_0)\phi)(t) - (K_+(y)\phi)(t)$. Then, $w^+(t)$ is a solution of (3.17) satisfying $|w^+|_{\rho+} < \infty$ and $P(y_0)\, w^+(0) = P(y_0)\, (\int_0^\infty [J_+(y_0)(s) - J_+(y)(s)]\, \phi(s)\, ds)$. By Lemma 3.2, such a solution is unique. Combining (3.10), (3.18), (3.20), we have

$$|w^+|_{\rho+} \leqslant M_2\, d(y_0, y)\, |\phi|_{\eta+}, \tag{3.23}$$

where $M_2 = KM_1(\delta - N + \eta)^{-1} + 4CK^2(\delta-\rho)^{-1}(\delta-\eta)^{-1}$. Therefore,

$$\| K_+(y_0) - K_+(y)\|_\infty \leqslant M_2\, d(y_0, y). \tag{3.24}$$

Similarly,

$$\| K_-(y_0) - K_-(y)\|_\infty \leqslant M_2\, d(y_0, y) \tag{3.25}$$

for any $y_0, y \in Y$.

(3) To simplify the proofs, for any $y \in Y$, we identify an open neighborhood U of y with an open set in the Banach space V associated to Y.

For any $h \in V$, such that $y + h \in V$, and any $\xi \in R^n$, we consider

$$Z' = A(y, t)\, Z + \partial_y A(y, t)\, h(S_+(y)\, \xi)(t). \tag{3.26}$$

It is clear that $|\partial_y A(y, \cdot)\, h(S_+(y)\,\xi)(t)|_{\rho_0+} \leqslant CK\,|h|\,|\xi|$ for $0 \leqslant \rho_0 \leqslant \delta - N$. Let $Z^+(y, h, \xi)(t)$ be the solution of (3.26) such that $P(y)\, Z^+(y, h, \xi)(0) = P(y)\, DP(y)\, h\xi$ and $|Z^+|_{\rho_0+} < \infty$. Lemma 3.2 implies that $|Z^+|_{\rho_0+} \leqslant M_3\, |h|\,|\xi|$ for some constant $M_3 > 0$. Since Z^+ is linear in h, ξ, if we define $Y^+(y)$ by $Z^+(y, h, \xi) = Y^+(y)\, h\xi$, then

$$\| Y^+(y)\| \leqslant M_3. \tag{3.27}$$

Next, we consider the equation

$$W' = A(y, t)\, W + f(t), \tag{3.28}$$

where $f(t) = [A(y+h, t) - A(y, t) - \partial_y A(y, t)\, h](S_+(y+h)\,\xi)(t) + \partial_y A(y, t)\, h[(S_+(y+h)\,\xi)(t) - (S_+(y)\,\xi)(t)]$. For $\rho_1 \in [0, \delta - 2N]$, since $e^{-(\delta-\rho_1)t}\, |\partial_y A(y, t)| \to 0$ uniformly as $t \to +\infty$, then

$$\begin{aligned} &|[A(y+h, t) - A(y, t) - \partial_y A(y, t)\, h](S_+(y)\,\xi)(t)|_{\rho_1 +} \\ &\quad \leqslant K\, |h|\, |\xi| \sup_{t \geqslant 0} \int_0^1 e^{-(\delta-\rho_1)t}\, |\partial_y A(y+sh, t) - \partial_y A(y, t)|\, ds \\ &\quad = o(|h|)\, |\xi|. \end{aligned} \tag{3.29}$$

Now (3.29) together with (3.18) then implies $|f|_{\rho_1 +} = o(|h|)$. Let $W^+(t) = (S_+(y+h)\,\xi)(t) - (S_+(y)\,\xi)(t) - Y^+(y)(h\xi)(t)$. Then $W^+(t)$ is a solution of (3.28) such that $|W^+|_{\rho_1 +} < \infty$ and $P(y)\, W^+(0) = P(y)(P(y+h) - P(y) - DP(y)\, h)\, \xi$. By Lemma 3.2, Theorem 3.1(3), we have $|W^+|_{\rho_1 +} = o(|h|)\, |\xi|$, that is, $S_+ : Y \to L(R^n, X_{\rho_1 +})$ is differentiable with $DS_+(y) = Y^+(y) \in L(T_y Y \times R^n, X_{\rho_1 +})$.

We now prove that $DS_+(y) \in L(T_y Y \times R^n, X_{\rho +})$ is continuous if $0 \leqslant \rho < \delta - 2N$. To do so, for any h, $v \in V$, and any $\xi \in R^n$ we define $Y(t) = Y^+(y+h)\, v\xi(t) - Y^+(y)\, v\xi(t) \equiv Z^+(y+h, v, \xi)(t) - Z^+(y, v, \xi)(t)$. Then, $Y(t)$ satisfies

$$Y' = A(y, t)\, Y + g(t), \tag{3.30}$$

where

$$\begin{aligned} g(t) &= [A(y+h, t) - A(y, t)]\, Z^+(y+h, v, \xi)(t) \\ &\quad + [\partial_y A(y+h, t) - \partial_y A(y, t)]\, v(S_+(y)\,\xi)(t) \\ &\quad + \partial_y A(y+h, t)\, v[(S_+(y+h)\,\xi)(t) - (S_+(y)\,\xi)(t)]. \end{aligned}$$

For any $\rho \in [0, \delta - 2N)$, take $\rho_1 \in (\rho + N, \delta - N)$. We have

$$\begin{aligned} &|[A(y+h, \cdot) - A(y, \cdot)]\, Z^+(y, v, \xi)|_{\rho +} \\ &\quad \leqslant (\sup_{t \geqslant 0} e^{-(\rho_1 - \rho)t}\, |A(y+h, t) - A(y, t)|)\, |Z^+(y, v, \xi)|_{\rho_1 +}, \\ &|[\partial_y A(y+h, \cdot) - \partial_y A(y, \cdot)]\, v(S_+(y)\,\xi)|_{\rho +} \\ &\quad \leqslant K\, |v|\, |\xi| \sup_{t \geqslant 0} e^{-(\delta-\rho)t}\, |\partial_y A(y+h, t) - \partial_y A(y, t)|, \end{aligned}$$

and

$$|\partial_y A(y+h,\cdot)\, v[S_+(y+h)\,\xi - S_+(y)\,\xi]|_{\rho+}$$
$$\leqslant C\,|v|\,|\xi|\,\|S_+(y+h)-S_+(y)\|_{L(R^n,\, X_{(\delta-N)+})}$$
$$\leqslant CM_1\,|v|\,|\xi|\,|h|.$$

Notice also $\rho_1-\rho>N$, $\delta-\rho>N$. We see that $\forall\,\varepsilon>0$, $\exists\,\delta>0$ such that $|h|<\delta$ implies

$$|g|_{\rho_+}<\varepsilon\,|v|\,|\xi|. \tag{3.31}$$

Again, by Lemma 3.2, Theorem 3.1(3), and the facts that $|Y|_{\rho+}<\infty$, $P(y)\,Y(0)=P(y)(DP(y+h)-DP(y))\,v\xi$, we have

$$|Y|_{\rho+}\leqslant M_4\varepsilon\,|v|\,|\xi| \qquad \text{as} \quad |h|<\delta_0,$$

where $M_4>0$ is a constant, $\delta_0=\min\{\delta,\varepsilon\}$. Therefore $DS_+=Y^+$ is continuous.

We claim that $S^1_+=Y^+: Y\to L(R^n\times TY, X_{\rho_0+})$ is well defined for $0\leqslant\rho_0\leqslant\delta-N$. For any $h, y\in A$, $\xi\in R^n$, let $Z^+(y,h,\xi)(t)\equiv(Y^+(y)\,h\xi)(t)\in X_{\rho_0+}$ be defined by (3.26). Since $S_+: A\to L(R^n, X_{\rho_1})$ $(0\leqslant\rho_1\leqslant\delta-2N)$ is differentiable with $DS_+(y)=Y^+(y)\in L(R^n\times TY, X_{\rho_1+})$, then $(DS_+(y)\,h\xi)(0)\equiv(Y^+(y)\,h\xi)(0)$. However, by definition, $(DS_+(y)\,h\xi)(0)=(S^1(y)\,h\xi)(0)=DP_+(y)\,h\xi$. It follows that $(S^1_+(y)\,h\xi)(0)\equiv(Y^+(y)\,h\xi)(0)$. Since $(S^1_+(y)\,h\xi)(t)$ solves (3.26), by uniqueness, then $(S^1_+(y)\,h\xi)(t)\equiv(Y^+(y)\,h\xi)(t)$, that is, $S^1_+(y)\equiv Y^+(y)\in L(R^n\times TY, X_{\rho_0+})$ for $0\leqslant\rho_0\leqslant\delta-N$. The continuity of $S^1_+\equiv Y^+: Y\to L(R^n\times TY, X_{\rho_0})$ with $0\leqslant\rho_0<\delta-N$ follows simply from (3.26) and arguments in (1).

By similar arguments, we can prove the results for S_-, S^1_-.

To prove the results for $K_+(y)$, we consider for each $\phi\in X_{\eta+}$ $(2N\leqslant\eta<\delta)$, and each y, $h\in V$ the equation

$$W'=A(y,t)\,W+\partial_y A(y,t)\,h(K_+(y)\,\phi)(t). \tag{3.32}$$

For $\rho_0\in[0,\eta-N]$, let $W^+(t)=W(y,h,\phi)(t)$ be a solution of (3.32) with properties that $|W^+|_{\rho_0}<\infty$, $P(y)\,W^+(0)=P(y)\int_0^\infty J^1_+(y)(t)\,h\phi(t)\,dt$. Since $|\partial_y A(y,\cdot)\,hK_+(y)\,\phi|_{\rho_0+}\leqslant 2CK(\delta-\eta)^{-1}\,|h|\,|\phi|_{\eta+}$, by Lemma 3.2, such a solution $W^+(t)$ exists and is unique. Furthermore, by Lemmas 3.2 and 3.3, we have

$$|W^+|_{\rho_0+}\leqslant M_5\,|h|\,|\phi|_{\eta+} \tag{3.33}$$

for some constant $M_5>0$. Note that W^+ is linear in h and ϕ. We hereby define $K^*(y): T_yY\to L(X_{\eta+}, X_{\rho_0+})$ by $W^+=K^*(y)\,h\phi$. Then $\|K^*(y)\|\leqslant M_5$. Consider now $V(t)=(K_+(y+h)\,\phi)(t)-(K_+(y)\,\phi)(t)-K^*(y)\,h\phi(t)$.

For $2N<\eta<\delta$, $\rho_1\in[0,\eta-2N]$ then $|V|_{\rho_1+}<\infty$ and $P(y)\,V(0)=P(y)\,\xi$, where $\xi=\int_0^\infty [J_+(y+h)(t)-J_+(y)(t)-\partial_y J_+(y)\,h]\,\phi(t)\,dt$. Furthermore, $V(t)$ satisfies

$$V'=A(y,t)\,V+H(t), \tag{3.34}$$

where

$$\begin{aligned}H(t)=&[A(y+h,t)-A(y,t)-\partial_y A(y,t)\,h](K_+(y+h)\,\phi)(t)\\&+\partial_y A(y,t)\,h[(K_+(y+h)\,\phi)(t)-(K_+(y)\,\phi)(t)].\end{aligned}$$

By Using Lemma 3.3, parts (1) or (2) of this theorem, we can easily see that $|\xi|=o(|h|)\,|\phi|_{\eta+}$, $|H|_{\rho_1+}=o(|h|)\,|\phi|_{\eta+}$. It then follows from Lemma 3.2 that $|V|_{\rho_1+}=o(|h|)\,|\phi|_{\eta+}$, that is, $\|K_+(y+h)-K_+(y)-K^*(y)\,h\|_\infty=o(|h|)$. Hence, $K_+(y)$ is differentiable with $DK_+(y)=K^*(y)\in L(T_yY\times X_{\eta+},X_{\rho_1+})$ $(2N<\eta<\delta,\ 0\leqslant\rho_1\leqslant\eta-2N)$.

To prove the continuity of $DK_+(y)=K^*(y)\in L(T_yY\times X_{\eta+},X_{\rho+})$ for $2N<\eta<\delta$, $0\leqslant\rho<\eta-2N$, we note that for any h, $v\in V$ and any $\phi\in X_{\eta+}$, $W(t)=: K^*(y+h)\,v\phi(t)-K^*(y)\,v\phi(t)$ satisfies

$$W'=A(y,t)\,W+g(t), \tag{3.35}$$

where

$$\begin{aligned}g(t)=&[A(y,t)-A(y+h,t)](K^*(y+h)\,v\phi(t))\\&+[\partial_y A(y,t)-\partial_y A(y+h,t)]\,vK_+(y)(\phi)(t)\\&+\partial_y A(y+h,t)\,v[K_+(y)-K_+(y+h)](\phi)(t);\end{aligned}$$

moreover, $|W|_{\rho+}<\infty$, $P(y)\,W(0)=P(y)\int_0^\infty[\partial_y J_+(y+h)(t)-\partial_y J_+(y)(t)]\,v\phi(t)\,dt$. For any $\varepsilon>0$, we first observe that there is $\delta>0$ such that $|g|_{\rho+}<\varepsilon\,|v|\,|\phi|_{\eta+}$ as $|h|<\delta$. The continuity of K^* then follows from Lemma 3.2 and Lemma 3.3 by using arguments similar to those above. Similar to arguments for S^1_+, one can show that $K^1_+\equiv K^*: Y\to L(X_{\eta+}\times TY,\ X_{\rho+})$ is well defined if $2N\leqslant\eta<\delta$, $0\leqslant\rho\leqslant\eta-N$, and continuous if $2N<\eta<\delta$, $0\leqslant\rho<\eta-N$. The results for K_- can also be proved in the same fashion.

For $1<p\leqslant r$, one is concerned only with elaborations of the above arguments and applications of inductions on p. We omit the details. ▮

LEMMA 3.5. *Let $f: R^n\to R^n: (x,t)\mapsto f(x,t)$ be C^r in x. Assume that $f(0,t)\equiv 0$, and that $\partial_x^p f(x,t)$ $(1\leqslant p\leqslant r)$ is uniformly continuous and uniformly bounded in x on $K\times R$ for any compact subset $K\subset R^n$. Define $G: X_{\rho\pm}\to X_{\rho\pm}$ $(\rho\geqslant 0)$ by $G(\phi)(t)=f(\phi(t),t)$. Then, G is C^p $(1\leqslant p\leqslant r)$ and bounded.*

Proof. We only prove the case when $p=1$. The case $1<p\leqslant r$ can be carried out by the same arguments.

Take $\rho\geqslant 0$. For any $\phi\in X_{\rho+}$, let $K_0\subset R^n$ be a compact set which contains the set $\{\phi_*(t)\mid t\in R,\ \sup_{t\geqslant 0}|\phi_*-\phi|\leqslant 1\}$. For any $\varepsilon>0$, by uniform continuity of f in x on $K_0\times R$, we can find δ, $0<\delta<1$, such that $|\partial_x f(x,t)-\partial_x f(x_0,t)|<\varepsilon$ for any $t\in R$ and any x, $x_0\in K_0$ with $|x-x_0|<\delta$.

Now, let $\phi_0\in X_{\rho+}$, $|\phi_0|_{\rho+}<\delta$. Then $|\phi_0(t)|<\delta$ for any $t\in R$. Hence $|\partial_x f(\phi(t)+s\phi_0(t),t)-\partial_x f(\phi(t),t)|<\varepsilon$ for any $t\in R^+$, $s\in[0,1]$. Therefore, $|G(\phi+\phi_0)-G(\phi)-\partial_x f(\phi,\cdot)\,\phi_0|_{\rho+}\leqslant \sup_{t\geqslant 0} e^{\rho t}\int_0^1|\partial_x f(\phi(t)+s\phi_0(t),t)-\partial_x f(\phi(t),t)|\,|\phi_0(t)|\,ds<\varepsilon\,|\phi_0|_{\rho+}$. This implies that G is differentiable with $DG(\phi)=\partial_x f(\phi,\cdot)\in L(X_{\rho+},X_{\rho+})$. The continuity of $DG(\phi)$ follows from the uniform continuity of $\partial_x f(\phi,\cdot)$. Similarly, $G\colon X_{\rho-}\to X_{\rho-}$ $(\rho\geqslant 0)$ is C^1 and bounded. ∎

LEMMA 3.6. *Let Σ be a locally compact Finsler manifold with metric d. Consider $f\colon R^n\times\Sigma\times R\to R^n : (x,\sigma,t)\mapsto f(x,\sigma,t)$. Assume that $f(0,\sigma,t)\equiv 0$, and that, for any compact set $K\subset R^n$, $\partial_x f$ exists and is bounded on $K\times\Sigma\times R$. Define $G\colon X_{\rho\pm}\times\Sigma\to X_{\rho\pm}$ $(\rho\geqslant 0)$ by $G(\phi,\sigma)(t)=f(\phi(t),\sigma,t)$. If f continuous, then $G\colon X_{\rho\pm}\times\Sigma\to X_{\eta\pm}$ $(0\leqslant\eta<\rho)$ is continuous.*

Proof. Let ρ,η be fixed so that $0\leqslant\eta<\rho$. For any $\phi_0\in X_{\rho+}$ and $\sigma_0\in\Sigma$, denote by $K_0\subset R^n$ the compact set which contains $\{\phi_*(t)\mid t\in R,\ \sup_{t\geqslant 0}|\phi_*-\phi_0|\leqslant 1\}$. Denote $\Sigma_0=\{\sigma\in\Sigma\mid d(\sigma,\sigma_0)\leqslant 1\}$. For any $\varepsilon>0$, we first choose $R_0>0$ such that

$$2\sup_{\substack{x\in K_0\\ \sigma\in\Sigma_0\\ t\in R}}|\partial_x f(x,\sigma,t)|\,(|\phi_0|_{\rho+}+1)\,e^{-(\rho-\eta)R_0}<\varepsilon.$$

