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Large spin strings in AdS_3

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Abstract

We consider strings with large spin in $AdS_3 \times S^3 \times \mathcal{M}$ with NS–NS background. We construct the string configurations as solutions of $SL(2,R)$ WZW theory. We compute the relation between the space–time energy and spin, and show that the anomalous correction is constant, and not logarithmic in the spin. This is in contrast to the S-dual background with R–R charge where the anomalous correction is logarithmic.

1. Introduction

Examples where gauge theory perturbative computations can be extrapolated to strong coupling and compared to classical supergravity results are rare when the calculated quantities are not protected by supersymmetry. An important example has been studied by Gubser, Klebanov and Polyakov [1]. They showed that the energy of a spinning string with spin S in AdS_5 in the limit that its size is much larger than the AdS_5 radius is

$$E = S + \frac{\sqrt{\lambda}}{\pi} \ln\bigl(S/\sqrt{\lambda}\bigr). \tag{1}$$

In global coordinates the energy is identified with the dimension of the dual operator. GKP proposed that this string configuration is dual to a twist-2 operator in $\mathcal{N} = 4$ SYM such as

$$\mathcal{O}_{(\mu_1\cdots\mu_n)} = \operatorname{Tr}\phi^* \mathcal{D}_{(\mu_1} \cdots \mathcal{D}_{\mu_n)}\phi. \tag{2}$$

In perturbative gauge theory this operator has an anomalous dimension that grows as $\ln n$, which is in agreement with the strong coupling result derived from supergravity. Quantum corrections to the energy of the spinning string were analyzed in [2,3] with no $\ln^2 S$ corrections found. One therefore expects that the leading term in the anomalous

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dimension to all orders in perturbation theory (and probably non-perturbatively as well) is logarithmic, i.e.,

$$E = S + f(\lambda)\ln\bigl(S/\sqrt{\lambda}\bigr). \tag{3}$$

Following the work of GKP many other string and membrane configurations were studied [4–12].

In this Letter we will study spinning strings in $AdS_3 \times S^3 \times \mathcal{M}$ with NS–NS 2-form background. This is of interest, since string theory on this background can be exactly solved in terms of the $SL(2,R)$ WZW model [13,14]. We will see that the "planetoid" spinning string solution of AdS_5 is not a solution in AdS_3 because of the NS–NS 2-form. In Section 2 we will find the explicit form of the classical spinning string configuration, and calculate the energy–spin relation. We will see that the leading behavior is the same as in AdS_5, but the anomalous correction is constant rather than logarithmic. This is in contrast with the S-dual background with R–R flux (for a review see [15]), where the anomalous correction is logarithmic. Indeed, it confirms the fact that the near horizon limits of D1–D5 and F1–NS5 systems lead to very different theories. In Section 3 we will analyze the conditions under which the classical string configuration is valid as a quantum state. In Section 4 we consider more general string configurations that rotate on the S^3 part of the metric as well. In the last section we discuss the results.

2. Classical spinning strings in AdS_3

String theory on $AdS_3 \times S^3 \times \mathcal{M}$ has many classical solutions. We will be interested in a class of classical solutions that correspond to spinning closed strings [16–18]. Similar string configurations were also analyzed in [19] in the context of open strings.

The NS–NS AdS_3 background, in global coordinates, is given by

$$ds^2 = R^2\left(-(1+r^2)\,dt^2 + \frac{dr^2}{1+r^2} + r^2\,d\phi^2\right), \quad \text{with} \quad B_{t\phi} = R^2 r^2. \tag{4}$$

We will consider the following ansatz for a time dependent embedding of a closed string in AdS_3

$$t = c_1\tau + \tilde{t}(\sigma), \qquad \phi = c_2\tau + \tilde{\phi}(\sigma), \qquad r = r(\sigma). \tag{5}$$

c_1 and c_2 are constants, and c_1 is assumed to be positive to insure forward propagation in time. τ and σ denote the world-sheet coordinates and $\sigma \simeq \sigma + 2\pi$.

