Centralizer and faithful groupoid

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ARTICLE INFO

Article history:
Received 10 March 2010
Available online 16 October 2010
Communicated by Michel Van den Bergh

MSC:
primary 20J05, 17B55, 18G50
secondary 18D35, 18D99, 18C99

Keywords:
Mal’cev
Protomodular and action representative
category
Split extension classifier
Internal groupoid
Commutator and centralizer

ABSTRACT

We correlate existence and aspect of some properties of centralizers with the conceptual notion of $\Theta$-faithful objects. We produce a large choice of examples from standard algebraic situations to topological models, going through less classical cases as the new notions of algebras introduced by Loday (Leibniz algebras, associative dialgebras and trialgebras), or the dual of any boolean topos.

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Introduction

Recent works [4,3] paid a special attention to the universal property of objects as the group $\text{Aut} G$ in the category $\text{Gp}$ of groups, or as the Lie-algebra $\text{Der} A$ in the category $K\text{-Lie}$ of Lie-algebras: namely, the split extension:

$$1 \rightarrow G \rightarrow \text{Aut} G \times G \rightarrow \text{Aut} G \rightarrow 1$$

determines, via the pulling back, a bijection between the set $\text{Gp}(K, \text{Aut} G)$ of group homomorphisms and the set of isomorphic classes of split extensions:

$$1 \rightarrow G \rightarrow H \rightarrow K \rightarrow 1$$

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On the model of the category $Gp$, in which we know to associate with any normal subgroup $G' \hookrightarrow G$ a group homomorphism $\psi : G \rightarrow \text{Aut} G'$ such that $\text{Ker} \psi$ is the centralizer of $G'$ in $G$, it was then possible to give a conceptual construction of the centralizer in any pointed category having objects with such a universal property [3].

But what was rather awkward and uncomfortable was that this universal property apparently did not give rise to any larger functorial process. Moreover, parallel kinds of considerations were extended, on one side, to the non-pointed context [8] and, on the other side, to pointed categories as those of (non unitary) rings or of associative algebras with the notion of action accessible category [11]. So that there was the growing up feeling that there should be a general scheme hidden behind those scattered situations.

Eventually the heart of the question is concentrated in the notion of $\Theta$-faithful object in a category $E$, where $\Theta$ is a class of maps stable under composition; namely, an object $X$ is said to be $\Theta$-faithful when, given any monomorphism $m : U \rightarrow V$ in $\Theta$ and any morphism $f : U \rightarrow X$ in $\Theta$, there is at most one morphism $g : V \rightarrow X$ in $\Theta$ making the following diagram commute:

\[
\begin{array}{ccc}
U & \xrightarrow{m} & V \\
\downarrow{f} & \downarrow{g} \\
X & & 
\end{array}
\]

With this notion, the functorial process is restored, but only with respect to the class $\Theta$. On the other hand all the previous scattered situations can be understood as contexts where there are enough $\Theta$-faithful objects. More precisely, these are categories $D$, now called groupoid accessible categories, such that the category $\text{Grd} D$ of internal groupoids in $D$ has enough $\Theta$-faithful groupoids, where $\Theta$ is the class of discrete fibrations. In a way, getting out of the pointed contexts of [3] and [11] leads us to a more lucid understanding of the general process which correlates existence of centralizers and existence of some kind of split epimorphisms (see Section 4.1 for the details).

More precisely, the main consequences of the groupoid accessibility are:

1) on the model of the previously quoted works [3,8,11] (and in particular of the category $Gp$, as recalled above), the existence, for any equivalence relation $R$ on an object $X$, of a largest equivalence relation $S$ on $X$ such that $[R, S] = 0$, namely its centralizer $Z(R)$, see Theorem 3.1; among other things, this allows us to assert the existence of centralizers in some of the new notions of algebras introduced by Loday, as Leibniz algebras [18], associative dialgebras [19] and trialgebras [20];

2) the intrinsic characterization of the faithful groupoids as those groupoids $Z_1$ which are such that $Z(R(d_0)) = R(d_1)$, where $d_0$ and $d_1$ are the domain and codomain mappings, see Proposition 3.9; the stability of faithful groupoids under product, see Proposition 3.10 and Theorem 4.2;

3) the reflection under pullback of the extensions with abelian kernel equivalence relation, in the stricter protomodular setting, see Theorem 4.1.