Since f is uniformly continuous on $K_0\times\Sigma_0\times[0,R_0]$, there is δ, $0<\delta<1$, with the property that if $x_1,x_2\in K_0$, $|x_1-x_2|<\delta$, and if $\sigma_1,\sigma_2\in\Sigma_0$, $d(\sigma_1,\sigma_2)<\delta$, then $|f(x_1,\sigma_1,t)-f(x_2,\sigma_2,t)|<e^{-\eta R}\varepsilon$ for any $t\in[0,R_0]$. Now, for any $\phi\in X_{\rho+}$, $\sigma\in\Sigma$ such that $|\phi-\phi_0|_{\rho+}<\delta$, $d(\sigma,\sigma_0)<\delta$, we have $|\phi(t)-\phi_0(t)|<\delta$ for any $t\in R$. Therefore,

$$\begin{aligned}|G(\phi,\sigma)-G(\phi_0,\sigma_0)|_{\eta+}&=\sup_{t\geqslant 0}e^{\eta t}\,|f(\phi(t),\sigma,t)-f(\phi_0(t),\sigma_0,t)|\\&\leqslant\max\Big\{\sup_{t\in[0,R_0]}e^{\eta R_0}\,|f(\phi(t),\sigma,t)-f(\phi_0(t),\sigma_0,t)|,\\&\qquad 2\sup_{\substack{x\in K_0\\ \sigma\in K_0\\ t\in R}}|\partial_x f(x,\sigma,t)|\,(|\phi_0|_{\rho+}+1)\,e^{-(\rho-\eta)R_0}\Big\}<\varepsilon.\end{aligned}$$

Similarly, $G\colon X_{\rho-}\times\Sigma\to X_{\eta-}$ $(0\leqslant\eta<\rho)$ is continuous. ∎

LEMMA 3.7. *Let Σ be a C^r Finsler manifold. Consider $f: R^n \times \Sigma \times R \to R^n : (x, \sigma, t) \mapsto f(x, \sigma, t)$. Assume that $f(0, \sigma, t) \equiv 0$, f is C^r in x, σ, and, for compact sets $K \subset R^n$, $I \subset R$, f as well as all of its partial derivatives are uniformly continuous on $K \times \Sigma \times I$. Assume further that there are constants $N \geqslant 0$, $C \geqslant 0$ such that for all integers $i \geqslant 0$, $j \geqslant 0$ with $1 \leqslant i + j \leqslant r$,*

$$|\partial_x^i \partial_\sigma^j f| \leqslant \begin{cases} C\,|x|\,e^{jN|t|} & (i=0), \\ Ce^{jN|t|} & (i>0). \end{cases} \tag{3.36}$$

(1) *Define $G: X_{\rho\pm} \times \Sigma \to X_{\eta\pm}$ by $G(\phi, \sigma)(t) = f(\phi(t), \sigma, t)$. If $\eta < \rho - pN$, then $G: X_{\rho\pm} \times \Sigma \to X_{\eta\pm}$ is C^p and bounded, where $p = 1, 2, ..., r$.*

(2) *For any integer $i \geqslant 0$, $j \geqslant 0$ with $1 \leqslant i + j \leqslant r$ and any $k_1, k_2, ..., k_i \geqslant 0$, define $G^{i,j}: X_{\rho\pm} \times \Sigma \to L(\prod_{s=1}^i X_{k_s\pm} \times (T\Sigma)^j, X_{\eta_1\pm})$ by $G^{i,j}(\phi_0, \sigma) = \partial_x^i \partial_\sigma^j f(\phi_0(\cdot), \sigma, \cdot)$. Then, $G^{i,j}$ is well defined if*

$$\eta_1 \leqslant \begin{cases} \rho - jN & (i=0) \\ \sum_{s=1}^i k_s - jN & (i>0); \end{cases}$$

it is continuous if

$$\eta_1 < \begin{cases} \rho - jN & (i=0) \\ \sum_{s=1}^i k_s - jN & (i>0). \end{cases}$$

Proof. For simplicity, we only do the case when $i = 0, j = 1$. The general case follows simply by similar arguments.

(1) We want to show that $G: X_{\rho\pm} \times \Sigma \to X_{\eta\pm}$ is C^p $(1 \leqslant p \leqslant r)$ with $\partial_\phi^i \partial_\sigma^j G(\phi, \sigma) = \partial_x^i \partial_\sigma^j f(\phi(\cdot), \sigma, \cdot) \in L(X_{\rho\pm}^i \times (T\Sigma)^j, X_{\eta\pm})$ if $\eta < \rho - pN$, $i + j = p$. As before, for any $\sigma \in \Sigma$, we identify an open neighborhood U of σ with an open set V of the Banach space equipped to Σ. Now, for any $h \in V$ such that $\sigma + h \in V$, observe by (3.36) that if $\eta < \rho - N$, then

$$\begin{aligned} &|G(\phi, \sigma + h) - G(\phi, \sigma) - \partial_\sigma f(\phi, \sigma, \cdot)\,h|_{\eta\pm} \\ &\leqslant \sup_{t \in R^\pm} \int_0^1 |\partial_\sigma f(\phi, \sigma + sh, t) - \partial_\sigma f(\phi, \sigma, t)|\, e^{\pm\eta t}\,|h|\,ds \\ &\leqslant |h| \max\left\{ \int_0^1 \sup_{t \in [-T, T]} |\partial_\sigma f(\phi, \sigma + sh, t) \right. \\ &\quad \left. - \partial_\sigma f(\phi, \sigma, t)| e^{\eta|t|},\ 2C\,|\phi|_{\rho\pm}\, e^{-(\rho - N - \eta)|T|} \right\} \qquad \text{for any } T \in R. \end{aligned}$$

It then follows from the uniform continuity of $\partial_\sigma f$ on $\overline{\text{Range}\{\phi\}} \times \Sigma \times [-T, T]$ that

$$|G(\phi, \sigma + h) - G(\phi, \sigma) - \partial_\sigma f(\phi, \sigma, \cdot)\,h|_{\eta\pm} = o(|h|),$$

that is, G is differentiable in σ with $\partial_\sigma G(\phi, \sigma) = \partial_\sigma f(\phi, \sigma, \cdot)$. The continuity of $\partial_\sigma G$ follows from the uniform continuity assumptions on f and the above arguments.

(2) If $\eta \leqslant \rho - jN$, then for any ϕ, $\phi_0 \in X_{\rho+}$, any σ, $\sigma_0 \in \Sigma$, and any $h \in (T\Sigma)^j$, we have

$$
\begin{aligned}
|G^{0,j}(\phi, \sigma) h|_{\eta_1 +} &= \sup_{t \geqslant 0} e^{\eta_1 t} |\partial_\sigma^j f(\phi(t), \sigma, t) h| \\
&\leqslant C |\phi|_{\rho+} |h| \sup_{t \geqslant 0} e^{-(\rho - \eta_1 - jN)t} \\
&\leqslant C |\phi|_{\rho+} |h|,
\end{aligned}
\tag{3.37}
$$

and if $\eta_1 < \rho - jN$, then we also have

$$
\begin{aligned}
&|[G^{0,j}(\phi, \sigma) - G^{0,j}(\phi_0, \sigma_0)] h|_{\eta_1 +} \\
&\quad \leqslant |h| \max\{ \sup_{t \in [0, T]} e^{\eta_1 T} |\partial_0^j f(\phi(t), \sigma, t) \\
&\qquad - \partial_0^j f(\phi_0(t), \sigma_0, t)|, 2C |\phi|_{\rho+} e^{-(\rho - \eta_1 - jN)T}\}
\end{aligned}
\tag{3.38}
$$

for any $T > 0$.

By using the arguments in Lemma 3.6, we see that $G^{0,j}: X_{\rho\pm} \times \Sigma \to L(X^i_{\eta\pm} \times (T\Sigma)^j, X_{\eta_1\pm})$ is well defined if $\eta_1 \leqslant \rho - jN$, and continuous if $\eta_1 < \rho - jN$. ∎

4. Uniform Contractions on Scale of Banach Spaces

We give in this section generalizations of the classical uniform contraction mapping principal.

Lemma 4.1. *Consider Banach spaces Y_0, Y such that $Y_0 \xrightarrow{J} Y$ is a continuous embedding. Let Λ be a Finsler manifold. Assume that:*

(1) *$T: Y \times \Lambda \to Y$ is a uniform contraction, that is, there is a $0 \leqslant \theta < 1$ such that $|T(y, \lambda) - T(y_1, \lambda)| \leqslant \theta |y - y_1|$ for any y, $y_1 \in Y$ and any $\lambda \in \Lambda$. Let $y: \Lambda \to Y$ be the fixed point of T. There is $y_0: \Lambda \to Y_0$ such that $y(\lambda) = Jy_0(\lambda)$.*

(2) *The mapping $T_0: Y_0 \times \Lambda \to Y: (y_0, \lambda) \mapsto T(Jy_0, \lambda)$ is (Lipschitz) continuous in λ uniformly in y_0.*

Then $y: \Lambda \to Y$ is (Lipschitz) uniformly continuous.

Proof. We only prove the uniform continuity part. For any $\varepsilon > 0$, because of (2), there is a $\delta > 0$, such that $|T_0((y_0, \lambda_0), \lambda) - T_0(y_0(\lambda_0), \lambda_0)| < (1-\theta)\varepsilon$ provided that $d(\lambda, \lambda_0) < \delta$. Here d is a fixed metric on Λ.

Now, for any $\lambda, \lambda_0 \in \Lambda$, such that $d(\lambda, \lambda_0) < \delta$,

$$\begin{aligned} |y(\lambda) - y(\lambda_0)| &= |T(y(\lambda), \lambda) - T(y(\lambda_0), \lambda_0)| \\ &\leqslant |T(y(\lambda), \lambda) - T(y(\lambda_0), \lambda)| \\ &\quad + |T_0(y_0(\lambda_0), \lambda) - T_0(y_0(\lambda_0), \lambda_0)| \\ &\leqslant \theta\, |y(\lambda) - y(\lambda_0)| + (1-\theta)\varepsilon; \end{aligned}$$

hence, $|y(\lambda) - y(\lambda_0)| < \varepsilon$. That is, $y\colon \Lambda \to Y$ is uniformly continuous. ∎

THEOREM 4.2. *Let $r \geqslant 1$ be an integer. Consider Banach spaces $Y, Y_1, ..., Y_r, Y_{d1}, Y_{d2}, ..., Y_{dr}$ such that $Y \to Y_1$, $Y_i \to Y_{i+1}$, $Y_i \to Y_{di}$, $Y_{di} \to Y_{d(i+1)}$, $i = 1, 2, ..., r$, are continuously embedded ($Y_{d(r+1)} := Y_{dr}$). We denote by $J_{a,b}\colon Y_a \to Y_b$, $J_b\colon Y \to Y_b$ the embedding operators for $a, b \in \{1, ..., r\} \cup \{d1, ..., dr\}$ if they exist. Let Λ be a C^r Finsler manifold associated with a Banach space Σ. We denote by $\varphi\colon \Sigma \to \Lambda$ the coordinate system. Assume the following:*

(1) *$T\colon Y_1 \times \Lambda \to Y_1$ is a uniform contraction with contraction constant $\theta \in [0, 1)$. Moreover, if $y_1\colon \Lambda \to Y_1$ is the fixed point of T, then there is a continuous $y_0\colon \Lambda \to Y$ such that $y_1 = J_1 y_0$.*

(2) *For any integer $i, j \geqslant 0$, $1 \leqslant p \leqslant r$ with $0 \leqslant i+j \leqslant p$ and any $v \in N^i$ with $0 \leqslant |v| \leqslant p-j$ (if $i = 0$, then $|v| := 0$), there are well-defined mappings*

$$T^{i,j,p}_{|v|+j}\colon Y \times \Lambda \to L\left(\prod_{s=1}^{i} Y_{v(s)} \times (T\Lambda)^j, Y_p\right)$$

$$T^{i,j,p}_{d(|v|+j)}\colon Y \times \Lambda \to L\left(\prod_{s=1}^{i} Y_{dv(s)} \times (T\Lambda)^j, Y_{dp}\right)$$

$$T^{i,j,p}_{c(|v|+j)}\colon Y \times \Lambda \to L\left(\prod_{s=1}^{i} Y_{v(s)} \times (T\Lambda)^j, Y_{dp}\right)$$

such that for any $y, y_1, y_2 \in Y$, $\sigma_1, \sigma_2 \in \Sigma$, $\lambda \in \Lambda$, we have

(a) $J_{p,dp} T^{i,j,p}_{|v|+j}(y, \lambda) = T^{i,j,p}_{d(|v|+j)}(y, \lambda)(\prod_{s=1}^{i} J_{v(s),dv(s)} \times I^j) = T^{i,j,p}_{c(|v|+j)}(y, \lambda)$, *where $I\colon T\Lambda \to T\Lambda$ is the identity;*

(b) $|T^{1,0,p}_{p}(y, \lambda)|_{L(Y_p, Y_p)} \leqslant \theta$, $|T^{1,0,p}_{dp}(y, \lambda)|_{L(Y_{dp}, Y_{dp})} \leqslant \theta$;

(c) $T^{i,j,p}_{c(|v|+j)}$ *is continuous;*

(d) $$T^{i,j,p}_{c(|v|+j)}(y_1,\lambda)-T^{i,j,p}_{c(|v|+j)}(y_2,\lambda)$$
$$=\int_0^1 T^{i+1,j,p}_{c(|v|+j+1)}(sy_1+(1-s)\,y_2,\lambda)(y_1-y_2)\,ds,$$
$$\times\, T^{i,j,p}_{c(|v|+j)}(y,\varphi(\sigma_1))-T^{i,j,p}_{c(|v|+j)}(y,\varphi(\sigma_2))$$
$$=\int_0^1 T^{i,j+1,p}_{c(|v|+j+1)}(y,\varphi(s\sigma_1+(1-s)\,\sigma_2))$$
$$\times\, D\varphi(s\sigma_1+(1-s)\,\sigma_2)(\sigma_1-\sigma_2)\,ds.$$

Then $y_{dr}:=J_{1,dr}\,y_1: \Lambda\to Y_{dr}$ *is* C^r.

Remarks. (1) If $i=j=0$, then $T_0^{0,0,p}:=J_{1,p}T_0$, $T_{c0}^{0,0,p}=T_{d0}^{0,0,p}:=J_{1,dp}T_0$, where $T_0\colon Y\times\Lambda\to Y_1: T_0(y,\lambda)=T(J_1\,y,\lambda)$.

(2) In applications, $T^{i,j,p}_{(|v|+j)}, T^{i,j,p}_{c(|v|+j)}, T^{i,j,p}_{d(|v|+j)}$ are usually restrictions of formal partial derivatives of T to some spaces.

Proof. We prove it by induction. We first claim that $y_1: \Lambda\to Y_1$ is Lipschitz. Without loss of generality, we identify Λ as an open convex set of Σ. Notice that, for any $\lambda_1,\lambda_2\in\Lambda$, and any $y_0\in Y$,

$$T_{c1}^{0,0,1}(y_0,\lambda_1)-T_{c1}^{0,0,1}(y_0,\lambda_2)$$
$$=\int_0^1 T_{c1}^{0,1,1}(y_0,s\lambda_1+(1-s)\,\lambda_2)(\lambda_1-\lambda_2)\,ds. \tag{4.1}$$

Since $T_{c1}^{0,0,1}=J_{1,d1}T_0$, $T_{c1}^{0,1,1}=J_{1,d1}T_1^{0,1,1}$, then (4.1) implies that

$$T_0(y_0,\lambda_1)-T_0(y_0,\lambda_2)=\int_0^1 T_1^{0,1,1}(y_0,s\lambda_1+(1-s)\,\lambda_2)(\lambda_1-\lambda_2)\,ds,$$

that is, T_0 is Lipschitz in λ uniformly in y_0. It then follows from Lemma 4.1 that $y_1: \Lambda\to Y_1$ is Lipschitz.

Let $A_1: \Lambda\to L(\Lambda, Y_1)$ be the unique fixed point of

$$A=T_1^{1,0,1}(y_0(\lambda),\lambda)\,A+T_1^{0,1,1}(y_0(\lambda),\lambda)=:f(A,\lambda).$$

Then, $A_{d1}:=J_{1,d1}A_1: \Lambda\to L(\Lambda, Y_{d1})$ solves

$$A=T_{d1}^{1,0,1}(y_0(\lambda),\lambda)\,A+T_{d1}^{0,1,1}(y_0(\lambda),\lambda)\;=:F(A,\lambda).$$

Since

$$F_0\colon L(\Lambda, Y_1)\times\Lambda\to L(\Lambda, Y_{d1}): (A,\lambda)\mapsto F(J_{1,d1}A,\lambda)$$
$$=T_{c1}^{1,0,1}(y_0(\lambda),\lambda)\,A+T_{c1}^{0,1,1}(y_0(\lambda),\lambda)$$

is continuous, Lemma 4.1 implies that $A_{d1} = J_{1,d1} A_1 : \Lambda \to L(\Lambda, Y_{d1})$ is continuous.