The GKP ansatz is the one in which $\tilde{t} = \tilde{\phi} = 0$. At first sight this seems to be a solution of the classical equations of motion. The energy–spin relation for such a configuration will be as in [1]. For strings larger than the AdS radius, it takes the form (3), with $\sqrt{\lambda} = R^2/\alpha'$. However, we will see that while this configuration solves the constraints $T_{\pm\pm} = 0$, the $SL(2,R)$ Kac–Moody currents associated with it are not holomorphic. Note, however, that for a fundamental spinning string in a R–R AdS_3 background, given by the near-horizon limit of the D1–D5 system, the simpler ansatz is perfectly valid.

The world-sheet action for a closed string embedding of the form (5) reads

$$\begin{aligned} S = \frac{R^2}{2\pi\alpha'}\int d\tau\,d\sigma\Biggl[&-(1+r^2)\left(-c_1^2 + \left(\frac{d\tilde{t}}{d\sigma}\right)^2\right) + \frac{1}{1+r^2}\left(\frac{dr}{d\sigma}\right)^2 \\ &+ r^2\left(-c_2^2 + \left(\frac{d\tilde{\phi}}{d\sigma}\right)^2\right) - 2r^2\left(c_1\left(\frac{d\tilde{\phi}}{d\sigma}\right) - c_2\left(\frac{d\tilde{t}}{d\sigma}\right)\right)\Biggr]. \end{aligned} \tag{6}$$

From (6) we get two conservation laws

$$\frac{d\tilde{t}}{d\sigma} = \frac{c_2 r^2 - k_1}{1+r^2}, \qquad \frac{d\tilde{\phi}}{d\sigma} = \frac{c_1 r^2 - k_2}{r^2}, \tag{7}$$

where k_1 and k_2 are integration constants. The constraint on the energy–momentum tensor of the system is

$$T_{\pm\pm} = T_{\sigma\sigma} + T_{\tau\tau} \pm 2T_{\tau\sigma} = 0. \tag{8}$$

In (8) we assumed that there are no other contributions to the energy–momentum tensor from the internal CFT on $S^3 \times \mathcal{M}$. It is, however, easy to generalize this calculation to include contributions from the internal CFT. In such a case the constraint is on the sum of energy–momentum tensors

$$T^{AdS}_{\pm\pm} + T^{S^3\times\mathcal{M}}_{\pm\pm} = 0. \tag{9}$$

We will discuss this generalization in Section 4. From the requirement that $T_{\sigma\tau} = 0$ we get that $c_1k_1 = c_2k_2$. To simplify the notation we define

$$\alpha = k_1^2 - k_2^2, \qquad \beta = c_1^2 + 2c_1k_2. \tag{10}$$

With these definitions the energy–momentum constraint reduces to

$$\left(\frac{dr}{d\sigma}\right)^2 = \frac{\alpha\beta}{k_2^2r^2}\left(r^2 - \frac{k_2^2}{\beta}\right)\left(\frac{k_2^2}{\alpha} - r^2\right). \tag{11}$$

The solution to this equation is given by

$$r^2(\sigma) = \frac{k_2^2}{2\alpha\beta}\left((\alpha+\beta) - (\alpha-\beta)\sin\left[\frac{\sqrt{4\alpha\beta}}{k_2}\sigma\right]\right). \tag{12}$$

For a closed string we need to impose $r(2\pi) = r(0)$ which leads to the quantization of n.

$$n = \frac{\sqrt{4\alpha\beta}}{k_2} = \text{integer}. \tag{13}$$

The angular coordinate can be deduced from (5) and (7) to be

$$\phi(\sigma,\tau) = \frac{c_1k_1}{k_2}\tau + c_1\sigma - \arctan\left[\frac{(\alpha+\beta)\tan\left[\frac{n}{2}\sigma\right] - (\alpha-\beta)}{\sqrt{4\alpha\beta}}\right]. \tag{14}$$

Note that (14) is discontinuous whenever $\tan[\frac{n}{2}\sigma]$ diverges. In order to get a continuous function we should add π after each discontinuous point ($n\pi$ altogether). In order to ensure the periodicity $\phi(2\pi,\tau) = \phi(0,\tau) \bmod 2\pi$, we should have either (i) c_1 an integer and n even, or (ii) c_1 half-integer and n odd. A few examples are plotted in Fig. 1 for $\tau = 0$.