This article is organized along the following lines:

Section 1: presentation of the general scheme; making of a functorial process; particular case of groupoid accessible categories.

Sections 2 and 3: relationship with the existence of centralizers in general and in the stricter Mal’tsev context where, actually, the notion of commutator $[R, S]$ has its full meaning, see [27] and [9]; characterization of the faithful groupoids; characterization of abelian equivalence relations; and more generally, in the exact Mal’tsev context, characterization of the existence of centralizers.

Section 4 is devoted to the stricter protomodular setting, where groupoid accessibility coincides with action accessibility, namely existence of enough $\Theta$-faithful objects, where $\Theta$ is the class of cartesian maps with respect to the fibration of points $PtD \rightarrow D$: here we get the reflection under pullback of the extensions with abelian kernel equivalence relation; and in the pointed case, the coincidence with the action accessibility in the sense of [11].
Section 5, which is well prepared by Section 4, is devoted to the production of examples of groupoid accessible categories, among which: the non-pointed category $Rg^*$ of unitary rings, the categories $GpTop$ and $RgTop^*$ of topological groups and topological unitary rings, and other examples chosen inside the two extremal cases of Mal’cev categories (among which, on one side, the additive categories and, on the other side, the dual of any boolean topos) which emphasize the two opposite discriminating positions of the internal groups and the undiscrete equivalence relations among the internal groupoids.

1. Faithful objects and $\Theta$-accessible categories

Let $E$ be a finitely complete category. Given the following right-hand side commutative square, we denote the kernel equivalence relation of a map $f$ by $R[f]$ and the induced map between the kernel equivalences by $R(x)$:

\[
\begin{array}{ccc}
R[f] & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
R(x) & \xrightarrow{x} & Y
\end{array}
\]

Recall (see [1]):

**Theorem 1.1** (Barr–Kock). Given any diagram of the previous form, suppose that any of the left-hand side commutative diagram is a pullback. When the map $f$ is a pullback stable regular epimorphism, then the right-hand side square is a pullback as well.

1.1. General scheme

First, we shall introduce the following very simple general scheme which is the core of this work. Let $\Theta$ be a class of morphisms in $E$ which is stable under composition:

1) this class is said to be **proper** when moreover it contains the isomorphisms, is stable under pullback and such that, whenever $g.f$ and $g$ are in $\Theta$, the map $f$ is in $\Theta$;
2) a proper class is said to be **regular**, when the regular epimorphisms in $\Theta$ are stable under pullback and when the kernel equivalence $R[h]$ of a map $h$ in $\Theta$ admits a quotient $q$ in $\Theta$ such that the induced (monomorphic) factorization $m$ satisfying $h = m.q$ is itself in $\Theta$.

A relation $R \rightrightarrows X$ on the object $X$ will be said to be in $\Theta$, when its two legs are in $\Theta$. Accordingly, when this class is proper, the kernel equivalence relation $R[f]$ of a map $f$ in $\Theta$ is in $\Theta$.

**Definition 1.1.** We call $\Theta$-faithful an object $X$ in $E$ when it is such that, given any monomorphism $m: U \rightarrow V$ in $\Theta$ and any morphism $f: U \rightarrow X$ in $\Theta$, there is at most one morphism $g: V \rightarrow X$ in $\Theta$ making the following diagram commute:

\[
\begin{array}{ccc}
U & \xrightarrow{m} & V \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & X
\end{array}
\]

Clearly this notion is stable under subobjects $X' \hookrightarrow X$ which are in $\Theta$. 