We claim that $y_{d1} = J_{1,d1} y_1 : \Lambda \to Y_{d1}$ is C^1 with $Dy_{d1} = A_{d1}$. For any $\lambda, h \in \Lambda$, denote $H(\lambda, h) = y_{d1}(\lambda + h) - y_{d1}(\lambda) - A_{d1}(\lambda) h$. Then

$$H(\lambda, h) = T_{d1}^{1,0,1}(y_0(\lambda), \lambda) H(\lambda, h) + R(\lambda, h), \tag{4.2}$$

where

$$\begin{aligned}
R(\lambda, h) &= T_{d1}^{0,0,1}(y_0(\lambda + h), \lambda + h) - T_{d1}^{0,0,1}(y_0(\lambda), \lambda) \\
&\quad - T_{d1}^{1,0,1}(y_0(\lambda), \lambda) y_{d1}(\lambda + h) \\
&\quad + T_{d1}^{1,0,1}(y_0(\lambda), \lambda) y_{d1}(\lambda) + T_{d1}^{0,1,1}(y_0(\lambda), \lambda) h \\
&= [T_{c1}^{0,0,1}(y_0(\lambda + h), \lambda) - T_{c1}^{0,0,1}(y_0(\lambda), \lambda) \\
&\quad - T_{c1}^{1,0,1}(y_0(\lambda), \lambda) y_1(\lambda + h) \\
&\quad + T_{c1}^{1,0,1}(y_0(\lambda), \lambda) y_1(\lambda)] \\
&\quad + [T_{c1}^{0,0,1}(y_0(\lambda + h), \lambda + h) - T_{c1}^{0,0,1}(y_0(\lambda + h), \lambda) \\
&\quad - T_{c1}^{0,1,1}(y_0(\lambda + h), \lambda) h] \\
&\quad + [T_{c1}^{0,1,1}(y_0(\lambda + h), \lambda) - T_{c1}^{0,1,1}(y_0(\lambda), \lambda)] h \\
&= \int_0^1 [T_{c1}^{1,0,1}(s y_0(\lambda + h) + (1 - s) y_0(\lambda), \lambda) \\
&\quad - T_{c1}^{1,0,1}(y_0(\lambda), \lambda)][y_1(\lambda + h) - y_1(\lambda)] \, ds \\
&\quad + \int_0^1 [T_{c1}^{0,1,1}(y_0(\lambda + h), s(\lambda + h) + (1 - s) \lambda) \\
&\quad - T_{c1}^{0,1,1}(y_0(\lambda + h), \lambda)] h \, ds \\
&\quad + [T_{c1}^{0,1,1}(y_0(\lambda + h), \lambda) - T_{c1}^{0,1,1}(y_0(\lambda), \lambda)] h.
\end{aligned}$$

Since y_1 is Lipschitz, and $T_{c1}^{1,0}$, $T_{c1}^{0,1}$ are continuous, then $R(\lambda, h) = o(|h|)$. By (4.2), then $H(\lambda, h) = o(|h|)$, that is, $y_{d1} : \Lambda \to Y_{d1}$ is differentiable with $Dy_{d1} = A_{d1}$. It follows from continuity of A_{d1} that $y_{d1} : \Lambda \to Y_{d1}$ is C^1.

By induction, we assume that $y_{dp} = J_{1,dp} y_1 : \Lambda \to Y_{dp}$ is C^p for $1 \leqslant p \leqslant r - 1$, and, there are $A_p : \Lambda \to L(\Lambda^p, Y_p)$ such that $A_{dp} := D^p y_{dp} = J_{p,dp} A_p$; moreover, A_{dp}, A_p satisfy

$$A_{dp} = T_{dp}^{1,0,p}(y_0(\lambda), \lambda) A_{dp} + H_p^d(\lambda), \tag{4.3}$$

$$A_p = T_p^{1,0,p}(y_0(\lambda), \lambda) A_p + H_p(\lambda), \tag{4.4}$$

where $H_p^d(\lambda)$ is defined as a finite sum of $H_p^{i,j}(\lambda) := T^{i,j,p}_{d(|v|+j)}(y_0(\lambda), \lambda) \prod_{s=1}^i A_{dv(s)}$ for some $v=(v(1), ..., v(i)) \in N^i$ with $1 \leqslant i+j \leqslant p$, $1 \leqslant |v| \leqslant p-j$ (if $i=0$, then we define $Y_{|v|} = \varnothing$), $H_p(\lambda)$ is given by exactly the same forms with $H_p^d(\lambda)$ except we drop "d" in $H_p^d(\lambda)$. It is clear by (2)(a) that $H_p^d(\lambda) \in L(\Lambda^p, Y_{dp})$, $H_p(\lambda) \in L(\Lambda^p, Y_p)$.

We first claim that $S_{i,j}^d : \Lambda \to L(\prod_{s=1}^i Y_{v(s)} \times \Lambda^j, Y_{d(p+1)}) : \lambda \to T^{i,j,p+1}_{c(|v|+j)}(y_0(\lambda), \lambda)$ is C^1. To do so, we let

$$\bar{S}_{i,j}^d(\lambda) = T^{i+1,j,p+1}_{c(|v|+j+1)}(y_0(\lambda), \lambda)\, A_1(\lambda) + T^{i,j+1,p+1}_{c(|v|+j+1)}(y_0(\lambda), \lambda). \tag{4.5}$$

For any $\tilde{R} \in \prod_{s=1}^i Y_{v(s)} \times \Lambda^i$, we have

$$\begin{aligned}
&[S_{i,j}^d(\lambda+h) - S_{i,j}^d(\lambda) - \bar{S}_{i,j}^d(\lambda)\, h]\, \tilde{R} \\
&\quad = [T^{i,j,p+1}_{c(|v|+j)}(y_0(\lambda+h), \lambda+h) - T^{i,j,p+1}_{c(|v|+j)}(y_0(\lambda), \lambda) \\
&\qquad - T^{i+1,j,p+1}_{c(|v|+j+1)}(y_0(\lambda), \lambda)\, A_1(\lambda)\, h - T^{i,j+1,p+1}_{c(|v|+j+1)}(y_0(\lambda), \lambda)\, h]\, \tilde{R} \\
&\quad = [T^{i,j,p+1}_{c(|v|+j)}(y_0(\lambda+h), \lambda+h) - T^{i,j,p+1}_{c(|v|+j)}(y_0(\lambda), \lambda) \\
&\qquad - T^{i+1,j,p+1}_{c(|v|+j+1)}(y_0(\lambda), \lambda)(y_1(\lambda+h) - y_1(\lambda)) \\
&\qquad - T^{i,j+1,p+1}_{c(|v|+j+1)}(y_0(\lambda), \lambda)\, h]\, \tilde{R} \\
&\qquad + T^{i+1,j,p+1}_{d(|v|+j+1)}(y_0(\lambda), \lambda)[y_{d1}(\lambda+h) - y_{d1}(\lambda) - A_{d1}(\lambda)\, h] \\
&\qquad \times \left(\prod_{s=1}^i J_{v(s), dv(s)} \times \Lambda^j\right) \tilde{R} \\
&\quad = \int_0^1 [T^{i+1,j,p+1}_{c(|v|+j+1)}(sy_0(\lambda+h) + (1-s)\, y_0(\lambda), \lambda+h) \\
&\qquad - T^{i+1,j,p+1}_{c(|v|+j+1)}(y_0(\lambda), \lambda+h)][y_1(\lambda+h) - y_1(\lambda)]\, \tilde{R}\, ds \\
&\qquad + \int_0^1 [T^{i,j+1,p+1}_{c(|v|+j+1)}(y_0(\lambda), s(\lambda+h) + (1-s)\, \lambda) \\
&\qquad - T^{i,j+1,p+1}_{c(|v|+j+1)}(y_0(\lambda), \lambda)]\, h\tilde{R}\, ds \\
&\qquad + T^{i+1,j,p+1}_{d(|v|+j+1)}(y_0(\lambda), \lambda)[y_{d1}(\lambda+h) - y_{d1}(\lambda) - A_{d1}(\lambda)\, h] \\
&\qquad \times \left(\prod_{s=1}^i J_{v(s), dv(s)} \times \Lambda^j\right) \tilde{R}.
\end{aligned}$$

Since $T^{i+1,j,p+1}_{c(|v|+j+1)}$, $T^{i,j+1,p+1}_{c(|v|+j+1)}$ are continuous, y_1 is Lipschitz, y_{d1} is C^1, from (4.6), we see that $S_{i,j}^d : \Lambda \to L(\prod_{s=1}^i Y_{v(s)} \times \Lambda^j, Y_{d(p+1)})$ is differentiable with $DS_{i,j}^d = \bar{S}_{i,j}^d$. Now, for any $\lambda_1, \lambda_2 \in \Lambda$, and any $\tilde{R} \in \prod_{s=1}^i Y_{v(s)} \times \Lambda^j$, we notice that

$$[\bar{S}^d_{i,j}(\lambda_1)-\bar{S}^d_{i,j}(\lambda_2)]\,h\tilde{R}$$
$$=[T^{i+1,j,p+1}_{c(|v|+j+1)}(y_0(\lambda_1),\lambda_1)-T^{i+1,j,p+1}_{c(|v|+j+1)}(y_0(\lambda_2),\lambda_2)]\,A_1(\lambda_1)\,h\tilde{R}$$
$$+\,T^{i+1,j,p+1}_{d(|v|+j+1)}(y_0(\lambda_2),\lambda_2)[A_{d1}(\lambda_1)-A_{d2}(\lambda_2)]$$
$$\times h\left(\prod_{s=1}^{i} J_{v(s),dv(s)}\times I^j\right)\tilde{R}$$
$$+\,[T^{i,j+1,p+1}_{c(|v|+j+1)}(y_0(\lambda_1),\lambda_1)-T^{i,j+1,p+1}_{c(|v|+j+1)}(y_0(\lambda_2),\lambda_2)]\,h\tilde{R}.$$

Then, $\bar{S}^d_{i,j}$ is continuous. This proves that $S^d_{i,j}$ is C^1. Since $J_{dp,d(p+1)}\,H^{i,j}_p(\lambda)=T^{i,j,p+1}_{c(|v|+j)}(y_0(\lambda),\lambda)\ \prod_{s=1}^i A_{v(s)} = T^{i,j,p+1}_{d(|v|+j)}(y_0(\lambda),\lambda)\ \prod_{s=1}^i A_{dv(s)}$, then $\tilde{H}^{i,p}_p:=J_{dp,d(p+1)}\,H^{i,j}_p$ is C^1 with

$$D\tilde{H}^{i,j}_p(\lambda)=\bar{S}^d_{i,j}(\lambda)\left(\prod_{s=1}^{i} A_{v(s)}\right)$$
$$+\,S^d_{i,j}(\lambda)\left(\sum_{s_0=1}^{i}\prod_{s\neq s_0} A_{dv(s)}(\lambda)\,A_{d(v(s_0)+1)}(\lambda)\right). \tag{4.7}$$

Thus, $\tilde{H}^d_p:=J_{dp,d(p+1)}H^d_p$ is C^1.

Now, we define $\bar{S}_{i,j}(\lambda)$, $\delta H_p(\lambda)$ by simply dropping "d" in $\bar{S}^d_{i,j}(\lambda)$ and $D\tilde{H}^{i,j}_p(\lambda)$, respectively. Let $A_{p+1}:\Lambda\to L(\Lambda^{p+1},\ Y_{p+1})$, $A_{d(p+1)}:\Lambda\to L(\Lambda^{p+1},\ Y_{d(p+1)})$ be fixed points of

$$A=T^{1,0,p+1}_{p+1}(y_0(\lambda),\lambda)\,A+\bar{S}_{1,0}(\lambda)\,A_p(\lambda)+\delta H_p(\lambda)$$

and

$$A=T^{1,0,p+1}_{d(p+1)}(y_0(\lambda),\lambda)\,A+\bar{S}^d_{1,0}(\lambda)\,A_p(\lambda)+D\tilde{H}^d_p(\lambda)=:Q(A,\lambda),$$

respectively. Since

$$Q_0: L(\Lambda^{p+1},Y_{p+1})\times\Lambda$$
$$\to L(\Lambda^{p+1},Y_{d(p+1)}):(A_0,\lambda)\mapsto Q(J_{p+1,d(p+1)}A_0,\lambda)$$
$$=T^{1,0,p+1}_{c(p+1)}(y_0(\lambda),\lambda)\,A_0+\bar{S}^d_{1,0}(\lambda)\,A_p(\lambda)+D\tilde{H}^d_p(\lambda)$$

is continuous, $(\bar{S}^d_{1,0}(\lambda_1)\,A_p(\lambda_1)-\bar{S}^d_{1,0}(\lambda_2)\,A_p(\lambda_2))=[\bar{S}^d_{1,0}(\lambda_1)-\bar{S}^d_{1,0}(\lambda_2)]\,A_p(\lambda_1)+[T^{2,0,p+1}_{d(p+1)}(y_1(\lambda_2),\lambda_2)\,A_{d1}(\lambda_2)+T^{1,2,p+1}_{d(p+1)}(y,(\lambda_2),\lambda_2)][A_{dp}(\lambda_1)-A_{dp}(\lambda_2)]$), it follows from Lemma 4.1 again that $A_{d(p+1)}:\Lambda\to L(\Lambda^{p+1}, y_{d(p+1)})$ is continuous. Let $R(\lambda,h)=J_{dp,d(p+1)}A_{dp}(\lambda+h)-J_{dp,d(p+1)}A_{dp}(\lambda)-A_{d(p+1)}(\lambda)\,h$. Then, $R(\lambda,h)$ satisfies

$$R(\lambda,h)=T^{1,0,p+1}_{d(p+1)}(y_0(\lambda),\lambda)\,R(\lambda,h)$$
$$+\,[S^d_{1,0}(\lambda+h)-S^d_{1,0}(\lambda)-\bar{S}^d_{1,0}(\lambda)\,h]\,A_p(\lambda)$$
$$+\,[\tilde{H}^d_p(\lambda+h)-\tilde{H}^d_p(\lambda)-D\tilde{H}^d_p(\lambda)\,h].$$

It follows that $R(\lambda, h) = o(|h|)$, that is,

$$\tilde{A}_{dp} := J_{dp, d(p+1)} A_{dp} : \Lambda \to L(\Lambda^p, Y_{d(p+1)})$$

is differentiable with $D\tilde{A}_{dp} = A_{d(p+1)}$ being continuous. This proves that $y_{d(p+1)} = J_{1, d(p+1)} y_1 : \Lambda \to Y_{d(p+1)}$ is C^{p+1}. The theorem is now complete. ∎

We now consider an application of this theorem.

Let $\{\Sigma_\lambda\}_{\lambda \in [0,1]}$ be a family of C^r Finsler manifolds varying continuously in λ such that $\Sigma_{\lambda_1} \subset \Sigma_{\lambda_2}$ if $\lambda_1 \leqslant \lambda_2$.

Consider

$$\begin{aligned} u' &= \tilde{B}(\sigma, t)\, u + \tilde{F}(u, \beta, \sigma, t) \\ \beta' &= \tilde{R}(u, \beta, \sigma, t), \end{aligned} \tag{4.9}$$

where $u \in R^n$, $\beta \in R^m$, $\sigma \in \Sigma_1$.

Hypothesis. (I) For any compact set $I \subset R$, $\tilde{B}$ is uniformly continuous on $\Sigma_1 \times I$, and, the equation

$$u' = \tilde{B}(\sigma, t)\, u \tag{4.10}$$

has ED on R uniformly for $\lambda \in \Sigma_1$ with projections $P(\sigma)$ and ED constants δ and K.

(II) $\tilde{F}$, $\tilde{R}$ are C^r in u and β such that for any compact set $I \subset R$, $\tilde{F}$, $\tilde{R}$ as well as their partial derivatives are uniformly bounded on $R^{n+m} \times \Sigma_1 \times R$ and uniformly continuous on $R^{n+m} \times \Sigma_1 \times I$. For fixed $\sigma \in \Sigma_1$, $\tilde{F}$, $\tilde{R}$ are also uniformly continuous on $R^{n+m} \times \{\sigma\} \times R$.

(III) $\tilde{F}(0, 0, \sigma, t) \equiv 0$, $\tilde{R}(0, 0, \sigma, t) \equiv 0$. Let $c_1(\lambda) = \sup_{E(\lambda)} |\partial_u \tilde{F}|$, $c_2(\lambda) = \sup_{E(\lambda)} |\partial_\beta \tilde{F}|$, $c_3(\lambda) = \sup_{E(\lambda)} |\partial_u \tilde{R}|$, where $E(\lambda) \equiv R^{n+m} \times \Sigma_\lambda \times R$. Then $c_i(\lambda) \to 0$ as $\lambda \to 0$ $(i = 1, 2, 3)$.

COROLLARY 4.3. *Consider* (4.9) *and assume* (I), (II), (III) *above. Denote*

$$L(\lambda) = \sup_{E(\lambda)} |\partial_\beta \tilde{R}|,$$

$L_0 := L(0)$. *If* $(r+1) L_0 < \delta$, *then for any* $0 < \varepsilon < (\delta - (r+1) L_0)/2$, *there exists* $\lambda_* \in (0, 1]$ *such that the following hold*:

(1) *Equation* (4.9) *has for each* $\xi \in R^n$, $\sigma \in \Sigma_{\lambda_*}$ unique solutions $\phi_\pm(\sigma, \xi)(t) \equiv (u_\pm(\sigma, \xi)(t), \beta_\pm(\sigma, \xi)(t))$ *such that* $\phi_\pm(\sigma, \xi)(t) \to 0$ *exponentially as* $t \to \pm\infty$ *with exponential rate* $\rho \in [L_0 + \varepsilon, \delta - \varepsilon]$. $\phi_\pm(\sigma, \xi)(0)$ *is* C^r *in* ξ *and is continuous in* σ. *Furthermore,* $\phi_\pm(\sigma, 0)(t) \equiv 0$, $P(\sigma)\, u_+(\sigma, \xi)(0) = P(\sigma)\, \xi$, $(I - P(\sigma))\, u_-(\sigma, \xi)(0) = (I - P(\sigma))\, \xi$.