The time coordinate is given by

$$t(\sigma,\tau) = c_1\tau + c_2\sigma - \frac{(c_2+k_1)\sqrt{4\alpha\beta}}{n\sqrt{\left(k_2^2+\alpha\right)\left(k_2^2+\beta\right)}}\arctan\left[\frac{\left(1+\frac{k_2^2(\alpha+\beta)}{2\alpha\beta}\right)\tan\left[\frac{n}{2}\sigma\right] - \frac{k_2^2(\alpha-\beta)}{2\alpha\beta}}{\sqrt{\frac{(k_2^2+\alpha)(k_2^2+\beta)}{\alpha\beta}}}\right]. \tag{15}$$

The periodic boundary condition $t(2\pi,\tau) = t(0,\tau)$ is obeyed if k_2 satisfies (again note that $\arctan[\cdots]$ gets shifted by $n\pi$)

$$c_2 = \frac{(c_2+k_1)\sqrt{\alpha\beta}}{\sqrt{\left(k_2^2+\alpha\right)\left(k_2^2+\beta\right)}}. \tag{16}$$

This can be solved using the definitions of α and β, and gives the allowed value for k_2

$$k_2 = \frac{2c_1^3}{n^2 - 4c_1^2}. \tag{17}$$

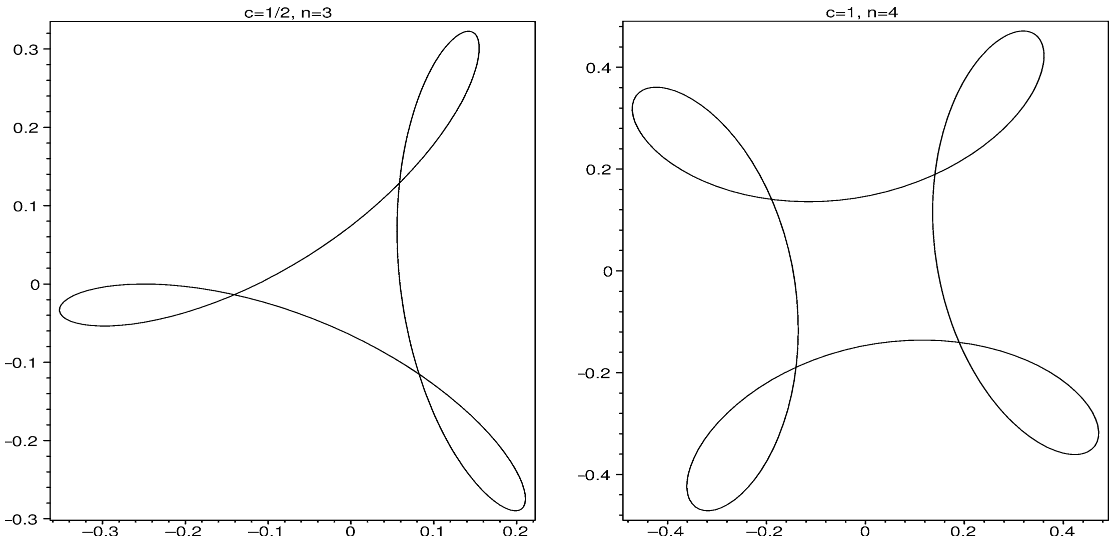

Fig. 1. Parametric polar plots depicting the string at $\tau = 0$ for (i) $n = 3$, $c_1 = 1/2$. (ii) $n = 4$, $c_1 = 1$. The radial coordinate is $r(\sigma)$ and the angular coordinate is $\phi(\sigma)$. Note that these plots show the string at constant world-sheet time, which is not the same as target space–time because of (5).