**Proposition 1.1.** Suppose that \( \Theta \) is a proper class and \( X \) is \( \Theta \)-faithful. Any map \( h : U \to X \) in \( \Theta \) is such that its kernel equivalence relation \( R[h] \) contains any reflexive relation \( R \), on the object \( U \), which is in \( \Theta \). Accordingly any map \( X \to Y \) in \( \Theta \) with domain \( X \) is a monomorphism.

**Proof.** Consider the following diagram where the map \( s_0 \) is given by the reflexivity of the relation \( R \):

\[
\begin{array}{ccc}
U & \xrightarrow{s_0} & R \\
\downarrow{p_0} & & \downarrow{p_1} \\
X & \xrightarrow{h} & Y
\end{array}
\]

Since \( R \) is in \( \Theta \) and the class \( \Theta \) is proper, the map \( s_0 \) is in \( \Theta \). Since \( X \) is \( \Theta \)-faithful, we get \( h.p_0 = h.p_1 \) and \( R \subset R[h] \). This is true in particular for \( h = 1_X \); suppose \( f : X \to Y \) is in \( \Theta \), the reflexive relation \( R[f] \) on \( X \) is in \( \Theta \), so that we have \( R[f] \subset R[1_X] = \Delta_X \). Accordingly \( f \) is a monomorphism. \( \square \)

**Corollary 1.1.** Suppose that \( \Theta \) is a proper class. Let \( Y \) be an object in \( \mathcal{E} \) with a pair \( X \xleftarrow{h} Y \xrightarrow{h'} X' \) of maps in \( \Theta \) with their codomains \( X \) and \( X' \) \( \Theta \)-faithful. Then we have \( R[h] = R[h'] \).

**Proof.** The equivalence relations \( R[h] \) and \( R[h'] \) are both in \( \Theta \) and, since \( X \) and \( X' \) are \( \Theta \)-faithful, they are included in each other, so that \( R[h] = R[h'] \). \( \square \)

**Proposition 1.2.** Suppose that \( \Theta \) is a regular proper class. Let \( Y \) be an object in \( \mathcal{E} \) with a pair \( X \xleftarrow{h} Y \xrightarrow{h'} X' \) of maps in \( \Theta \) with their codomains \( X \) and \( X' \) \( \Theta \)-faithful. Then the maps \( h \) and \( h' \) have their regular epic part \( q \) in common and the codomain of \( q \) is \( \Theta \)-faithful.

**Proof.** From the previous corollary we know that we have \( R[h] = R[h'] \) and that this equivalence relation in is \( \Theta \). Accordingly its quotient \( q_Y : Y \to Q(Y) \), which lies in \( \Theta \), is the common regular epic part of \( h \) and \( h' \). Let \( m : Q(Y) \to X \) be the factorization of \( h \). It is a monomorphism in \( \Theta \) and, accordingly, the object \( Q(Y) \) is \( \Theta \)-faithful. \( \square \)

In this way, the object \( Q(Y) \) appears as the universal \( \Theta \)-faithful object associated with the object \( Y \).

**Definition 1.2.** We shall call \( \Theta \)-accessible any category \( \mathcal{E} \) which has “enough” \( \Theta \)-faithful objects, namely such that any object \( Y \) admits a map in \( \Theta \) toward a \( \Theta \)-faithful object. Such a map (or, for short, its codomain) will be called a \( \Theta \)-index of the object \( Y \).

**Proposition 1.3.** Suppose that the proper class \( \Theta \) is regular and that the category \( \mathcal{E} \) is \( \Theta \)-accessible. With any object \( Y \) there is associated a universal \( \Theta \)-faithful object \( Q(Y) \) which will be called the \( \Theta \)-index of \( Y \); with any map \( f : Y \to Y' \) in \( \Theta \) is functorially associated a monomorphism \( Q(f) : Q(Y) \to Q(Y') \) in \( \Theta \).