(2) *If* $\tilde{B}$, $\tilde{F}$, $\tilde{R}$ *are also* C^r *in* $\sigma \in \Sigma_1$, *and there are positive constants* C, C_4, C_5, N_λ *with* $N_\lambda \geq L_0$, $N_\lambda \to L_0$ *as* $\lambda \to 0$ *such that for each* $i \geq 0, j \geq 0$ *with* $0 \leq i+j \leq r-1$

$$\sup_{\Sigma_\lambda \times R} |\partial_\sigma^j \tilde{B}| \leq Ce^{jN_\lambda |t|},$$

$$\sup_{E(\lambda)} |\partial_{(u,\beta)}^i \partial_\sigma^j \tilde{F}| \leq \begin{cases} C_4(|u|+|\beta|)\, e^{jN_\lambda |t|} & (i=0), \\ C_4 e^{jN_\lambda |t|} & (i>0), \end{cases}$$

and

$$\sup_{E(\lambda)} |\partial_{(u,\beta)}^i \partial_\sigma^j \tilde{R}| \leq \begin{cases} C_5(|u|+|\beta|)\, e^{jN_\lambda |t|} & (i=0), \\ C_5 e^{jN_\lambda |t|} & (i>0), \end{cases}$$

then $\phi_\pm(\sigma, \xi)(0)$ *is* C^{r-1} *in* $\sigma \in \Sigma_{\lambda_*}$. *In fact,* $\phi_\pm(\sigma, \xi)(0)$ *is a* C^{r-1} *function.*

Proof. Denote for each $p>0$, $X_{\rho\pm} = \{x: R \to R^n \mid |x|_{\rho\pm} < \infty\}$, $Y_{\rho\pm} = \{y: R \to R^m \mid |y|_{\rho\pm} < \infty\}$. Let $Z_{\rho\pm} \equiv X_{\rho\pm} \times Y_{\rho\pm}$ with norm $|(x,y)|_{\rho\pm} \equiv |x|_{\rho\pm} + |y|_{\rho\pm}$.

Let $K_\pm, S_\pm$ be defined as in (3.2)–(3.5) for Eq. (4.10) with $Y \equiv \Lambda \equiv \Sigma_1 \times R^n$. For any $\rho \in [L_0+\varepsilon, \delta-\varepsilon]$, define $G: Z_{\rho\pm} \times \Sigma_1 \to Z_{\rho\pm}$ by $G(\phi, \sigma)(t) = (\tilde{F}(u(t), \beta(t), \sigma, t), \tilde{R}(u(t), \beta(t), \sigma, t)) =: (G_1(\phi, \sigma)(t), G_2(\phi, \sigma)(t))$, $\tilde{K}_\pm: \Sigma_1 \to L(Z_{\rho\pm}, Z_{\rho\pm})$ by $\tilde{K}_\pm(\sigma)\phi \equiv (K_\pm(\sigma)u, \hat{K}_\pm \beta)$; here $(\hat{K}_+\beta)(t) = -\int_t^\infty \beta(s)\, ds$, $(\hat{K}_-\beta)(t) = \int_{-\infty}^t \beta(s)\, ds$. We also define $\tilde{S}_\pm: \Sigma_1 \to L(R^n, Z_{\rho\pm})$ by $\tilde{S}_\pm(\sigma)\xi = (S_\pm(\sigma)\xi, 0)$. Clearly, the above mappings are well defined.

Now, for any $\phi \in X_{\rho\pm}$, and any $(\sigma, \xi) \in \Lambda$, define

$$f_\pm(\phi, \lambda) = \tilde{K}_\pm(\sigma) \circ G(\sigma, \phi) + \tilde{S}_\pm(\sigma)\xi. \tag{4.11}$$

Then fixed points of

$$\phi = f_\pm(\phi, \sigma, \xi) \tag{4.12}$$

solve Eq. (4.9).

By our conditions, it is easy to verify that

$$|f_\pm(\phi_1, \sigma, \xi) - f_\pm(\phi_2, \sigma, \xi)|_{\rho\pm} \leq \theta(\lambda)\, |\phi_1 - \phi_2|_{\rho\pm} \tag{4.13}$$

for any $\phi_1, \phi_2 \in X_{\rho\pm}$ and any $(\sigma, \xi) \in \Lambda$, where $\theta(\lambda) = \max\{(2K/\varepsilon)\, c_1(\lambda) + c_3(\lambda)/(L_0+\varepsilon),\ (2K/\varepsilon)\, c_2(\lambda) + L(\lambda)/(L_0+\varepsilon)\}$. Hence, there is a $\lambda_* > 0$ such that $\theta(\lambda_*) < 1$, that is, (4.12) are uniform contractions on $Z_{\rho\pm} \times \Sigma_{\lambda_*} \times R^n$ for any $\rho \in [L_0+\varepsilon, \delta-\varepsilon]$. We denote the fixed points by $\phi_\pm(\sigma, \xi) \equiv (u_\pm(\sigma, \xi), \beta_\pm(\sigma, \xi))$.

For any $\rho \in (L_0+\varepsilon, \delta-\varepsilon)$, let us take ρ_1, ρ_2 such that $L_0+\varepsilon < \rho_1 < \rho_2 < \rho$. Then by Lemma 3.6 and Theorem 3.4(1), $G: Z_{\rho\pm} \times \Sigma_{\lambda_*} \to Z_{\rho_2\pm}$,

$K_\pm : \Sigma_{\lambda_*} \to L(X_{\rho_2\pm}, X_{\rho_1\pm})$, $S_\pm : \Sigma_{\lambda_*} \to L(R^n, X_{\rho_1\pm})$ are continuous. Therefore, $f_\pm : Z_{\rho\pm} \times \Sigma_{\lambda_*} \times R^n \to Z_{\rho_1\pm}$ are continuous. It then follows from Lemma 4.1 that $\phi_\pm : \Sigma_{\lambda_*} \times R^n \to Z_{\rho_1\pm}$ are continuous. Specifically, $\phi_\pm(\sigma, \xi)(0)$ are continuous in σ and ξ. As $\xi = 0$, since 0 is clearly a fixed point of (4.12), by uniqueness, we have $\phi_\pm(\sigma, 0)(t) \equiv 0$. The identities $P(\sigma)\, u_+(\sigma, \xi)(0) = P(\sigma)\, \xi$, $(I - P(\sigma))\, u_-(\sigma, \xi)(0) = (I - P(\sigma))\, \xi$ follow directly from the definitions of $f_\pm$.

To prove that $\phi_\pm(\sigma, \xi)(0)$ are C^r in ξ for fixed $\sigma \in \Sigma_{\lambda_*}$, we apply the classical uniform contraction mapping principal on $Y^\pm \times \Lambda$; here $Y^\pm := Z_{\rho\pm}$ for some $\rho \in (L_0 + \varepsilon, \delta - \varepsilon)$, $\Lambda := R^n$. Since $G : Y^\pm \to Y^\pm$ are C^r by Lemma 3.5, then $f_\pm : Y^\pm \times \Lambda \to Y^\pm$ are C^r. But $f_\pm$ are uniform contractions; hence the fixed points $\phi_\pm(\cdot, \sigma) : R^n \to Z_{\rho\pm}$ are C^r. In particular, $\phi_\pm(\xi, \sigma)(0)$ are C^r in ξ.

(2) We now verify the conditions of Theorem 4.2. Let $N := N_{\lambda_*}$. For any fixed $\eta \in ((r+1)\, N + \varepsilon, \delta - \varepsilon)$, take a $\rho \in ((r+1)\, N + \varepsilon, \eta)$. Define $\rho_0(s) = \eta - sN$, $\rho(s) = \rho - sN$, $s = 1, 2, ..., r-1$. Let $Y = X_{\eta\pm}$, $Y_s = X_{\rho_0(s)\pm}$, $Y_{ds} = X_{\rho(s)\pm}$. Then $Y \to Y_1$, $Y_s \to Y_{s+1}$, $Y_{ds} \to Y_{d(s+1)}$, $Y_s \to Y_{ds}$, $s = 1, ..., r-1$, are continuously embedded ($Y_{dr} := Y_{d(r-1)}$). Since $N + \varepsilon < \rho_0(1) < \eta < \delta - \varepsilon$, by (1), $T : f_\pm$ are uniformly contractions on both Y and Y_1. We denote by $y_0 := \phi_0^\pm$, $y_1 := \phi^\pm$ the fixed points of T in Y, T_1, respectively. Then, $y_1 = J_1\, y_0$, where $J_1 : Y \to Y_1$ is the embedding. It also follows from arguments of (1) that $y_0 : \Lambda \to Y$ is continuous. Now, for any i, j with $1 \leqslant i + j \leqslant r - 1$ we differentiate T formally i times with respect to ϕ, j times with respect to σ. We then have

$$T^{i,j}(\phi_0, \sigma) = \tilde{S}^j_\pm(\sigma)\, \xi + \sum_{m=0}^{j} \binom{j}{m} \tilde{K}^m_\pm(\sigma) \circ \tilde{G}^{i,j-m}(\phi_0, \sigma), \qquad (4.14)$$

for any $\phi_0 \in Y$, $\sigma \in \Lambda$, where $\tilde{S}^j_\pm = (S^j_\pm, 0)$, $\tilde{K}^m_\pm = (K^m_\pm, 0)$ $(m \geqslant 0,\ i \geqslant 0)$ with $S^j_\pm$, $K^m_\pm$ being defined in Theorem 3.4. ($S^0_\pm := S_\pm$, $K^0_\pm := K_\pm$), and, $\tilde{G}^{i,j-m}$ is defined in Lemma 3.7 with $f = (\tilde{F}, \tilde{R})$.

For any m, i, j, $p \geqslant 0$, with $1 \leqslant p \leqslant r - 1$, $0 \leqslant m \leqslant j$, $0 \leqslant i + j \leqslant p$ and any $v \in N^i$ with $0 \leqslant |v| \leqslant p - j$, since

$$\begin{aligned} \rho_0(j - m + |v|) &= \eta - (j - m + |v|)\, N \\ &\leqslant \begin{cases} \eta - (j-m)\, N & (i = 0) \\ (i\eta - |v|\, N) - (j - m)\, N & (i > 0), \end{cases} \\ \rho(j - m + |v|) &= \rho - (j - m + |v|)\, N \\ &\leqslant \begin{cases} \eta - (j-m)\, N & (i = 0) \\ (i\rho - |v|\, N) - (j - m)\, N & (i > 0), \end{cases} \end{aligned}$$

it follows from Lemma 3.7 that $\tilde{G}^{i,j-m} : Y \times \Lambda \to L(\prod_{s=1}^{i} Y_{v(s)} \times (T\Lambda)^{j-m},$

$Y_{j-m+|v|})$, $\tilde{G}^{i,j-m}: Y\times \Lambda \to L(\prod_{s=1}^{i} Y_{dv(s)} \times (T\Lambda)^{j-m}, Y_{d(j-m+|v|)})$ are well defined. Since $(m+1)N \leqslant \rho_0(j-m+|v|) < \delta$, $0 \leqslant \rho_0(p) \leqslant \rho_0(j-m+|v|) - mN$, $(m+1)N \leqslant \rho(j-m+|v|) < \delta$, $0 \leqslant \rho(p) \leqslant \rho(j-m+|v|) - mN$, $0 \leqslant \rho_0(p) \leqslant \delta - jN$, $0 \leqslant \rho(p) \leqslant \delta - jN$, Theorem 3.4 implies that $\tilde{K}_{\pm}^{m}: \Lambda \to L(Y_{j-m+|v|} \times (T\Lambda)^m, Y_p)$, $\tilde{K}_{\pm}^{m}: \Lambda \to L(Y_{d(j-m+|v|)} \times (T\Lambda)^m, Y_{dp})$, $\tilde{S}_{\pm}^{j}: \Lambda \to L((T\Lambda)^j, Y_p)$, $\tilde{S}_{\pm}^{j}: \Lambda \to L((T\Lambda)^j, Y_{dp})$ are well defined. Therefore, $T^{i,j}: Y\times \Lambda \to (\prod_{s=1}^{i} Y_{v(s)} \times (T\Lambda)^j, Y_p)$, $T^{i,j}: Y\times \Lambda \to (\prod_{s=1}^{i} Y_{dv(s)} \times (T\Lambda)^j, Y_{dp})$ are well defined. We now take $\eta_{m,j}$ such that

$$\rho(p) + mN < \eta_{m,j} < \begin{cases} \eta - (j-m)N & (i=0) \\ (i\eta - |v|N) - (j-m)N & (i>0). \end{cases}$$

By Lemma 3.7, $\tilde{G}^{i,j-m}: Y\times \Lambda \to (\prod_{i=1}^{i} Y_{v(s)} \times (T\Lambda)^j, X_{\eta_{m,j}\pm})$ are continuous, and, by Theorem 3.4, $\tilde{S}_{\pm}^{j}: \Lambda \to L((T\Lambda)^j, Y_{dp})$, $\tilde{K}_{\pm}^{m}: \Lambda \to L(X_{\eta_{n,j}\pm} \times (T\Lambda)^m, Y_{dp})$ are continuous, that is, $T^{i,j}: Y\times \Lambda \to L\ (\prod_{s=1}^{i} Y_{v(s)} \times (T\Lambda)^j, Y_{dp})$ are continuous. Similar to (1), we also have

$$|T^{1,0}(\phi_0, \sigma)|_{L(Y_p, Y_p)} \leqslant \theta(\lambda_*), \qquad |T^{1,0}(\phi_0, \sigma)|_{L(Y_{dp}, Y_{dp})} \leqslant \theta(\lambda_*).$$

Thus, conditions (1), (2)(a)–(c) of Theorem 4.2 are verified. Condition (2)(d) of Theorem 4.2 is trivial in this case because $\tilde{S}_{\pm}^{j}$, $\tilde{K}_{\pm}^{m}$, $\tilde{G}^{j-m}$ are all formal partial derivatives. We then conclude from Theorem 4.2 that $\phi_{\pm}: \Lambda \to Y_{d(r-1)} = X_{(\rho-(r-1)N)\pm}$ are C^{r-1}. Therefore $\phi_{\pm}(\sigma, \xi)(0)$ are C^{r-1} in σ. By using exactly the same arguments, one proves that $\phi_{\pm}(\sigma, \xi)$ are C^{r-1} functions. ▮

5. Stability of Integral Manifolds

Consider systems

$$\begin{aligned} z' &= A(\theta, t, \lambda)z + F(z, \theta, t, \lambda) \\ \theta' &= Q(\theta, t, \lambda) + G(z, \theta, t, \lambda), \end{aligned} \tag{5.1$_\lambda$}$$

where $z \in R^n$, $\theta \in T^k$, $\lambda \in R^m$.

Hypotheses. (I) The equation

$$z' = A(\hat{\theta}(t), t, 0)z \tag{5.2}$$

has ED on R uniformly for each $\theta_0 \in T^k$ with projection $P(\theta_0)$ and ED constants which are independent of $\theta_0 \in T^k$, where $\hat{\theta}(t) \equiv \hat{\theta}(\theta_0, t)$ is the solution of

$$\theta' = Q(\theta, t, 0) \tag{5.3}$$

such that $\hat{\theta}(0) = \theta_0$.

(II) Denote $I_1 = \{\lambda \in R^m \mid |\lambda| \leqslant 1\}$. Assume that A, Q are C^r in θ, λ such that A, Q as well as all of their partial derivatives are uniformly bounded and uniformly continuous on $T^k \times R \times I_1$, F, G are C^r in z, θ, λ such that F, G and all their partial derivatives are uniformly bounded and uniformly continuous on $E \times T^k \times R \times I_1$, where $E \subset R^n$ is an arbitrary compact set. Furthermore, $F(0, \theta, t, 0) \equiv 0$, $\partial_z F(0, \theta, t, 0) \equiv 0$, $G(0, \theta, t, 0) \equiv 0$ for any $\theta \in T^k$, $t \in R$.

(III) Denote by δ^0 the smallest Lyapounov exponent of (5.2) in absolute value, and $L^0 := \sup_{\theta \in T^k, t \in R} |\partial_\theta Q(\theta, t, 0)| < \delta^0/(r+1)$.

For functions A, Q, F, G which satisfy hypothesis (II), it has been shown in *Yi* [35] that for any $\varepsilon_0 > 0$, there exists a flow (Ω, R), with (Ω, d) compact metric such that (i) $\Omega = \mathrm{cl}\{\omega_0 \cdot t \mid t \in R\}$ for some $\omega_0 \in \Omega$, (ii) $d(\omega_1 \cdot t, \omega_2 \cdot t) \leqslant e^{\varepsilon_0 |t|} d(\omega_1, \omega_2)$ for any $\omega_1, \omega_2 \in \Omega$ and $t \in R$, (iii) there are continuous functions $a: T^k \times \Omega \times I_1 \to M_n$, $f: E_1 \times T^k \times \Omega \times I_1 \to R^n$, $q: T^k \times \Omega \times I_1 \to T^k$, $g: E_1 \times T^k \times \Omega \times I_1 \to T^k$, with $a(\theta, \omega_0 \cdot t, \lambda) \equiv A(\theta, t, \lambda)$, $f(z, \theta, \omega_0 \cdot t, \lambda) \equiv F(z, \theta, t, \lambda)$, $q(\theta, \omega_0 \cdot t, \lambda) \equiv Q(\theta, t, \lambda)$, $g(z, \theta, \omega_0 \cdot t, \lambda) \equiv G(z, \theta, t, \lambda)$, where $E_1 = \{z \in R^n \mid |z| \leqslant 1\}$, $I_1 = \{\lambda \in R^n \mid |\lambda| \leqslant 1\}$, (iv) a, q, f, g are C^r continuous in z, θ, λ and uniform Lipschitz in ω with Lipschitz constants less than or equal to ε_0.

In the language of topological dynamical systems, Ω is referred as the hull of $\{A, Q, F, G\}$ in the compact open topology [32]. Note that $\Omega_0 =: \{\omega_0 \cdot t \mid t \in R\}$ is a one-dimensional topological manifold with compact closure Ω.