Note that the values of n and c_1 should be restricted to $n > 2c_1$. At the point $n = 2c_1$ the size of the string diverges. We will analyze this limit below. For $n < 2c_1$ we have the unphysical result $r^2(\sigma) < 0$. Substituting k_2 back in Eqs. (12), (14) and (15) we get

$$r^2(\sigma) = \frac{2}{n^2}\left(\frac{c_1^2(n^2 + c_1^2)}{n^2 - 4c_1^2} + \frac{c_1^2(n^2 - c_1^2)}{n^2 - 4c_1^2}\sin[n\sigma]\right), \tag{18}$$

$$\phi(\sigma, \tau) = \frac{n}{2}\tau + c_1\sigma - \arctan\left[\frac{n^2 + c_1^2}{2c_1 n}\tan\left[\frac{n}{2}\sigma\right] + \frac{n^2 - c_1^2}{2c_1 n}\right], \tag{19}$$

$$t(\sigma, \tau) = c_1\tau + \frac{n}{2}\sigma - \arctan\left[\frac{n^4 - 2c_1^2 n^2 + 2c_1^4}{n^2(n^2 - 2c_1^2)}\tan\left[\frac{n}{2}\sigma\right] + \frac{2c_1^2(n^2 - c_1^2)}{n^2(n^2 - 2c_1^2)}\right]. \tag{20}$$

The space–time energy E and spin S of the system are given by

$$E = \frac{R^2}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,\frac{\delta L}{\delta \dot{t}} = \frac{R^2}{2\pi\alpha'}\int_0^{2\pi} d\sigma\left(c_1(1 + r^2) - r^2\frac{d\phi}{d\sigma}\right),$$

$$S = \frac{R^2}{2\pi\alpha'}\int_0^{2\pi} d\sigma\,\frac{\delta L}{\delta \dot{\phi}} = \frac{R^2}{2\pi\alpha'}\int_0^{2\pi} d\sigma\left(-c_2 r^2 + r^2\frac{dt}{d\sigma}\right). \tag{21}$$

Using (7), the integrals can be calculated explicitly, and we can express E and S in terms of c_1 and n as

$$E = \frac{R^2(c_1 + k_2)}{\alpha'} = \frac{R^2 c_1}{\alpha'}\frac{n^2 - 2c_1^2}{n^2 - 4c_1^2}, \qquad S = \frac{R^2 k_1}{\alpha'} = \frac{R^2}{\alpha'}\frac{c_1^2 n}{n^2 - 4c_1^2}. \tag{22}$$

It is straightforward to solve this system and get the energy–spin relation

$$E(S,n) = \frac{R^2}{\alpha'}\frac{n^2+2n^2\alpha' S/R^2}{n^3}\sqrt{\frac{n^2\alpha' S/R^2}{n+4\alpha' S/R^2}}. \tag{23}$$

Let us examine this relation in the limits where the string size is much smaller or much larger than the AdS_3 radius. These two limits will correspond to the $2c_1 \ll n$ and the $2c_1 \to n$ limits, respectively. When $2c_1 \ll n$ the string is mostly concentrated near the origin of AdS_3, as can be seen from taking the limit of (18)

$$r^2(\sigma) \to \frac{2c_1^2}{n^2}\bigl(1+\sin[n\sigma]\bigr). \tag{24}$$

In other words, the string size is much less than the AdS_3 radius. In this limit we get the flat space relation between the energy of the string and its spin

$$S = \frac{\alpha' E^2}{R^2 n}. \tag{25}$$

On the other hand, if we take $\epsilon = n^2 - 4c_1^2 \to 0$ the asymptotic form of (18) is

$$r^2(\sigma) \to \frac{n^2}{8\epsilon}\bigl(5+3\sin[n\sigma]\bigr). \tag{26}$$

In this limit the size of the string is much larger than the AdS_3 radius, and the leading relation between E and S is given by $E = S$. This linear behavior was also found for open rotating strings in [19]. Using (22) we can actually calculate all the classical corrections to the leading linear behavior

$$E = S + \frac{3}{8}\frac{R^2 n}{\alpha'} - \frac{10}{256}\frac{R^4 n^2}{\alpha'^2 S} + \mathcal{O}\left(\frac{R^6 n^3}{\alpha'^3 S^2}\right). \tag{27}$$

The first correction is a constant. Note in comparison that in AdS_5 or AdS_3 with R–R background the leading correction is $\ln S$.