**Proof.** Given an object \( Y \), take any index \( h : Y \to X \) and its regular epic part \( q_Y : Y \to Q(Y) \), which lies in \( \Theta \), it is the index of \( Y \). Given any map \( f : Y \to Y' \) in \( \Theta \), and \( q_{Y'} : Y' \to Q(Y') \) the index of \( Y' \), we have \( R[q_{Y'}, f] = R[q_Y] \) and thus, since \( q_Y \) is a regular epimorphism in \( \Theta \), the monomorphic factorization \( Q(f) : Q(Y) \to Q(Y') \) in \( \Theta \). \( \square \)
A very easy useful result is the following:

**Proposition 1.4.** Let $U : \mathbb{E} \to \mathbb{E}'$ be a left exact and faithful functor. Let $\Theta$ (resp. $\Theta'$) a class of maps stable under composition in $\mathbb{E}$ (resp. $\mathbb{E}'$). Suppose $U(\Theta) \subset \Theta'$. Then the functor $U$ reflects the relative faithful objects.

1.2. Internal groupoids

We shall be mainly interested in the case where $\mathbb{E} = \text{Grd}\mathbb{D}$ is the category of internal groupoids in a finitely complete category $\mathbb{D}$, and $\Theta$ is the class $\text{DiF}$ of the discrete fibrations between groupoids. This class is a proper class, and a regular proper class whenever the category $\mathbb{D}$ is regular [1].

An internal groupoid $X_1$ in any category $\mathbb{E}$ will be presented as a reflexive graph $(d_0, d_1) : X_1 \rightrightarrows X_0$ endowed with an operation $d_2$:

\[
\begin{array}{ccc}
R(d_2) & & X_1 \\
p_2 & \searrow & \Downarrow & \nearrow & p_0 \\
R[d_0]^2 & \rightarrow & R[d_0] & \rightarrow & X_0 \\
p_1 & \downarrow & & \downarrow & \rightarrow & p_0 \\
R(d_0) & & X_0
\end{array}
\]

making the previous diagram satisfy all the simplicial identities, including the ones concerning the degeneracies. In the set theoretical context, this operation $d_2$ associates the composite $\psi \cdot \phi^{-1}$ with any pair $(\phi, \psi)$ of arrows of $X_1$ with same domain.

A groupoid $X_1$ will be said to be **totally disconnected** when the maps $d_0$ and $d_1$ are equal.

Any equivalence relation $R$ on an object $X$ in $\mathbb{E}$ provides a special kind of internal groupoid (namely one of those groupoids whose pair $(d_0, d_1)$ is jointly monic):

\[
\begin{array}{ccc}
R(p_0) & & X \\
p_3 & \searrow & \Downarrow & \nearrow & p_0 \\
R[p_0]^2 & \rightarrow & R[p_0] & \rightarrow & X \\
p_2 & \downarrow & & \downarrow & \rightarrow & p_0 \\
R(p_0) & & X
\end{array}
\]

which, in some formal circumstances, will be denoted by $R_1$.

Let $\text{Grd}\mathbb{E}$ denote the category of internal groupoids and internal functors in $\mathbb{E}$, and $(\cdot)_0 : \text{Grd}\mathbb{E} \to \mathbb{E}$ the forgetful functor associating with the groupoid $X_1$ its “object of objects” $X_0$. This functor is a left exact fibration. Any fibre $\text{Grd}_X\mathbb{E}$ (above a given object $X$) has a terminal object $\nabla_1(X)$ which is the undiscrete relation on the object $X$:

\[
\begin{array}{ccc}
p_0 & \rightarrow & X \\
p_1 & \searrow & \Downarrow & \nearrow & p_0 \\
X \times X & \rightarrow & X
\end{array}
\]

and an initial object $\Delta_1(X)$ which is the discrete equivalence relation on $X$:

\[
\begin{array}{ccc}
p_0 & \rightarrow & X \\
1_X & \searrow & \Downarrow & \nearrow & 1_X \\
X & \rightarrow & X
\end{array}
\]