For each ε_0, $0 < \varepsilon_0 \ll 1$, as remarked in the introduction, we shall consider a class of equations based on the hull Ω, namely,

$$\begin{aligned} z' &= a(\theta, \omega \cdot t, \lambda)\, z + f(z, \theta, \omega \cdot t, \lambda) \\ \theta' &= q(\theta, \omega \cdot t, \lambda) + h(z, \theta, \omega \cdot t, \lambda) \end{aligned} \tag{5.4$_\omega$}$$

for $\omega \in \Omega$. Note that $(5.4)_{\omega_0}$ coincides with $(5.1)_\lambda$.

Let $(z(t), \theta(t))$ be solutions of $(5.4)_\omega$ such that $z(0) = z_0$, $\theta(0) = \theta_0$. Then

$$A_t^\lambda(z_0, \theta_0, \omega) = (z(t), \theta(t), \omega \cdot t) \tag{5.5$_\lambda$}$$

defines a (nonlinear) skew product flow on $E_1 \times T^k \times \Omega$ for each $\lambda \in I_1$.

The following invariant manifold theorem can be found in Yi [35].

THEOREM 5.1. *Consider* $(5.5)_\lambda$. *Assume that* (I), (II), *and* (III) *hold. Then for each* ε_0, $0 < \varepsilon_0 < (\delta - (r+1) L_0)/2$, *there is a* $\lambda^0 =: \lambda^0(\varepsilon_0)$, $0 < \lambda^0 \leqslant 1$, *such that for each* $\lambda_0 \in (0, \lambda^0]$ we have:

(1) *The skew product flow* $(5.5)_\lambda$ *has for each* $\lambda \in I_{\lambda_0}$ *a unique invariant manifold of type*

$$\mathcal{S}_\lambda = \{(p_\lambda(\theta, \omega), \theta, \omega) \mid \theta \in T^k, \omega \in \Omega_0\}, \tag{5.6$_\lambda$}$$

where $p_\lambda(\theta, \omega)$ is C^r in θ, λ and is uniformly Lipschitz in ω, $p_\lambda(\theta, \omega) \to 0$ uniformly as $\lambda \to 0$.

(2) *For any $\theta_0 \in T^k$, let $\hat{\Theta}(t) \equiv \theta(\theta_0, \lambda, \omega, t)$ be the solution of*

$$\theta' = q(\theta, \omega \cdot t, \lambda) + g(p_\lambda(\theta, \omega \cdot t), \theta, \omega \cdot t, \lambda) \tag{5.7}$$

with $\hat{\Theta}(0) = \theta_0$. Then the equation

$$z' = a(\hat{\Theta}(t), \omega \cdot t, \lambda)\, z \qquad (5.8)_\lambda$$

has ED on R uniformly for each $\theta_0 \in T^k$, $\omega \in \Omega_0$, and $\lambda \in I_{\lambda_0}$ with projections $P_\lambda(\theta_0, \omega)$ and ED constants $K > 0$, $\delta_{\lambda_0} > 0$. Furthermore, $\delta_{\lambda_0} \in (0, \delta^0 - \varepsilon_0/3)$, $\delta_{\lambda_0} \to \delta^0 - \varepsilon_0/3$ as $\lambda_0 \to 0$.

(3) *There are constants $M_1 > 0$, $M_2 > 0$, $M_3 > 0$, and $N_{\lambda_0} \geqslant L_0 + \varepsilon_0$, with $N_{\lambda_0} \to L_0 + \varepsilon_0$ as $\lambda_0 \to 0$, such that*

$$\sup_{T^k \times \Omega_0 \times I_{\lambda_0}} |\partial^i_\theta \hat{\Theta}(t)| \leqslant M_1 e^{iN_{\lambda_0}|t|}, \tag{5.9}$$

$$\sup_{T^k \times \Omega_0 \times I_{\lambda_0}} |\partial^i_\lambda \hat{\Theta}(t)| \leqslant M_2 e^{iN_{\lambda_0}|t|}, \tag{5.10}$$

$$\sup_{T^k \times \Omega_0 \times I_{\lambda_0}} \mathrm{Lip}_\omega \hat{\Theta}(t) \leqslant M_3 e^{N_{\lambda_0}|t|}, \tag{5.11}$$

for all $i = 1, 2, ..., r$.

(4) *Let $f_\lambda(\theta, t) = p_\lambda(\theta, \omega_0 \cdot t)$. Then*

$$S_\lambda = \{(f_\lambda(\theta, t), \theta, t) |\ \theta \in T^k, t \in R\}$$

defines an integral manifold of $(5.1)_\lambda$ for each $\lambda \in I_{\lambda^0}$.

Note. Let $\hat{\Theta}(t)$ be defined in the theorem. Then the flow on $\mathscr{S}_\lambda$ is given by

$$\Lambda^\lambda_t(p_\lambda(\theta_0, \omega), \theta_0, \omega) = (p_\lambda(\hat{\Theta}(t), \omega \cdot t), \hat{\Theta}(t), \omega \cdot t).$$

This is equivalent to the flow

$$\Gamma_t(\theta_0, \omega) := (\hat{\Theta}(t), \omega \cdot t) \tag{5.12}$$

on $T^k \times \Omega_0$.

We are now ready to state our main theorems in this section.

THEOREM 5.2. *Under conditions of Theorem 5.1, for any ε_0 with $0 < \varepsilon_0 < (\delta^0 - (r+1)L^0)/2$, there is a $\lambda_0 =: \lambda_0(\varepsilon_0)$, $0 < \lambda_0 \leqslant 1$, such that for each $\lambda \in I_{\lambda_0}$, there are invariant topological manifolds $\mathscr{W}^+_\lambda(\mathscr{S}_\lambda)$, $\mathscr{W}^-_\lambda(\mathscr{S}_\lambda)$ to the* *skew product flow* Λ_t^λ *referred to as the stable and unstable manifolds of* $\mathscr{S}_\lambda$, *respectively. Moreover, we have the following*:

(1) $\mathscr{W}_\lambda^\pm(\mathscr{S}_\lambda)=\{(z,\theta,\omega)\in R^n\times T^k\times\Omega_0 \mid$ *there is unique* $(p_\lambda(\theta_0,\omega),\theta_0,\omega)\in\mathscr{S}_\lambda$ *such that* $\sup_{t\in R^\pm}|\Lambda_t^\lambda(z,\theta,\omega)-\Lambda_t^\lambda(p_\lambda(\theta_0,\omega),\theta_0,\omega)|\,e^{\pm\eta t}<\infty$ *for some* $\eta\in[L_0+\varepsilon_0,\delta^0-\varepsilon_0]\}=\{(z,\theta,\omega)\in R^n\times T^k\times\Omega_0\mid$ *the solution* $(\tilde{z}(t),\tilde{\theta}(t))$ *of* $(5.4)_\omega$ *with* $(\tilde{z}(0),\tilde{\theta}(0))=(z,\theta)$ *is such that* $\sup_{t\in R^\pm}|\tilde{z}(t)-p_\lambda(\tilde{\theta}(t),\omega\cdot t)|\,e^{\pm\eta t}<\infty$ *for some* $\eta\in[L_0+\varepsilon_0,\delta^0-\varepsilon_0]\}$.

(2) $\mathscr{W}_\lambda^\pm(\mathscr{S}_\lambda)$ *are foliated by disjoint immersed* C^r-*submanifolds*

$$\mathscr{W}_\lambda^\pm(\theta_0,\omega)=\{(z,\theta,\omega)\in R^n\times T^k\times\{\omega\}\mid \sup_{t\in R^\pm}|\Lambda_t^\lambda(z,\theta,\omega)-\Lambda_t^\lambda(p_\lambda(\theta_0,\omega),\theta_0,\omega)|\,e^{\pm\eta t}<\infty \text{ for some } \eta\in[L_0+\varepsilon_0,\delta^0-\varepsilon_0]\},$$

that is, $\mathscr{W}_\lambda^\pm(\mathscr{S}_\lambda)=\bigcup_{\theta_0\in T^k,\,\omega\in\Omega_0}\mathscr{W}_\lambda^\pm(\theta_0,\omega)$. *Furthermore,* $\mathscr{W}_\lambda^\pm(\theta_0,\omega)$ *are invariant in the sense that* $\Lambda_t\mathscr{W}_\lambda^\pm(\theta_0,\omega)=\mathscr{W}_\lambda^\pm(\Gamma_t(\theta_0,\omega))$.

(3) *There are small tubular neighborhoods* $D_a=\{(\xi,\theta,\omega)\in R^n\times T^k\times\Omega_0\mid|\xi-p_\lambda(\theta,\omega)|\leqslant a\}$ *of* $\mathscr{S}_\lambda$ *such that* $\mathscr{W}_\lambda^+(\theta_0,\omega)\cap D_a$, $\mathscr{W}_\lambda^-(\theta_0,\omega)\cap D_a$ *are* C^r *diffeomorphic to the stable subspace* $V_\lambda^+(\theta_0,\omega)$ *and unstable subspace* $V_\lambda^-(\theta_0,\omega)$ *of* (5.8), *respectively*.

(4) $\mathscr{W}_\lambda^\pm(\theta_0,\omega)$ *vary continuously in* $\omega\in\Omega_0$, C^{r-1} *smoothly in* θ_0 *and* λ. *Furthermore,* $\mathscr{W}_\lambda^+(\theta_0,\omega)$ *and* $\mathscr{W}_\lambda^-(\theta_0,\omega)$ *intersect transversally at* $(p_\lambda(\theta_0,\omega),\theta_0,\omega)\in\mathscr{S}_\lambda$.

To state our next theorem, we let $N_t^\lambda(z,\theta,\tau)$ be solution map of $(5.1)_\lambda$ (namely, $N_t^\lambda(z,\theta,\tau)$ solves $(5.1)_\lambda$ with $N_\tau^\lambda(z,\theta,\tau)=(z,\theta)$). We also denote by $n_t(\theta,\tau)$ the solution map of

$$\theta'=Q(\theta,t,\lambda)+G(f_\lambda(\theta,t),\theta,t,\lambda) \tag{5.13}$$

(namely, $n_t(\theta,\tau)$ solves (5.13) and $n_\tau(\theta,\tau)=0$).

THEOREM 5.3. *Consider* $(5.1)_\lambda$, *assuming the conditions of Theorem* 5.1. *Then, for any* ε_0 *with* $0<\varepsilon_0<(\delta^0-(r+1)L^0)/2$, *there is* $\lambda_0:=\lambda_0(\varepsilon)$, $0<\lambda_0\leqslant 1$, *such that for each* $\lambda\in I_{\lambda_0}$, *there are integral manifolds* $W_\lambda^+(S_\lambda)$, $W_\lambda^-(S_\lambda)$ *to Eq.* $(5.1)_\lambda$ *referred to as the stable and unstable manifolds of* S_λ, *respectively. Moreover, we have the following*:

(1) $W_\lambda^\pm(S_\lambda)=\{(z,\theta,\tau)\in R^n\times T^k\times R\mid$ *there is unique* $(f_\lambda(\theta_0,\tau),\theta_0,\tau)\in S_\lambda$ *such that* $\sup_{t\in R^\pm}|N_t^\lambda(z,\theta,\tau)-N_t^\lambda(f_\lambda(\theta_0,\tau),\theta_0,\tau)|\,e^{\pm\eta t}<\infty$ *for some* $\eta\in[L_0+\varepsilon_0,\delta^0-\varepsilon_0]\}=\{(z,\theta,\tau)\in R^n\times T^k\times R\mid$ *the solution* $(\tilde{z}(t),\tilde{\theta}(t))$ *of* $(5.1)_\lambda$ *through* (z,θ) *at time* τ *is such that* $\sup_{t\in R^\pm}|\hat{z}(t)-f_\lambda(\hat{\theta}(t),t)|\,e^{\pm\eta t}<\infty$ *for some* $\eta\in[L_0+\varepsilon_0,\delta^0-\varepsilon_0]\}$.

(2) *$W_\lambda^\pm(S_\lambda)$ are foliated by disjoint immersed C^r-submanifolds*

$$W_\lambda^\pm(\theta_0,\tau)=\{(z,\theta,\tau)\in R^n\times T^k\times\{\tau\}\mid \sup_{t\in R^\pm}|N_t^\lambda(z,\theta,\tau)-N_t^\lambda(f_\lambda(\theta_0,\tau),\theta_0,\tau)|\,e^{\pm\eta t}<\infty \text{ for some } \eta\in[L_0+\varepsilon_0,\delta^0-\varepsilon_0]\},$$

that is, $W_\lambda^\pm(S_\lambda)=\bigcup_{\theta_0\in T^k,\tau\in R}W_\lambda^\pm(\theta_0,\tau)$. *Furthermore,* $W_\lambda^\pm(\theta_0,\tau)$ *are invariant in the sense that*

$$N_t^\lambda(W_\lambda^\pm(\theta_0,\tau))=W_\lambda^\pm(n_t(\theta_0,\tau)).$$

(3) $W_\lambda^+(\theta_0,\tau)$, $W_\lambda^-(\theta_0,\tau)$ *are locally (in a tubular neighborhood of* S_λ*)* C^r *diffeomorphic to the stable and unstable subspaces of*

$$z'=A(n_t(\theta_0,\tau),t,\lambda)\,z, \tag{5.14}$$

respectively.

(4) $W_\lambda^\pm(\theta_0,\tau)$ *vary continuously in* τ, C^{r-1} *smoothly in* θ_0 *and* λ. Furthermore, $W_\lambda^+(\theta_0,\tau)$, $W_\lambda^-(\theta_0,\tau)$ *intersect transversally at* $(f_\lambda(\theta_0,\tau),\theta_0,\tau)\in S_\lambda$.

Remark. (1) λ_0 in Theorems 5.1, 5.2, 5.3 depends on ε_0; however, we could fix a small ε_0 to begin with, for instance, $\varepsilon_0=(\delta-(r+1)L_0)/4$.

(2) Theorem 5.3 is a consequence of Theorem 5.2 by simply setting $W_\lambda^\pm(\theta_0,\tau)=\mathscr{W}_\lambda^\pm(\theta_0,\omega_0\cdot\tau)$, $W_\lambda^\pm(S_\lambda)=\sup_{\theta_0\in T^k,\tau\in R}W_\lambda^\pm(\theta_0,\tau)$.

Proof of Theorem 5.2. Let λ^0 be defined as in Theorem 5.1. For any $\lambda\in I_{\lambda^0}$ and any $\theta_0\in T^k$, $\omega\in\Omega_0$, let $\hat{\Theta}(t)$ be the solution of (5.7) such that $\hat{\Theta}(\theta)=\theta_0$. Consider the transformation

$$\begin{aligned}\rho&=z-\rho_\lambda(\hat{\Theta}(t),\omega\cdot t)\\ \beta&=\theta-\hat{\Theta}(t)\end{aligned} \tag{5.15}$$

to Eq. $(5.4)_\omega$. Then

$$\begin{aligned}\rho'&=\tilde{B}(\theta_0,\lambda,\omega,t)\,\rho+F_0(\rho,\beta,\theta_0,\lambda,\omega,t)\\ \beta'&=R_0(\rho,\beta,\theta_0,\lambda,\omega,t)\end{aligned} \tag{5.16}$$

with

$$\begin{aligned}\tilde{B}&=a(\hat{\Theta})\\ F_0&=f(\rho+p_\lambda,\beta+\hat{\Theta},\omega\cdot t,\lambda)\\ &\quad-f(p_\lambda,\hat{\Theta},\omega\cdot t,\lambda)+(a(\beta+\hat{\Theta})-a(\hat{\Theta}))(\rho+p_\lambda)\\ R_0&=(q(\beta+\hat{\Theta})-q(\hat{\Theta}))\\ &\quad+(g(\rho+p_\lambda,\beta+\hat{\Theta},\omega\cdot t,\lambda)-g(p_\lambda,\hat{\Theta},\omega\cdot t,\lambda)),\end{aligned}$$

where $\hat{\Theta} =: \hat{\Theta}(t)$, $p_\lambda =: p_\lambda(\hat{\Theta}(t), \omega \cdot t)$, $a(\theta) =: a(\theta, \omega \cdot t, \lambda)$, $q(\theta) =: q(\theta, \omega \cdot t, \lambda)$.

It then follows from Theorem 5.1(2) that

$$u' = \tilde{B}(\theta_0, \lambda, \omega, t)\, u \tag{5.17}$$

has ED on R uniformly for any $\theta_0 \in T^k$, $\lambda \in I_{\lambda_1}$ $(0 < \lambda_1 \leqslant \lambda^0)$, and any $\omega \in \Omega_0$ with projections $P_\lambda(\theta_0, \omega)$ and uniform ED constants K, δ_{λ_1} with $\delta_{\lambda_1} < \delta^0 - \varepsilon_0/3$, $\delta_{\lambda_1} \to \delta^0 - \varepsilon_0/3$ as $\lambda_1 \to 0$.

We choose $\lambda_1 \leqslant \lambda^0$ first so that $\delta := \delta_{\lambda_1} > \delta^0 - 2\varepsilon_0/3$.

Let $\phi: R^n \to [0, 1]$ be a C^∞ function equal to 1 if $|x| \leqslant 1$, equal to 0 $|x| \geqslant 2$. For any $\lambda_0 \in (0, \lambda_1)$, denote $\eta^0 := \eta^0(\lambda_0) = \sup_{\theta \in T^k, \omega \in \Omega_0, \lambda \in I_{\lambda_0}} |p_\lambda(\theta, \omega)|$. Since p_λ is C^1 in λ and $p_0 \equiv 0$, then

$$\eta^0 \leqslant M\lambda_0 \tag{5.18}$$

for some constant $M > 0$.