3. Quantum spinning strings

The spectrum of strings on AdS_3 with NS–NS background was analyzed in [14]. In this section we will use a similar analysis in order to identify the $SL(2,R)$ representation corresponding to the spinning string solution. We will investigate when is this state expected to be part of the physical spectrum of strings on AdS_3.

In order to identify the corresponding representation of $SL(2,R)$, we switch to light-cone coordinates, and calculate the $SL(2,R)$ currents associated with the spinning string solution. The solution of the previous section satisfies $T^{AdS}_{\pm\pm} = 0$, which implies that the Casimir of $SL(2,R)$, $C_2 = -j(j-1) = 0$. We therefore should look for a representation with $j = 1$.

We define $k = R^2/\alpha'$. We also define new target space coordinates

$$u = \frac{1}{2}(t+\phi), \qquad v = \frac{1}{2}(t-\phi). \tag{28}$$

The light-cone coordinates on the world-sheet are: $x^\pm = \tau \pm \sigma$. Using the conservation laws (7) we evaluate the right and left $SL(2,R)$ currents. In the following we shall only need the explicit form of J^3

$$\begin{aligned} J_R^3 &= k\bigl(\partial_+ u + (1+2r^2)\partial_+ v\bigr) = k(c_1+k_2-k_1),\\ J_L^3 &= k\bigl(\partial_- v + (1+2r^2)\partial_- u\bigr) = k(c_1+k_2+k_1). \end{aligned} \tag{29}$$

The other two components, $J^{\pm}$, can be determined in the same way.

Note that had we used the ansatz with $\tilde{t} = \tilde{\phi} = 0$ the currents would have been

$$J_R^3 = k\big((c_1 + c_2) + \big(1 + 2r^2(\sigma)\big)(c_1 - c_2)\big), \qquad J_L^3 = k\big((c_1 - c_2) + \big(1 + 2r^2(\sigma)\big)(c_1 + c_2)\big). \tag{30}$$

By taking $c_1 = \pm c_2$ we can the make left/right current holomorphic/anti-holomorphic, but not both. In WZW theory the holomorphy of the currents is equivalent to the equations of motion, thus such a configuration does not solve the field equations of the $SL(2,R)$ WZW model.

The zero modes of the currents (29) are

$$J_0^3 = \int_0^{2\pi} \frac{dx^+}{2\pi}\, J_R^3, \qquad \bar{J}_0^3 = \int_0^{2\pi} \frac{dx^-}{2\pi}\, J_L^3. \tag{31}$$

We denote by m and $\bar{m}$ the eigenvalues of J_0^3 and $\bar{J}_0^3$ which are given by

$$m = k\frac{n^2 c_1 - 2c_1^3 - nc_1^2}{n^2 - 4c_1^2}, \qquad \bar{m} = k\frac{n^2 c_1 - 2c_1^3 + nc_1^2}{n^2 - 4c_1^2}. \tag{32}$$

Because the currents (29) are constant, the solution has only a zero mode and no other higher modes, since $J_\nu^3 \sim \int J_L^3 e^{i\nu x^+}\, dx^+$ vanishes unless $\nu = 0$. For this solution to be a valid quantum state it must be in a unitary representation of $SL(2,R)$. The full quantum spectrum will be associated with the corresponding representation of the affine $SL(2,R)$ subject to the physical state constraint.

Recall that the unitary representations of the $SL(2,R)$ Lie algebra are:

- Discrete representations: $\mathcal{D}_j^+$ and $\mathcal{D}_j^-$ for real j, for which $m = j, j \pm 1, j \pm 2, \ldots$;
- Continuous representations: $\mathcal{C}_j^a$ ($0 \leqslant a < 1$) for $j = 1/2 + is$ (s is real); or for $1/2 < j < 1$ and $j - 1/2 < |a - 1/2|$, for which $m = a, a \pm 1, a \pm 2, \ldots$;
- Identity representation: $j = 0$.