They produce respectively a right adjoint and a left adjoint to the forgetful functor $(\cdot)_0$. 
An internal functor \( f_1 : X_1 \to Y_1 \) is \( (0)_0 \)-cartesian if and only if the following square is a pullback in \( E \), in other words if and only if \( f_1 \) is internally \emph{fully faithful}:

\[
\begin{array}{c}
\begin{array}{r}
X_1 \\
\downarrow^{(d_0,d_1)}
\end{array} \\
\begin{array}{r}
X_0 \times X_0 \\
\downarrow_{f_0 \times f_0}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{r}
Y_1 \\
\downarrow^{(d_0,d_1)}
\end{array} \\
\begin{array}{r}
Y_0 \times Y_0 \\
\downarrow_{f_0 \times f_0}
\end{array}
\end{array}
\]

We need also the following classical definition:

**Definition 1.3.** The internal functor \( f_1 \) is said to be a \emph{discrete fibration} when any of the following commutative squares is a pullback:

\[
\begin{array}{c}
\begin{array}{r}
X_1 \\
\downarrow^{d_1}
\end{array} \\
\begin{array}{r}
X_0 \\
\downarrow^{d_0}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{r}
Y_1 \\
\downarrow^{d_1}
\end{array} \\
\begin{array}{r}
Y_0 \\
\downarrow^{d_0}
\end{array}
\end{array}
\]

The following is straightforward:

**Lemma 1.1.** Suppose \( f_1 \) is a discrete fibration. It is \( (0)_0 \)-cartesian if and only if it is monomorphic.

On the other hand, we denote by \( PtE \) the category whose objects are the split epimorphisms in \( E \) with a given splitting and morphisms the commutative squares between these data, and by \( j_E : PtE \to E \) the functor associating its codomain with any split epimorphism. As soon as the category \( E \) has pullbacks, the functor \( j_E \) is a fibration whose cartesian maps are the pullbacks between split epimorphisms. The fibre above \( Y \) will be denoted \( PtY(\mathcal{E}) \).

The internal groupoids are strongly related to the split epimorphisms. Recall that the forgetful functor: \( \Upsilon_E : GrdE \to PtE \), associating with any groupoid \( X_1 \) the split epimorphism \( (d_0, s_0) : X_1 \rightrightarrows X_0 \), is left exact, maps discrete fibrations onto \( j_E \)-cartesian maps, and, above all, is \emph{monadic} [6], which implies in particular that it is faithful and conservative. This is this strong relationship which will allow us in Section 4.1 to correlate, as indicated in the introduction, the existence of faithful groupoids (see next section) to the existence of some special kind of split epimorphisms. We denote by \( \text{Dec}_1 \) the endofunctor of the induced comonad on \( GrdE \). Its counit \( \epsilon_1 X_1 : \text{Dec}_1 X_1 \to X_1 \) is given by the following diagram in \( E \):

\[
\begin{array}{c}
\begin{array}{r}
R[d_0] \\
\downarrow{p_1}
\end{array} \\
\begin{array}{r}
X_1 \\
\downarrow{d_1}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{r}
X_1 \\
\downarrow{d_1}
\end{array} \\
\begin{array}{r}
X_0 \\
\downarrow{d_0}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{r}
X_1 \\
\downarrow{d_1}
\end{array} \\
\begin{array}{r}
X_0 \\
\downarrow{d_0}
\end{array}
\end{array}
\]

this counit is clearly a discrete fibration. Moreover the following diagram, in the category \( GrdE \), is a kernel equivalence relation with its quotient:
Dec\textsubscript{2}X\textsubscript{1} \xrightarrow{\epsilon_1 Dec\textsubscript{2}X\textsubscript{1}} Dec\textsubscript{1}X\textsubscript{1} \xrightarrow{\epsilon_1 X\textsubscript{1}} \xrightarrow{} X\textsubscript{1}

and this endofunctor \text{Dec}_1 preserves the discrete fibrations.