Denote

$$F_1(\rho, \beta, \theta_0, \lambda, \omega, t) = F_0(\phi(\rho)\, \rho, \beta, \theta_0, \lambda, \omega, t)$$
$$R_1(\rho, \beta, \theta_0, \lambda, \omega, t) = R_0(\phi(\rho)\, \rho, \beta, \theta_0, \lambda, \omega, t).$$

Then, the equation

$$\begin{aligned} \rho' &= \tilde{B}(\theta_0, \lambda, \omega, t)\, \rho + F_1(\rho, \beta, \theta_0, \lambda, \omega, t) \\ \beta' &= R_1(\rho, \beta, \theta_0, \lambda, \omega, t) \end{aligned} \tag{5.19}$$

is identical with (5.16) if $|\rho| \leqslant 1$.

We now rescale (5.19) by

$$\rho \to \sqrt{\lambda_0}\, u$$
$$\beta \to \beta.$$

Then the equation for (u, β) becomes

$$\begin{aligned} u' &= \tilde{B}(\theta_0, \lambda, \omega, t)\, u + \tilde{F}(u, \beta, \theta_0, \lambda, \omega, t) \\ \beta' &= R(u, \beta, \theta_0, \lambda, \omega, t), \end{aligned} \tag{5.20}$$

where

$$\tilde{F} = \frac{1}{\sqrt{\lambda_0}} F_1(\sqrt{\lambda_0}\, u, \beta, \theta_0, \lambda, \omega, t)$$

$$\tilde{R} = R_1(\sqrt{\lambda_0}\, u, \beta, \theta_0, \lambda, \omega, t).$$

Let $\Sigma_{\lambda_0}=T^k\times I_{\lambda_0}\times\Omega_0$, $E_{\lambda_0}=R^{n+m}\times\Sigma_{\lambda_0}\times R$, $C_1(\lambda_0)=:\sup_{E_{\lambda_0}}|\partial u\tilde{F}|$, $C_2(\lambda_0)=:\sup_{E_{\lambda_0}}|\partial_\beta\tilde{F}|$, $C_3(\lambda_0)=:\sup_{E_{\lambda_0}}|\partial u\tilde{R}|$, $L_{\lambda_0}=:\sup_{E_{\lambda_0}}|\partial_\beta\tilde{R}|$. It is clear by our assumptions that $C_i(\lambda_0)\to 0$, $L_{\lambda_0}\to L_0$, $L_{\lambda_0}\geqslant L_0$, as $\lambda_0\to 0$ $(i=1,2,3)$. Furthermore, if N_{λ_0} is defined as in Theorem 5.1, then it is easy to verify that there are constants C, $C_1, C_2>0$ such that for $\sigma=(\theta_0,\lambda)\in T^k\times I_{\lambda_0}$,

$$\sup_{E_{\lambda_0}}|\partial^i_{(u,\beta)}\partial^j_\sigma\tilde{F}|\leqslant\begin{cases}C_1(|u|+|\beta|)\,e^{jN_{\lambda_0}|t|} & (i=0),\\ C_1e^{jN_{\lambda_0}|t|} & (i>0),\end{cases}$$

$$\sup_{E_{\lambda_0}}|\partial^i_{(u,\beta)}\partial^j_\sigma\tilde{R}|\leqslant\begin{cases}C_2(|u|+|\beta|)\,e^{jN_{\lambda_0}|t|} & (i=0),\\ C_2e^{jN_{\lambda_0}|t|}, & (i>0)\end{cases}$$

$$\sup_{\Sigma_{\lambda_0}\times R}|\partial^j_\sigma\tilde{B}|\leqslant Ce^{jN_{\lambda_0}|t|}$$

hold true for all $i\geqslant 0$, $j\geqslant 0$ with $0\leqslant i+j\leqslant r-1$.

For $\varepsilon:=\varepsilon_0/3$, we conclude from Corollary 4.3 that there is $\lambda^*(\varepsilon)=:\lambda_0(\varepsilon_0)\equiv\lambda_0\leqslant\lambda_1$ such that (5.20) has for each $\lambda\in I_{\lambda_0}$, $\omega\in\Omega_0$, $\theta_0\in T^k$, $\xi\in R^n$ unique solutions $\phi^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0,t)\equiv(u^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0,t),\ \beta^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0,t))\in Z_{\pm\eta}$ with $\eta\in[L_0+\varepsilon_0/3,\ \delta_1-\varepsilon_0/3]$. Note that $[L_0+\varepsilon_0,\delta^0-\varepsilon_0]\subset[L_0+\varepsilon_0/3,\ \delta_1-\varepsilon_0/3]$; hence $\eta\in[L_0+\varepsilon_0,\delta^0-\varepsilon_0]$.

By applying Corollary 4.3 with $\Sigma_{\lambda_0}:=T^k\times I_{\lambda_0}$ or $\Sigma_{\lambda_0}:=\Omega_0$, respectively, we see that $\phi^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0,0)$ are C^r in $\xi\in R^n$, C^{r-1} in $\theta_0\in T^k$ and $\lambda\in I_{\lambda_0}$, and continuous in $\omega\in\Omega_0$.

Since $\partial_\xi u^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0,t)$ exist and are bounded for $t\in R^\pm$, and $u^\lambda_\pm(0,\theta_0,\omega,\lambda_0,t)\equiv 0$, there exists an $a_0:=a_0(\lambda_0)>0$, such that $|\sqrt{\lambda_0}\,u^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0,t)|\leqslant 1$ if $|\xi|\leqslant a_0$.

Now, we denote

$$\begin{aligned}
Z^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0)&=:\sqrt{\lambda_0}\,u^\lambda_\pm(p,\theta_0,\omega,\lambda_0,0)+p_\lambda(\theta_0,\omega),\\
\Theta^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0)&=:\beta^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0,0)+\theta_0,\\
D(a_0)&=:\{(\xi,\theta,\omega)\in R^n\times T^k\times\Omega_0\mid|\xi|\leqslant a_0\},\\
V^\pm_\lambda(\theta_0,\omega,a_0)&=:V^\pm_\lambda(\theta_0,\omega)\cap D(a_0).
\end{aligned}$$

Here $V^\pm_\lambda(\theta_0,\omega)$ are stable and unstable subspaces of (5.17). Let

$$\begin{aligned}
&\tilde{\mathscr{W}}^\pm_\lambda(\theta_0,\omega,a_0)\\
&\quad=\{(Z^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0),\tilde{\Theta}^\lambda_\pm(\xi,\theta_0,\omega,\lambda_0),\omega)\mid\xi\in V^\pm_\lambda(\theta_0,\omega,a_0)\},
\end{aligned}\tag{5.21}$$

and

$$\tilde{\mathscr{W}}^\pm(a_0)=\bigcup_{\substack{\theta\in T^k\\ \omega\in\Omega_0}}\tilde{\mathscr{W}}^\pm_\lambda(\theta_0,\omega,a_0).\tag{5.22}$$

CLAIM 1. *$\tilde{\mathscr{W}}_\lambda^\pm(\theta_0, \omega, a_0)$ are C^r diffeomorphic* to $V_\lambda^\pm(\theta_0, \omega, a_0)$.

Proof. Define for fixed $\theta_0, \omega, \lambda$, $f^\pm_{\theta_0, \omega, \lambda} : V_\lambda^\pm(\theta_0, \omega, a_0) \to \tilde{W}_\lambda^\pm(\theta_0, \omega, a_0)$ by $f^\pm_{\theta_0, \omega, \lambda}(\xi) = (Z_\pm^\lambda(\xi, \theta, \omega, \lambda_0),\ \Theta_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0), \omega)$. We only need to check that $f^\pm_{\theta_0, \omega, \lambda}$ are one-to-one. This follows from Corollary 4.3, that is, $P_\lambda(\theta_0, \omega)\, u_+^\lambda(\xi, \theta_0, \omega, \lambda_0, 0) = P_\lambda(\theta_0, \omega)\,\xi$, and $(I - P_\lambda(\theta_0, \omega))\, u_-^\lambda(\xi, \theta_0, \omega, \lambda_0, 0) = (I - P_\lambda(\theta_0, \omega))\,\xi$. Q.E.D.

CLAIM 2. *$\tilde{\mathscr{W}}_\lambda^+(\theta_0, \omega, a_0) \pitchfork \tilde{\mathscr{W}}_\lambda^-(\theta_0, \omega, a_0)$ at $(p_\lambda(\theta_0, \omega), \theta_0, \omega) \in \mathscr{S}_\lambda$.*

Proof. Define $\hat{\mathscr{W}}_\lambda^\pm(\theta_0, \omega, \lambda_0) = \{(u_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0, 0) + p_\lambda(\theta_0, \omega),\ \Theta_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0), \omega) \mid \xi \in V_\lambda^\pm(\theta_0, \omega)\}$. Then, to prove the claim, it is equivalent to show that

$$\hat{\mathscr{W}}_\lambda^+(\theta_0, \omega, \lambda_0) \pitchfork \hat{\mathscr{W}}_\lambda^-(\theta_0, \omega, \lambda_0).$$

From our constructions, we observe that

$$\partial_\xi u_\pm^0(\xi, \theta_0, \omega, 0, 0)|_{V^\pm(\theta_0, \omega)} \equiv I, \qquad \partial_\xi \beta_\pm^0(\xi, \theta_0, \omega, 0, 0)|_{V^\pm(\theta_0, \omega)} \equiv 0.$$

Then, $T_{\{\xi = 0\}} \hat{\mathscr{W}}_0^\pm(\theta_0, \omega, 0) = V_\lambda^\pm(\theta_0, \omega)$. Hence

$$\hat{\mathscr{W}}_0^+(\theta_0, \omega, 0) \pitchfork \hat{\mathscr{W}}_0^-(\theta_0, \omega, 0). \tag{5.23}$$

Therefore,

$$\hat{\mathscr{W}}_\lambda^+(\theta_0, \omega, \lambda_0) \pitchfork \hat{\mathscr{W}}_\lambda^-(\theta_0, \omega, \lambda_0) \tag{5.24}$$

for λ_0 small and all $\lambda \in I_{\lambda_0}$. Q.E.D.

CLAIM 3. *$\tilde{\mathscr{W}}_\lambda^\pm(a_0)$ are overflowing invariant to Λ_t^λ.*

Proof. Take $(z, \theta, \omega) \in \tilde{\mathscr{W}}_\lambda^\pm(a_0)$. Then $(z, \theta, \omega) \in \tilde{\mathscr{W}}_\lambda^\pm(\theta_0, \omega, a_0)$ for some $(\theta_0, \omega) \in T^k \times \Omega_0$. It follows from Claim 1 that there is a unique $\xi \in V_\lambda^\pm(\theta_0, \omega, a_0)$ such that $(z, \theta, \omega) = f^\pm_{\theta_0, \omega, \lambda}(\xi)$. Let

$$\begin{aligned} \hat{z}_\pm(t) &= \sqrt{\lambda_0}\, u_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0, t) + p_\lambda(\Gamma_t(\theta_0, \omega)) \\ \hat{\theta}_\pm(t) &= \beta_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0, t) + \hat{\Theta}(\theta_0, \omega, t). \end{aligned} \tag{5.25}$$

Since $|\xi| \leqslant a_0$, then $|\sqrt{\lambda_0}\, u_\pm^\lambda(\xi, \theta_0, \omega, t)| \leqslant 1$ for $t \in R^\pm$. Therefore

$$\Lambda_t(z, \theta, \omega) = (\hat{z}_\pm(t), \hat{\theta}_\pm(t), \omega \cdot t), \qquad t \in R^\pm. \tag{5.26}$$

Denote by $\Phi_\lambda(\theta_0, \omega, t)$ the fundamental matrix of (5.17) with $\Phi_\lambda(\theta_0, \omega, 0) \equiv I$. It is not hard to verify by our constructions that

$$\begin{aligned} (\hat{z}_\pm(t), \hat{\theta}_\pm(t), \omega \cdot t) = (&Z_\pm^\lambda(\phi_\lambda(\theta_0, \omega, t)\,\xi, \Gamma_t(\theta_0, \omega), \lambda_0, 0), \\ &\Theta_\pm^\lambda(\phi_\lambda(\theta_0, \omega, t)\,\xi, \Gamma_t(\theta_0, \omega), \lambda_0, 0), \omega \cdot t). \end{aligned} \tag{5.27}$$

Combining (5.26), (5.27) we see that $\Lambda_t^\lambda(z, \theta, \omega) \in \tilde{\mathscr{W}}_\lambda^\pm(\Gamma_t(\theta_0, \omega), a_0)$ for $t \in R^\pm$. Q.E.D.

$\tilde{\mathscr{W}}_\lambda^+(a_0)$ and $\tilde{\mathscr{W}}_\lambda^-(a_0)$ are referred to as local stable and unstable manifolds to $\mathscr{S}_\lambda$ by the following:

CLAIM 4. $E_\lambda^\pm(\theta_0, \omega, r_0) \subset \tilde{\mathscr{W}}_\lambda^\pm(\theta_0, \omega, a_0) \subset E_\lambda^\pm(\theta_0, \omega, 1)$, *where*

$$E_\lambda^\pm(\theta_0, \omega, r) := \{(z, \theta, \omega) \in R^n \times T^k \times \Omega_0 \mid |z - p_\lambda(\theta_0, \omega)| \leqslant r,$$
$$\sup_{t \in R^\pm} |\Lambda_t(z, \theta, \omega) - \Lambda_t(p_\lambda(\theta_0, \omega), \theta_0, \omega)| e^{\pm \eta t} < \infty,$$
$$\text{for some } \eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]\} \qquad \text{for any} \quad r > 0 \quad (5.28)$$

and

$$r_0 \equiv \frac{\sqrt{\lambda_0}\, a_0}{\eta^0(\lambda_0)}.$$

Proof. For any $(z, \theta, \omega) \in \tilde{\mathscr{W}}_\lambda^\pm(\theta_0, \omega, a_0)$, there exists a unique $\xi \in V_\lambda^\pm(\theta_0, \omega, a_0)$ such that

$$z = \sqrt{\lambda_0}\, u_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0, 0) + p_\lambda(\theta_0, \omega),$$
$$\theta = \phi_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0, 0) + \theta_0.$$

Let $\hat{z}_\pm(t)$, $\hat{\theta}_\pm(t)$ be given in (5.25). By the same reasoning as before, $\Lambda_t(z, \theta, \omega) = (\hat{z}_\pm(t), \hat{\theta}_\pm(t), \omega \cdot t)$, and $|\sqrt{\lambda_0}\, u_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0, t)| \leqslant 1$ for $t \in R^\pm$. In particular, $|z - p_\lambda(\theta_0, \omega)| \leqslant 1$, $\Lambda_t(z, \theta, \omega) - \Lambda_t(p_\lambda(\theta_0, \omega), \theta_0, \omega) = (\sqrt{\lambda_0}\, u_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0, t), \beta_\pm^\lambda(\xi, \theta_0, \omega, \lambda_0, t), 0) \in Z_{\pm\eta}$ for $\eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]$. This proves that $\tilde{\mathscr{W}}^\pm(\theta_0, \omega, a_0) \subset E_\lambda^\pm(\theta_0, \omega, 1)$.

Next, we take $(z, \theta, \omega) \in E_\lambda^+(\theta_0, \omega, r_0)$ and define $\xi = (1/\sqrt{\lambda_0})\, P_\lambda(\theta_0, \omega)(z - p_\lambda(\theta_0, \omega))$. By the way that r_0 is chosen, we see that $|\xi| \leqslant a_0$. Let $(z(t), \theta(t))$ be the solution of $(5.4)_\omega$ with $(z(0), \theta(0)) = (z, \theta)$; define $u(t) = (1/\sqrt{\lambda_0})(z(t) - p_\lambda(\Gamma_t(\theta_0, \omega)))$, $\beta(t) = \theta(t) - \hat{\Theta}(\theta_0, \omega, t)$. Then $(u(t), \beta(t)) \in Z_{\eta+}$ for $\eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]$, and, $(u(t), \beta(t))$ solves (5.20). Since $P_\lambda(\theta_0, \omega)\, \xi = P_\lambda(\theta_0, \omega)\, u(0)$, by uniqueness (Corollary 4.3), then, $u(t) \equiv u_+^\lambda(\xi, \theta_0, \omega, \lambda_0, t)$ and $\beta(t) \equiv \beta_+^\lambda(\xi, \theta_0, \omega, \lambda_0, t)$. Specifically,

$$z = \sqrt{\lambda_0}\, u_+^\lambda(\xi, \theta_0, \omega, \lambda_0, 0) + p_\lambda(\theta_0, \omega)$$
$$\theta = \beta_+^\lambda(\xi, \theta_0, \omega, \lambda_0, 0) + \theta_0.$$

Recall that $|\xi| \leqslant a_0$. Hence, $(z, \theta, \omega) \in \tilde{\mathscr{W}}_\lambda^+(\theta_0, \omega, a_0)$. Therefore, $E_\lambda^+(\theta_0, \omega, r_0) \subset \tilde{\mathscr{W}}_\lambda^-(\theta_0, \omega, a_0)$. Similarly, $E_\lambda^-(\theta_0, \omega, r_0) \subset \tilde{\mathscr{W}}_\lambda^-(\theta_0, \omega, a_0)$.
Q.E.D.

CLAIM 5. $\tilde{\mathscr{W}}_{\lambda}^{\pm}(\theta_0, \omega, a_0) \cap \tilde{\mathscr{W}}_{\lambda}^{\pm}(\theta_1, \omega, a_0) = \phi$ *if* $\theta_0 \neq \theta_1$.