Our solution has $j = 1$ so it cannot correspond to a unitary continuous representation. It can correspond to one of the discrete representations but only if m and $\bar{m}$ are integers. If they are not integers they can still belong to a non-unitary continuous representation, but we will not consider them as physical quantum states. It is clear that to have integer m and $\bar{m}$, k must be rational. Let us examine the spectrum for $k = 1$ as an example. There do not seem to be solutions with integer m and $\bar{m}$ for odd n in this case. For even n there are solutions. The first one is $n = 24$, $c_1 = 6$. In this case the string state corresponds to the WZW state

$$|j = 1, m = 5\rangle \times |\bar{j} = 1, \bar{m} = 9\rangle.$$

For higher values of k there will be other solutions. It is not clear though whether all integers can be generated by (32) for a given value of k.

It was argued in [14] that using spectral flow one can generate new classical solutions

$$t(\tau,\sigma) \to t(\tau,\sigma) + \omega\tau, \qquad \phi(\tau,\sigma) \to \phi(\tau,\sigma) + \omega\sigma, \tag{33}$$

where ω is an integer. These new solutions were shown to be important in understanding the nature of long string states in AdS_3. The $SL(2,R)$ currents, and the energy–momentum tensor are modified by the spectral flow to

$$J_0^3 \to J_0^3 + \frac{k\omega}{2}, \qquad T_{++} \to T_{++} - \omega J_0^3 - \frac{k}{4}\omega^2, \tag{34}$$

and the same for $\bar{J}_0^3$, and T_{--}. The spin of the flowed state is the same as the unflowed state, since it is $S = m - \bar{m}$. The energy, however, is shifted by $k\omega$. Since the energy tensor is modified by the spectral flow one must impose a new physical state condition

$$T_{++}^{(\omega)} = T_{++} - \omega J_0^3 - \frac{k}{4}\omega^2 = 0, \tag{35}$$

and the same for $T_{--}^{(\omega)}$. Clearly, since $T_{\pm\pm} = 0$ for the original solutions, and $J_0^3 \neq \bar{J}_0^3$, there cannot be a solution to the physical state condition other than $\omega = 0$. In fact, this will also be true even if there are contributions from the CFT on S^3, as long as $T_{++}^{S^3} = T_{--}^{S^3}$.

We now describe the standard way in which the quantum spectrum of strings in $AdS_3 \times S^3 \times \mathcal{M}$ is built around the classical spinning string solution we presented. We consider the state $|\psi_0\rangle = |j, \bar{j}, m, \bar{m}, h, \bar{h}\rangle$ as the ground state, where $(h, \bar{h})$ are the conformal weights of some state of the internal CFT on $S^3 \times \mathcal{M}$. In our case $h = \bar{h} = 0$. The excited states are constructed by applying the operators $J_{-\nu}^a$, with $\sum \nu_i = N$ and $L_{-\mu_i}$ with $\sum \mu_i = M$ (and the same for the anti-holomorphic sector)

$$|\psi\rangle = \prod L_{-\mu_i} \prod \bar{L}_{-\bar{\mu}_i} \prod J_{-\nu_i}^{a_i} \prod \bar{J}_{-\bar{\nu}_i}^{a_i} |\psi_0\rangle, \tag{36}$$

and imposing the physical state conditions [20]

$$\begin{aligned} &\big(L_0^{AdS_3} - 1 + N + M + h\big)|\psi\rangle = \left(-\frac{j(j-1)}{k-2} - 1 + N + M + h\right)|\psi\rangle = 0, \\ &J_\nu|\psi\rangle = L_\nu|\psi\rangle = 0, \quad \nu > 0, \end{aligned} \tag{37}$$

and a similar constraint for the anti-holomorphic part. For the state to be invariant under arbitrary translation of the world-sheet coordinate σ, one must also impose a level matching condition

$$\big(L_0^{\text{Total}} - \bar{L}_0^{\text{Total}}\big)|\psi\rangle = 0, \tag{38}$$

which can also be stated as

$$-\frac{j(j-1)}{k-2} + N + M + h = -\frac{\bar{j}(\bar{j}-1)}{k-2} + \bar{N} + \bar{M} + \bar{h}. \tag{39}$$