Proof. If not, there is a $(z, \theta, \omega) \in \tilde{W}^{\pm}(\theta_0, \omega, a_0) \cap \tilde{W}^{\pm}(\theta_1, \omega, a_0)$. By Claim 4, $\hat{\theta}(t) \equiv \hat{\Theta}_{\lambda}(\theta_0, \omega, t) - \hat{\Theta}_{\lambda}(\theta_1, \omega, t) \in X_{\eta\pm}$ with $\eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]$. Consider

$$\beta' = f_{\lambda}(\beta, t), \qquad \beta \in T^k, \tag{5.29}$$

where $f_{\lambda}(\beta, t) = q(\beta + \hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t, \lambda) + g(p_{\lambda}(\beta + \hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t), \beta + \hat{\Theta}(\theta_0, \omega, t), \lambda) - q(\hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t, \lambda) - g((p_{\lambda}(\hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t), \hat{\Theta}(\theta_0, \omega, t), \lambda), \lambda)$. It is clear that

$$f_{\lambda}(0, t) \equiv 0 \qquad \text{and} \qquad \sup_{\substack{\lambda \in I \\ \beta \in T^k \\ t \in R}} |\partial_{\beta} f_{\lambda}(\beta, t)| =: c(\lambda_0) \to L_0$$

as $\lambda_0 \to 0$. Without loss of generality, we assume that $c(\lambda_0) < L_0 + \varepsilon_0$. For $\eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]$, we define mappings $F_{\pm} : X_{\eta\pm} \to X_{\eta\pm}$ by $F_{+}(\beta) = -\int_t^{\infty} f_{\lambda}(\beta(s), s)\, ds$, $F_{-}(\beta) = \int_{-\infty}^{t} f_{\lambda}(\beta(s), s)\, ds$. It follows from the contraction mapping principle that there is a unique fixed point $\beta_{\pm} \in X_{\eta\pm}$ to $F_{\pm}$ which is clearly a solution of (5.29). Since $\beta \equiv 0 \in X_{\eta\pm}$ and $\beta \equiv \hat{\theta} \in X_{\eta\pm}$ are both solutions of (5.29). By uniqueness of the fixed point, then $\hat{\theta}(t) \equiv 0$; in particular, $\theta_1 = \theta_0$. Q.E.D.

We now define

$$\mathscr{W}_{\lambda}^{\pm}(\theta_0, \omega) = \bigcup_{t \in R} \Lambda_t(\tilde{\mathscr{W}}_{\lambda}^{\pm}(\theta_0, \omega, a_0)),$$

and

$$\mathscr{W}_{\lambda}^{\pm}(\mathscr{S}_{\lambda}) = \bigcup_{\substack{\theta_0 \in T^k \\ \omega \in \Omega_0}} \mathscr{W}^{\pm}(\theta_0, \omega).$$

CLAIM 6.

$$\begin{aligned} \mathscr{W}_{\lambda}^{\pm}(\theta_0, \omega) = \{ & (z, \theta, \omega) \in R^n \times T^k \times \Omega_0 \mid \\ & \times \sup_{t \in R^{\pm}} |\Lambda_t(z, \theta, \omega) - \Lambda_t(p_{\lambda}(\theta_0, \omega), \theta_0, \omega)|\, e^{\pm \eta t} < \infty \\ & \textit{for some} \quad \eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0] \}. \end{aligned}$$

Proof. For any $(z, \theta, \omega) \in \mathscr{W}_{\lambda}^{\pm}(\theta_0, \omega)$, we have

$$(z, \theta, \omega) \in \Lambda_{\tau}(\tilde{\mathscr{W}}_{\lambda}^{\pm}(\theta_0, \omega, a_0)) \subset \Lambda_{\tau}(E_{\lambda}^{\pm}(\theta_0, \omega, 1))$$

for some $\tau \in R^{\pm}$. Therefore,

$$\sup_{t \in R^{\pm}} |\Lambda_{t+\tau}(z, \theta, \omega) - \Lambda_{t+\tau}(p_{\lambda}(\theta_0, \omega), \theta_0, \omega)|\, e^{\pm \eta t} < \infty. \tag{5.30}$$

Hence

$$\sup_{t \in R^{\pm}} |\Lambda_t(z, \theta, \omega) - \Lambda_t(p_\lambda(\theta_0, \omega), \theta_0, \omega)| \, e^{\pm \eta t} < \infty. \tag{5.31}$$

On the other hand, if (z, θ, ω) is such that (5.31) holds, then there is $\tau \in R^{\pm}$ so that $|\Lambda_\tau(z, \theta, \omega) - \Lambda_\tau(p_\lambda(\theta_0, \omega), \theta_0, \omega)| \leqslant r_0$, that is,

$$\begin{aligned}(z, \theta, \omega) \in E_\lambda^{\pm}(\Gamma_\tau(\theta_0, \omega), r_0) &\subset \tilde{\mathscr{W}}_\lambda^{\pm}(\Gamma_\tau(\theta_0, \omega), a_0)\\ &= \Lambda_\tau(\tilde{\mathscr{W}}_\lambda^{\pm}(\theta_0, \omega, a_0)) \subset \mathscr{W}_\lambda^{\pm}(\theta_0, \omega).\end{aligned}$$

Q.E.D.

CLAIM 7.

$$\begin{aligned}\mathscr{W}_\lambda^{\pm}(\mathscr{S}_\lambda) = \{(z, \theta, \omega) \in R^n \times T^k \times \Omega_0 \mid\ & \textit{the solution } (\tilde{z}(t), \tilde{\theta}(t))\\ & \textit{of } (5.4)_\omega \textit{ with } (\tilde{z}(0), \tilde{\theta}(0)) = (z, \theta)\\ & \textit{is such that } \sup_{t \in R^{\pm}} |\tilde{z}(t) - p_\lambda(\tilde{\theta}(t), \omega \cdot t)| \, e^{\pm \eta t} < \infty\\ & \textit{for some } \eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]\}.\end{aligned} \tag{5.32}$$

Proof. Denote by $\hat{\mathscr{W}}_\lambda^{\pm}$ the right-hand side of (5.32). It is clear that $\mathscr{W}_\lambda^{\pm}(\mathscr{S}_\lambda) \subset \hat{\mathscr{W}}_\lambda^{\pm}$.

Now, for any $(z, \theta, \omega) \in \hat{\mathscr{W}}_\lambda^{\pm}$, let $(\tilde{z}(t), \hat{\theta}(t))$ be the solution of $(5.4)_\omega$ with $(\tilde{z}(0), \tilde{\theta}(0)) = (z, \theta)$. Then $\sup_{t \in R^{\pm}} |\tilde{z}(t) - p_\lambda(\tilde{\theta}(t), \omega \cdot t)| e^{\pm \eta t} < \infty$ for some $\eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]$.

Consider

$$\beta' = F_1(\beta, t) + F_2(t) \tag{5.33}$$

where

$$\begin{aligned}F_1(\beta, t) = q(\beta + \tilde{\theta}(t), \omega \cdot t, \lambda) &- q(\tilde{\theta}(t), \omega \cdot t, \lambda)\\ &+ g(p_\lambda(\beta + \tilde{\theta}(t), \omega \cdot t), \beta + \tilde{\theta}(t), \omega \cdot t, \lambda)\\ &- g(p_\lambda(\tilde{\theta}(t), \omega \cdot t), \tilde{\theta}(t), \omega \cdot t, \lambda),\\ F_2(t) = g(p_\lambda(\tilde{\theta}(t), \omega \cdot t), &\tilde{\theta}(t), \omega \cdot t, \lambda) - g(\tilde{z}(t), \tilde{\theta}(t), \omega \cdot t, \lambda).\end{aligned}$$

It is clear that $F_2 \in X_{\eta\pm}$, $F_1(0, t) \equiv 0$, $\sup_{\beta \in T^k, t \in R^{\pm}, \lambda \in I_{\lambda_0}} |\partial_\beta F_1(\beta, t)| := c(\lambda_0) \leqslant L_0$, and $c(\lambda_0) \to L_0$, as $\lambda_0 \to 0$. Without loss of generality, we assume that $c(\lambda_0) < L_0 + \varepsilon_0$. Define $T_{\pm} : X_{\eta\pm} \to X_{\eta\pm}$ by

$$T_+ \beta = -\int_t^\infty (F_1(\beta(s), s) + F_2(s)) \, ds,$$

$$T_- \beta = \int_{-\infty}^t (F_1(\beta(s), s) + F_2(s)) \, ds.$$

It then follows from the contraction mapping principle that $T_{\pm}$ has a unique fixed point $\beta_{\pm} \in X_{\eta\pm}$. Let $\theta_0 = \beta_{\pm}(0) + \theta$. Then $\hat{\Theta}_{\lambda}(\theta_0, \omega, t) = \beta_{\pm}(t) + \tilde{\theta}(t)$. Since

$$\tilde{z}(t) - p_{\lambda}(\hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t)$$
$$= [\tilde{z}(t) - p_{\lambda}(\tilde{\theta}(t), \omega \cdot t)] + [p_{\lambda}(\tilde{\theta}(t), \omega \cdot t) - p_{\lambda}(\hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t)],$$

then $\tilde{z}(t) - p_{\lambda}(\hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t) \in X_{\eta\pm}$ as well. Therefore

$$\Lambda_t(z, \theta, \omega) - \Lambda_t(p_{\lambda}(\theta_0, \omega), \theta_0, \omega)$$
$$= (\tilde{z}(t), \tilde{\theta}(t), \omega \cdot t) - (p_{\lambda}(\hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t), \hat{\Theta}_{\lambda}(\theta_0, \omega, t), \omega \cdot t)$$

goes to 0 exponentially with rate $\eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]$ as $t \to \pm\infty$, that is, $(z, \theta, \omega) \in \mathscr{W}_{\lambda}^{\pm}(\mathscr{S}_{\lambda})$. Q.E.D.

The proof of Theorem 5.2 is completed by Claims 1–7. ▮

COROLLARY 5.4. *Consider* $(5.4)_{\omega}$. *Assume that* (I), (II), (III) *hold. If* (5.2) *is linearly stable, that is,* $P(\theta_0) \equiv I$ *(identity), then there is a* $\lambda_0 > 0$ *sufficiently small such that for each* $\lambda \in I_{\lambda_0}$, $\mathscr{S}_{\lambda}$ *is orbitally and asymptotically stable in the sense that* $\exists \eta_0 = \eta_0(\lambda_0) > 0$, *and if* $d((z, \theta, \omega), \mathscr{S}_{\lambda}(\omega)) \leqslant \eta_0$ *then there is a unique* $\theta_0 \in T^k$ *such that* $\Lambda_t(z, \theta, \omega) - \Lambda_t(p_{\lambda}(\theta_0, \omega), \theta_0, \omega) \to 0$ *exponentially as* $t \to +\infty$ *with rate* $\eta = (\delta^0 + L_0)/2$, *where* $\mathscr{S}_{\lambda}(\omega) = \mathscr{S}_{\lambda} \cap (R^n \times T^k \times \{\omega\})$, *and d is a metric on* $R^n \times T^k \times \Omega_0$.

Proof. Since $P(\theta_0) = I$, that is, the S–S spectrum Σ_0 of $(5.8)_0$ is contained in $(-\infty, 0)$, by the S–S perturbation theorem $\exists \lambda_1 > 0$ such that for each $\lambda \in I_{\lambda_1}$, the S–S spectrums Σ_{λ} of $(5.8)_{\lambda}$ are all contained in $(-\infty, 0)$. Therefore, $P_{\lambda}(\theta, \omega) \equiv I$ in Theorem 5.2(2); hence, $V_+^{\lambda}(\theta_0, \omega) = R^n$ for $\lambda \in I_{\lambda_1}$.

Since $\eta := (\delta^0 + L_0)/2 \in (L_0, \delta^0)$, we fix a small $\varepsilon > 0$ so that $\eta \in [L_0 + \varepsilon_0, \delta^0 - \varepsilon_0]$. Let $\lambda_0 \leqslant \lambda_1$ be given by Theorem 5.2 so that the functions $u_{\pm}^{\lambda}(\xi, \theta_0, \omega, \lambda_0, 0)$, $\beta_{\pm}^{\lambda}(\xi, \theta_0, \omega, \lambda_0, 0)$ in the proofs of Theorem 5.2 are well defined for $\lambda \in I_{\lambda_0}$. Since $P_{\lambda}(\theta_0, \omega) \equiv I$, then $u_+^{\lambda}(\xi, \theta_0, \omega, \lambda_0, 0) \equiv \xi$, $u_-^{\lambda}(\xi, \theta_0, \omega, \lambda_0, 0) \equiv 0$. Let $f_{\lambda}(\xi, \theta, \omega) = (\sqrt{\lambda_0}\, \xi + p_{\lambda}(\theta, \omega), \theta + \beta_+^{\lambda}(\xi, \theta, \omega, \lambda_0, 0), \omega)$. Since $\partial_{\theta} \beta_+^{\lambda}(0, \theta, \omega, \lambda_0, 0) \equiv 0$, $\partial_{\theta} p_0(\theta, \omega) \equiv 0$, we can find $\eta_* > 0$, $\eta_* < a_0$ (for λ_0 sufficiently small) such that

$$C := \Big(\sup_{\{|\xi| \leqslant \eta_*\} \times T^k \times \Omega_0 \times I_{\lambda_0}} |\partial_{\xi} \beta_+^{\lambda}|\Big) \frac{\eta^0(\lambda_0)}{\sqrt{\lambda_0}} + \Big(\sup_{\{|\xi| \leqslant \eta_*\} \times T^k \times \Omega_0 \times I_{\lambda_0}} |\partial_{\theta} \beta_+^{\lambda}|\Big) < 1,$$

where $\eta^0(\lambda_0) \leqslant M\lambda_0$ is given by (5.18), and a_0 is defined in Theorem 5.2. Denote for $\lambda \in I_{\lambda_0}$, $E_{\lambda}(\eta_*) = \{(\xi, \theta, \omega) \in R^n \times T^k \times \Omega_0 \mid |\xi| \leqslant \eta_*,\ \theta \in T^k,$

$\omega \in \Omega_0\}$. For any $(\xi_1, \theta_1, \omega_1), (\xi_2, \theta_2, \omega_2) \in E_\lambda(\eta_0)$ such that $f_\lambda(\xi_1, \theta_1, \omega_1) = f_\lambda(\xi_2, \theta_2, \omega_2)$, then $\omega_1 = \omega_2 =: \omega$, $\sqrt{\lambda_0}\,\xi_1 + p_\lambda(\theta_1, \omega) = \sqrt{\lambda_0}\,\xi_2 + p_\lambda(\theta_2, \omega)$, $\theta_1 + \beta^\lambda_+(\xi_1, \theta_1, \omega, \lambda_0, 0) = \theta_2 + \beta^\lambda_+(\xi_2, \theta_2, \omega, \lambda_0, 0)$. Therefore, $\sqrt{\lambda_0}\,|\xi_1 - \xi_2| \leqslant \eta^0(\lambda_0)|\theta_1 - \theta_2|$, and, $|\theta_1 - \theta_2| \leqslant \sup|\partial_\xi \beta^\lambda_+| \cdot |\xi_1 - \xi_2| + \sup|\partial_\theta \beta^\lambda_+|\,|\theta_1 - \theta_1| \leqslant C\,|\theta_1 - \theta_2|$. Since $C < 1$, then $\theta_1 = \theta_2$; hence $\xi_1 = \xi_2$ as well. We then conclude from the above arguments that $f_\lambda\colon E_\lambda(\eta_*) \to f_\lambda(E_\lambda(\eta_*))$ is a homeomorphism. (This fact also follows from Claims 1, 5 in the proof of Theorem 5.1.) Since $f_\lambda(0, \theta, \omega) = (p_\lambda(\theta, \omega), \theta, \omega)$, then there is an $\eta_0 = \eta(\lambda_0) > 0$ such that

$$\mathscr{W}_\lambda(\eta_0) := \{(z, \theta, \omega) \in R^n \times T^k \times \Omega_0 \mid d((z, \theta, \omega), \mathscr{S}_\lambda(\omega)) \leqslant \eta_0\}$$
$$\subset f_\lambda(E_\lambda(\eta_*)) \subset \mathscr{W}^+_\lambda(\mathscr{S}_\lambda).$$

By Theorem 5.2, if $(z, \theta, \omega) \in \mathscr{W}_\lambda(\eta_0)$, there is a unique $\theta_0 \in T^k$ such that $\Lambda_t(z, \theta, \omega) - \Lambda_t(p_\lambda(\theta_0, \omega), \theta_0, \omega) \to 0$ exponentially as $t \to +\infty$ with rate $\eta = (L_0 + \delta^0)/2$. ∎

As a special case of $(5.1)_\lambda$ or $(5.4)_\omega$, we consider

$$\begin{aligned} z' &= B(\theta)\,z + F(z, \theta, \lambda) \\ \theta' &= \omega + G(z, \theta, \lambda), \end{aligned} \tag{5.34$_\lambda$}$$

where $z \in R^n$, $\theta \in T^k$, $\lambda \in R^m$, $\omega = (\omega_1, ..., \omega_k)^T \in R^k$, B, F, G are C^2 functions such that $G(0, \theta, 0) \equiv 0$, $F(0, \theta, 0) \equiv 0$, $\partial_z F(0, \theta, 0) \equiv 0$. It is clear that $\mathscr{S}_0 =: \{0\} \times T^k$ is an invariant torus to Eq. $(5.34)_0$.

The following corollary is an immediate consequence of Theorems 5.1, 5.2, and Corollary 5.4.

COROLLARY 5.5. *Consider* $(5.34)_\lambda$, *and denote by* Λ^λ_t *the flow generated by* $(5.34)_\lambda$. *Assume that*

$$z' = B(\theta \cdot t)\,z \tag{5.35}$$

has ED on T^k, *where* $\theta \cdot t \equiv \theta + \omega t$. *We denote by* δ^0 *the smallest Lyapounov exponent of* (5.35) *in absolute value. Then there is a* $\lambda_0 > 0$ *such that the following hold*:

(1) *Equation* $(5.34)_\lambda$ *possess for each* $\lambda \in I_{\lambda_0}$ *a unique invariant torus of type* $\mathscr{S}_\lambda = \{(p_\lambda(\theta), \theta) \mid \theta \in T^k\}$, *and* $\mathscr{S}_\lambda \to \mathscr{S}_0$ *as* $\lambda \to 0$.