4. Adding momentum on S^3

In this section we would like to generalize the discussion of spinning strings in AdS_3 to include rotation in the S^3 part of the metric. The simplest ansatz is to assume that the string is point-like on S^3. The S^3 background is given by

$$ds^2 = R^2\big(\cos^2 x\, d\theta^2 - \sin^2 x\, d\tilde{\theta}^2 - dx^2\big), \qquad B_{\theta\tilde{\theta}} = 2R^2 \cos^2 x, \tag{40}$$

with the angular variables taken to be periodic

$$\theta \sim \theta + 2\pi, \qquad \tilde{\theta} \sim \tilde{\theta} + 2\pi. \tag{41}$$

Consider a string embedding in S^3 of the form $\theta = \omega\tau$, and $\tilde{\theta} = x = 0$. The contributions to $T_{++}^{S^3}$ equals that for $T_{--}^{S^3}$ since the string embedding is σ-independent

$$T_{\pm\pm}^{S^3} = R^2\omega^2. \tag{42}$$

The physical state condition reads

$$T_{\pm\pm}^{AdS} + R^2\omega^2 = 0. \tag{43}$$

This modifies (11) to be

$$\left(\frac{dr}{d\sigma}\right)^2 = \frac{\alpha\beta}{k_2^2 r^2}\left(r^2 - \frac{k_2^2}{\beta}\right)\left(\frac{k_2^2}{\alpha} - r^2\right) - \omega^2(1+r^2). \tag{44}$$

In order to get a closed string solution, we must assume that

$$\Delta = (\alpha-\beta)^2 - \omega^2\bigl(2\alpha + 2\beta + 4k_2^2 - \omega^2\bigr) > 0. \tag{45}$$

For $\omega^2 = 0$ this is automatically satisfied. However, for generic values of ω it is not the case. We shall proceed to solve (44) under this assumption, and at the end check if (45) is obeyed. The solution for $r(\sigma)$ is similar to the one found for $\omega = 0$

$$r^2(\sigma) = \frac{k_2^2/2}{\alpha\beta + k_2^2\omega^2}\left((\alpha+\beta-\omega^2) + \sqrt{\Delta}\sin\left[\sqrt{\frac{4\alpha\beta}{k_2^2} + 4\omega^2}\,\sigma\right]\right). \tag{46}$$

For the string to close $r(\sigma) = r(\sigma + 2\pi)$, we need

$$n_\omega = \sqrt{\frac{4\alpha\beta}{k_2^2} + 4\omega^2} = \text{integer}. \tag{47}$$

The solution to the other two coordinates can be found in the same way as for the $\omega = 0$ case. In particular, the periodic boundary conditions on the angular coordinate will lead to the same result as in the $\omega = 0$ case with n replaced by n_ω. The time coordinate is given by

$$\begin{aligned} t(\tau,\sigma) = c_1\tau - c_2\sigma + &\frac{(c_2+k_1)\sqrt{4\alpha\beta + 4k_2^2\omega^2}}{n_\omega\sqrt{\bigl(k_2^2+\alpha\bigr)\bigl(k_2^2+\beta\bigr)}} \\ &\times \arctan\left[\frac{\Bigl(1 + \frac{k_2^2/2}{\alpha\beta + k_2^2\omega^2}(\alpha+\beta-\omega)\Bigr)\tan\bigl[\frac{n_\omega}{2}\sigma\bigr] + \frac{k_2^2/2}{\alpha\beta+k_2^2\omega^2}\sqrt{\Delta}}{\sqrt{\frac{(k_2^2+\alpha)(k_2^2+\beta)}{\alpha\beta + k_2^2\omega^2}}}\right]. \end{aligned} \tag{48}$$

Imposing $t(\tau,0) = t(\tau,2\pi)$ gives

$$k_2 = \frac{2c_1^3 - 2c_1\omega^2}{n_\omega^2 - 4c_1^2}. \tag{49}$$

We now return to (45), and check if this assumption is correct. We first note that from the definition of n_ω it is now bounded from below, $n_\omega > 2\omega$. To ensure $k_2 \geqslant 0$ we must have $c_1 \geqslant \omega$. Combining all these relations we get $n_\omega > 2c_1 \geqslant 2\omega$. It can be seen by direct substitution of k_2 in (45) that this is enough to ensure $\Delta \geqslant 0$. When $c_1 = \omega$, $\Delta = 0$. We will shortly see what is the physical meaning of this point.

Using this result and (22) we can write down the energy and spin of the string

$$E = \frac{R^2(c_1+k_2)}{\alpha'} = \frac{R^2 c_1}{\alpha'}\frac{n_\omega^2 - 2c_1^2 - 2\omega^2}{n_\omega^2 - 4c_1^2}, \qquad S = \frac{R^2 k_1}{\alpha'} = \frac{R^2}{\alpha'}\frac{n_\omega\bigl(c_1^2 - \omega^2\bigr)}{n_\omega^2 - 4c_1^2}. \tag{50}$$

It is again straightforward, though lengthy, to write down $E = E(S, n_\omega, \omega)$. Several important limits of our result can, however, be easily analyzed. The first one is by setting $\omega = 0$ and retrieving the earlier result (22). The second one is by taking $c_1 = \omega$. We get a string state with vanishing spin in AdS_3. At this point the energy of the string depends only on ω in the expected way:

$$E = \frac{R^2\omega}{\alpha'}. \tag{51}$$

The last interesting limit is that of "long" strings, which corresponds to $\epsilon = n_\omega^2 - 4c_1^2 \to 0$. In this limit we get

$$E = S + \frac{3}{8}\frac{R^2 n_\omega}{\alpha'} + \frac{1}{2}\frac{R^2\omega^2}{\alpha' n_\omega} + \mathcal{O}\left(\frac{R^4 n_\omega^2}{\alpha'^2 S}\right). \tag{52}$$

Again a leading linear behavior followed by a constant correction.

5. Discussion

In this Letter we analyzed spinning strings in $AdS_3 \times S^3 \times \mathcal{M}$ with NS–NS 2-form background. We saw that the "planetoid" spinning string solution of AdS_5 is not a solution in AdS_3 because of the NS–NS 2-form. According to the AdS/CFT duality string theory on $AdS_3 \times S^3 \times \mathcal{M}$ is dual to a superconformal field theory on a cylinder, which is the boundary of AdS_3 in global coordinates. In fact string theory on this background can be exactly solved in terms of the $SL(2, R)$ WZW model and the dual (space–time) conformal field theory can be constructed. There is an exact correspondence between the bulk string states and the boundary operators. We denote by $\mathcal{L}_n$ the Virasoro operators of the boundary superconformal theory, and by $\mathcal{J}^3$ the generator of $U(1) \subset SU(2)$ subgroup of the $(4, 4)$ superconformal algebra. These are not to be confused with L_0 and J^3 of the bulk WZW model. The relation to the bulk parameters E, S and ω is

$$E = \mathcal{L}_0 + \bar{\mathcal{L}}_0, \qquad S = \mathcal{L}_0 - \bar{\mathcal{L}}_0, \qquad \mathcal{J}^3 = \mathcal{J}^3 = \omega. \tag{53}$$

Using the known relation between the space–times energy in AdS_3 and the eigenvalues of J_0^3 and $\bar{J}_0^3$ from the WZW model we get that $\mathcal{L}_0 = J_0^3$ and $\bar{\mathcal{L}}_0 = \bar{J}_0^3$. Thus, for a string state with a given $(m, \bar{m})$ (32) we know the exact space–time energy $E = m + \bar{m}$ and spin $S = m - \bar{m}$. The classical string solutions that we found also give the information about the general semi-classical relation between the two, $E(S)$. We saw that the leading behavior of the relation is the same as in AdS_5, but the anomalous correction is constant rather than logarithmic.

In comparison, the S-dual background has R–R charge. The conformal theory in this case can be thought of the IR fixed point of a $(1 + 1)$-dimensional gauge theory on the D1–D5 system. A spinning fundamental string in this background is of the same, "planetoid", form as the GKP string, and is likely to be dual to a similar operator on the conformal theory side. In this case the relation between the energy and spin of the string exhibits the same $\ln S$ behavior in the "long" string limit. We saw that the conformal field theory dual to the AdS_3 NS–NS background is quite different in this respect.

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