(2) *There are invariant manifolds* $W^\pm_\lambda(\mathscr{S}_\lambda)$ *to Eq.* $(5.34)_\lambda$ *such that*

(a) $W^\pm_\lambda(\mathscr{S}_\lambda) = \{(z, \theta) \in R^n \times T^k \mid$ *there is a unique* $(p_\lambda(\theta_0), \theta_0) \in \mathscr{S}_\lambda$ *such that* $\sup_{t \in R^\pm} |\Lambda_t(z, \theta) - \Lambda_t(p_\lambda(\theta_0), \theta_0)|\, e^{\pm \eta t} < \infty$ *for some* $\eta \in [\delta^0/4, \delta^0/2]\} = \{(z, \theta) \in R^n \times T^k \mid$ *the solution* $(z(t), \theta(t))$ *of* $(5.34)_\lambda$ *with*

$(z(0), \theta(0)) = (z, \theta)$ is such that $\sup_{t \in R^{\pm}} |z(t) - p_\lambda(\theta(t))| \, e^{\pm \eta t} < \infty$ *for some* $\eta \in [\delta^0/4, \delta^0/2]\}$.

(b) *$W_\lambda^{\pm}(\mathcal{S}_\lambda)$ are foliated by disjoint immersed submanifolds* $W_\lambda^{\pm}(\theta_0) = \{(z, \theta) \in R^n \times T^k \mid \sup_{t \in R^{\pm}} |\Lambda_t(z, \theta) - \Lambda_t(p_\lambda(\theta_0), \theta_0)| \, e^{\pm \eta t} < \infty$ *for some* $\eta \in [\delta^0/4, \delta^0/2]\}$, *that is,* $W_\lambda^{\pm}(\mathcal{S}_\lambda) = \bigcup_{\theta_0 \in T^k} W_\lambda^{\pm}(\theta_0)$. *Moreover,* $W_\lambda^{+}(\theta_0) \pitchfork W_\lambda^{-}(\theta_0)$ *at* $(p_\lambda(\theta_0), \theta_0) \in \mathcal{S}_\lambda$, *and,* $W_\lambda^{\pm}(\theta_0)$ *are locally diffeomorphic to the stable and unstable subspaces* $V_\lambda^{\pm}(\theta_0)$ *of* (5.35).

(3) *If* (5.35) *is linearly stable, that is, its S–S spectrum* $\Sigma \subset (-\infty, 0)$, *then* $\mathcal{S}_\lambda$ *is orbitally and asymptotically stable. That is, there is an* $\eta_0 = \eta_0(\lambda_0) > 0$ *such that if* $d((z, \theta), \mathcal{S}_\lambda) \leqslant \eta_0$ *then there is a unique* $\theta_0 \in T^k$ *such that* $\lambda_t(z, \theta) - \Lambda_t(p_\lambda(\theta_0), \theta_0) \to 0$ *exponentially as* $t \to +\infty$ *with rate* $\eta \in [\delta^0/4, \delta^0/2]$.

(4) *If F, G, B are* C^r $(r \geqslant 2)$, *then* $\mathcal{S}_\lambda$ *is a* C^r *manifold which varies* C^r *continuously in* λ, $W_\lambda^{\pm}(\theta_0)$ *are* C^r *submanifolds which vary* C^{r-1} *continuously in* θ_0, λ; $W_\lambda^{\pm}(\mathcal{S}_\lambda)$ *are therefore* C^{r-1} *manifolds varying* C^{r-1} *continuously in* λ.

6. Orbital Stability of Quasi-Periodic Motions

Consider

$$x' = f(x, \mu), \qquad x \in R^n, \tag{6.1$_\mu$}$$

where $f \in C^2$, $\mu \in R^1$. If $(6.1)_0$ has a periodic solution $x = p(t)$, and the variational equation

$$y' = f_x(p(t), 0)\, y \tag{6.2}$$

has $n-1$ Floquet exponents $\lambda_1, ..., \lambda_{n-1}$ whose real parts lie in $(-\infty, 0)$ (note that 0 is a Floquet exponent), then it is known from the classical theory (see [2] or [9]) that there is a $\mu_0 > 0$ such that $(6.1)_\mu$ has for each μ, $0 \leqslant |\mu| \leqslant \mu_0$, a periodic solution $q(t, \mu)$ with $q(t, 0) = p(t)$. Moreover, $q(t, \mu)$ has asymptotic orbital stability. To be more precise, there is an $\varepsilon = \varepsilon(\mu) > 0$ such that if a solution ϕ of $(6.1)_\mu$ satisfies $|\phi(t_1) - q(t_2, \mu)| < \varepsilon$ for some t_0 and t_1, there exists a constant $c(\mu)$ (asymptotic phase) such that $\lim_{t \to +\infty} |\phi(t) - q(t + c, \mu)| = 0$.

In the language of S–S spectrum, what has been assumed for (6.2) is just that its normal spectrum $\Sigma_N = \{\mathrm{Re}\, \lambda_1, ..., \mathrm{Re}\, \lambda_{n-1}\} \subset (-\infty, 0)$ (the tangential spectrum $\Sigma_T = \{0\}$ is trivial), since the S–S spectrum of a linear periodic system conists of exactly the real parts of the Floquet exponents (see [27]).

We are now seeking for a generalization of the above theory to the quasi-periodic case. We will see in the next theorem that there similar results in the quasi-periodic case even though the S–S spectrum for a quasi-periodic linear system may no longer be discrete.

THEOREM 6.1. *Assume* $(6.1)_0$ *has a quasi-periodic solution* $\phi(t)$ *with k frequencies* $\omega_1, \ldots, \omega_k$ *(namely, there is a* $q \in C(T^k, R^n)$ *such that* $\phi(t) = q(\omega_1 t, \ldots, \omega_k t)$*), and the S–S spectrum of the variational equation*

$$y' = f_x(\phi(t), 0)\, y \tag{6.3}$$

satisfies $\Sigma_N \subset (-\infty, 0)$ *(*$\Sigma_T = \{0\}$ *is trivial). Then, there is a* $\mu_0 > 0$ *such that for* $0 < |\mu| \leqslant \mu_0$, *we have:*

(1) *Equation* $(6.1)_\mu$ *has invariant smooth k-tori* $S_\mu = \{q(\theta, \mu) \mid \theta \in T^k\}$, $q(\theta, 0) = q(\theta)$ *(*S_μ *is invariant in the sense that if* $q(\theta, \mu) \in S_\mu$, *then* $q(\theta \cdot t, \mu) \subset S_\mu$ *and* $x(t) := q(\theta \cdot t, \mu)$ *solves* $(6.1)_\mu$, *where* $\theta \cdot t$ *is a flow on* T^k*).*

(2) S_μ *has asymptotic orbital stability; that is, there is an* $\varepsilon = \varepsilon(\mu) > 0$ *such that if a solution* $x(t)$ *of* $(6.2)_\mu$ *satisfies* $|x(t_1) - q(\theta, \mu)| < \varepsilon$ *for some* $t_1 \in R$ *and* $\theta \in R^k$, *there exists a* $\theta_0 = \theta_0(\mu) \equiv (\theta_1, \ldots, \theta_k) \in R^k$ *such that* $\lim_{t \to +\infty} |x(t) - q(\theta_0 \cdot t, \mu)| = 0$ *exponentially.*

(3) *When* $\mu = 0$, *there is an* $\varepsilon > 0$ *such that if a solution* $x(t)$ *of* $(6.1)_0$ *satisfies* $|x(t_2) - q(\omega_1 t_1, \ldots, \omega_k t_1)| < \varepsilon$ *for some* $t_1, t_2 \in R$, *then there exists a constant vector* $(h_1, \ldots, h_k) \in R^k$, *such that* $\lim_{t \to +\infty} |x(t) - q(h_1 + \omega_1 t, \ldots, h_k + \omega_k t)| = 0$ *exponentially.*

Proof. Let $\Omega = \mathrm{cl}\{\phi(t) \mid t \in R\}$. Since our assumptions imply that Ω is "normally hyperbolic," it follows from [30] that Ω is diffeomorphic to T^k; hence $q \in C^1(T^k, R^n)$. By the well-known tubular neighborhood theorem, there is a family of C^1 diffeomorphisms $x = H_\mu(z, \theta)$ of a neighborhood U of $\{0\} \times T^k \subset R^{n-k} \times T^k$ onto an open neighborhood of Ω in R^n such that $q = H_0|_{\{0\} \times T^k}$, and, H_μ^{-1} takes $(6.1)_\mu$ in the vicinity of Ω to

$$\begin{aligned} z' &= B(\theta)\, z + F(z, \theta, \mu) \\ \theta' &= \omega + G(z, \theta, \mu), \end{aligned} \tag{6.4$_\mu$}$$

where $\omega = (\omega_1, \ldots, \omega_k)^T$, $F = O(|z|^2)$, $G = O(|z|)$ as $\mu = 0$ (see arguments in [18]). Furthermore

$$\begin{aligned} z' &= B(\theta)\, z \\ \theta' &= \omega \end{aligned} \tag{6.5}$$

has ED on T^k with projection $P(\theta) \equiv I$ (the S–S spectrum of (6.5) is just the normal spectrum Σ_N of (6.3)). Let $\mathscr{S}_\mu = \{(p_\mu(\theta), \theta) \mid \theta \in T^k\}$ ($|\mu| \leqslant \mu_0$ for some $\mu_0 > 0$) be the invariant torus of $(6.4)_\mu$ given by Theorem 5.1.

Define

$$q(\theta, \mu) = H_\mu(p_\mu(\theta), \theta). \tag{6.6}$$

Then $S_\mu = \{q(\theta, \mu) | \theta \in T^k\}$ is an invariant torus for $(6.1)_\mu$, that is, for any $\theta \in T^k$, $q(\theta \cdot t, \mu)$ solves $(6.1)_\mu$, where $\theta \cdot t$ is the flow generated by

$$\theta' = \omega + G(p_\mu(\theta), \theta, \mu). \qquad (6.7)_\mu$$

This proves (1).

To prove (2), we denote by $\Lambda_t(z, \theta)$ the flow generated by $(6.4)_\mu$. Let $\eta_0 = \eta_0(\mu_0)$ be defined by Corollary 5.4 for $(6.4)_\mu$. Choose $\varepsilon = \varepsilon(\mu)$ so that $|x - q(\theta, \mu)| < \varepsilon$ implies $d(H_\mu^{-1}(x), H_\mu^{-1}(q(\theta, \mu))) < \eta_0$, where d is a metric on $R^n \times T^k$. Suppose now that $x(t)$ is a solution of $(6.1)_\mu$ such that $|x(t_1) - q(\theta, \mu)| < \varepsilon$ for some $t_1 \in R$, $\theta \in R^k$. Let $(\bar{z}, \bar{\theta}) = H_\mu^{-1}(x(t_1))$, then $d((\bar{z}, \bar{\theta}), \mathscr{S}_\mu) < \eta_0$. By Corollary 5.4, there exists a unique $\theta_* \in T^k$ such that $\lim_{t \to +\infty} d(\Lambda_t(\bar{z}, \bar{\theta}), \Lambda_t(p_\mu(\theta_*), \theta_*)) \to 0$ exponentially. Let $\theta_0 = \theta_*(-t_1)$. Then $x(t) = H_\mu(\Lambda_{t-t_1}(\bar{z}, \bar{\theta}))$, $q(\theta_0 \cdot t, \mu) = H_\mu(\Lambda_t(p_\mu(\theta_0), \theta_0)) = H_\mu(\Lambda_{t-t_1}(p_\mu(\theta_*), \theta_*))$; hence $|x(t) - q(\theta_0 \cdot t, \mu)| = |H_\mu(\Lambda_{t-t_1}(\bar{z}, \bar{\theta})) - H_\mu(\Lambda_{t-t_1}(p_\mu(\theta_*), \theta_*))| \to 0$ exponentially as $t \to +\infty$. This proves (2).

(3) This is just a special case of (2) by noting that the flow $\{\theta \cdot t\}$ on T^k in this case is the twist flow $\theta \cdot t = \theta + \omega t$. ∎

Note. (1) To apply the tubular neighborhood theorem in our arguments, we need to assume that the normal bundle of tori Ω is trivial. However, if the normal bundle is non-trivial, as remarked in [18], we can add additional coordinates $u_1, ..., u_q$ and corresponding equations $u_j' = \lambda_j u_j$ ($\lambda_j < 0$), $j = 1, 2, ..., q$, for sufficiently large q, to Eq. $(6.1)_\mu$. It is a consequence of K-theory that the normal bundle will be trivial if the normal dimension is sufficiently large. We can therefore just prove our results with the modified equation then drop the extra coordinates and come back to our original equation $(6.1)_\mu$.

(2) Coppel [3] also discussed the orbital stability of quasi-periodic mitions under the assumption that (6.3) has an exponential tricotomy. We believe that our conditions here are weaker and more natural in speaking of a generalization of the periodic case.

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REFERENCES

1. J. CARR, "Appl. of Center Manifold Theory," Springer-Verlag, New York/Berlin, 1981.
2. E. A. CODDINGTON AND N. LEVINSON, Theory of ODE's Krieger, 1985.
3. W. A. COPPEL, "Dichotomies in Stability Theory," Lect. Notes in Math., Springer-Verlag, New York/Berlin, 1978.
4. S.-N. CHOW AND J. K. HALE, "Methods of Bifurcation Theory," Springer-Verlag, New York/Berlin, 1982.
5. S.-N. CHOW, X.-B. LIN AND K. LU, Smooth invariant foliations in finite dimensional spaces, preprint, Georgia Tech., 1990.
6. S.-N. CHOW AND K. LU, Invariant manifolds for flows in Banach spaces, *J. Differential Equations* **74** (1988A), 285–317.
7. N. FENICHEL, Persistence and smoothness of invariant manifolds for flows, *Indiana Univ. Math. J.* **21** (1971), 192–226.
8. N. FENICHEL, Asymptotic Stability with rate cond., *Indiana Univ. Math. J.* **23**, No. 12 (1974), 1109–1137.
9. J. K. HALE, "Ordinary Differential Equations," 2nd ed., Krieger, 1980.
10. J. K. HALE, Integral manifolds of perturbed differential systems, *Ann. of Math. (2)* **73** (1961), 496–531.
11. J. K. HALE, Periodic solutions of a class of hyperbolic equations containing a small parameter, *Arch. Rational Mech. Anal.* **23** (1967), 380–398.
12. J. K. HALE, "Theory of Functional Differential Equations," Springer-Verlag, New York/Berlin, 1977.
13. D. HENRY, "Geometric Theory of Semilinear Parabolic Equations," Lec. Notes in Math., Vol. 840, Springer-Verlag, New York/Berlin, 1981.
14. M. W. HIRSCH, C. C. PUGH, AND M. SHUB, Invariant manifolds, *Bull. Amer. Math. Soc. (N.S.)* **76** (1970), 1015–1019.
15. R. A. JOHNSON, *M*-functions and Floquet exponents for linear differential systems, *Annali di Mat. Pura ed Appl.* **147** (1987), 211–248.
16. R. A. JOHNSON, Concerning a theorem of Sell, *J. Differential Equations* **30** (1987), 324–339.
17. R. A. JOHNSON, K. J. PALMER, AND G. R. SELL, Ergodic properties of linear dynamical systems, *SIAM J. Math. Anal.* **18** (1987), 1–33.
18. R. A. JOHNSON AND G. R. SELL, Smoothness of spectral subbundles and reducibility of *Q*-periodic linear differential systems, *J. Differential Equations* **41** (1981), 262–288.
19. K. LU, A Hartman–Grobman theorem for scalar reaction–diffusion equations, *J. Differential Equations*, in press.
20. K. J. PALMER, Transversal heteroclinic points and Cherry's example of a non-integrable Hamiltonian system, *J. Differential Equations* **65** (1986), 321–360.
21. K. J. PALMER, Exponential dichotomies, the shadowing lemma and transversal homoclinic points, *in* "Dynamics Reported VI" (B. G. Teubner, Ed.), Wiley, New York, 1988.
22. K. J. PALMER, Exponential dichotomies and transversal homoclinic points, *J. Differential Equations* **55** (1984), 225–256.
23. K. J. PALMER, On the stability of center manifold, *J. Appl. Math. Phys.* **38** (1987), 273–278.
24. K. P. RYBAKOWSKI, An implicit-function theorem for a scale of Banach spaces and smoothness of invariant manifolds, CDSNS Preprint.
25. R. J. SACKER, A new approach to the perturbation theory of invariant surfaces, *Comm. Pure Appl. Math.* **18** (1965), 717–732.
26. R. J. SACKER AND G. R. SELL, Existence of dichotomies and invariant splitting for linear differential systems, I, *J. Differential Equations* **15** (1974), 429–458; part II, *J. Differential Equations* **22** (1976).

27. R. J. Sacker and G. R. Sell, A spectral theory for linear differential systems, *J. Differential Equations* **27** (1978), 320–358.
28. R. J. Sacker and G. R. Sell, The spectrum of an invariant submanifold, *J. Differential Equations* **38** (1980), 135–160.
29. K. Sakamoto, Invariant manifold in singular-perturbation problems for ODE's, *Proc. of Royal Society of Edinburgh* **166A** (1990), 45–78.
30. G. R. Sell, The structure of a flow in the vicinity of an almost periodic motion, *J. Differential Equations* **27** (1978), 359–393.
31. G. R. Sell, Bifurcation of higher dimension tori, *Arch. Rational Mech. Anal.* **69** (1979), 199–230.
32. G. R. Sell, "Topological Dynamical Systems and ODE," Van Nostrand, New York, 1971.
33. G. R. Sell, References on dyn. syst., presented to IMA Dyn. Syst. Year, 1990.
34. A. Vanderbauwhede and S. A. Van Gils, Center manifolds and contractions on a scale of *B*-spaces, *J. Funct. Anal.* **72** (1987), 209-224.
35. Y. Yi, Generalized integral manifold theorem, *J. Differential Equations*, in press.