1.3. Faithful groupoids and groupoid accessible categories

Definition 1.4. We call faithful any internal groupoid \(X_1\) in \(E\) which is \(DiF\)-faithful. We call groupoid accessible any category \(E\) such that \(\text{Grd}E\) is \(DiF\)-accessible.

We have a large choice of examples:

Examples. 1. Let \(\mathcal{A}\) be any finitely complete additive category [21]. The only faithful internal groupoids in \(\mathcal{A}\) are the internal groups. Any additive category is groupoid accessible. They are characterized in Section 5.4.2 as those pointed protomodular groupoid accessible categories whose only faithful groupoids are the internal groups.

2. Let \(E\) be the category \(Gp\) of groups. An internal groupoid \(Y_1\) is faithful if and only if the induced action of the group \(Y_0\) on the kernel of \(d_0 : Y_1 \rightarrow Y_0\) is faithful. The category \(Gp\) is groupoid accessible.

3. Let \(E\) be the category \(R\text{-Lie}\) of Lie \(R\)-algebras, where \(R\) is a commutative ring (with unit). An internal groupoid \(Y_1\) is faithful if and only if the induced morphism from \(Y_0\) to the derivation algebra of the kernel of \(d_0 : Y_1 \rightarrow Y_0\) is injective. The category \(R\text{-Lie}\) is groupoid accessible.

4. The three previous examples are actually special cases of pointed protomodular action representative categories \(\mathcal{D}\) in the sense of [4] and [3] which are necessarily groupoid accessible, see next example.

5. More generally, any pointed protomodular groupoid accessible category in the sense of [11] is groupoid accessible in our sense. The explanation of this point is detailed in Section 4.1.2. In [11], the pointed category \(Rg\) of (non unitary) rings was given as the guiding example of a groupoid accessible category which was not action representative. Moreover it is showed in [22] that any category of interest in the sense of [23] is groupoid accessible in the sense of [11]; this allows, in particular, to include the new notions of algebras introduced by Loday [18–20], as the Leibniz algebras and the associative dialgebras and trialgebras.

6. The first non-pointed examples of action representative categories were given in [8] with any fibre \(Grd_X\) of groupoids with a fixed set of objects \(X\) (in which the involved internal groupoids are 2-groupoids). These fibres generalized naturally the Example 2 insofar as \(Gp = Grd_1\).

7. We shall show in Section 5.1 that the non-pointed category \(Rg_*\) of unitary rings is groupoid accessible.

8. We shall show in Section 5.4.2 that the dual \(E^{op}\) of any boolean elementary topos \(E\) is groupoid accessible.

9. We shall show in Section 5.3 that the category \(\text{Top}(\mathbb{T})\) of topological \(\mathbb{T}\)-algebras of a protomodular theory \(\mathbb{T}\), whose corresponding variety \(\mathcal{V}(\mathbb{T})\) is groupoid accessible, is itself groupoid accessible. In particular, the category \(Gp\text{Top}\) of topological groups and \(Rg_*\text{Top}\) of topological rings with unit are groupoid accessible.

10. We shall show in Section 5.2 that any Birkhoff subcategory (subvariety) of a groupoid accessible Mal’cev category (variety) is groupoid accessible.

On the other hand, the notion of groupoid accessible category has good stability properties:

Proposition 1.5. Given a faithful groupoid \(X_1\) in \(E\), then, for any object \(Y\), the groupoid \(X_1 \times \Delta_1(Y)\) is faithful in \(E/Y\). Accordingly, any slice category \(E/Y\) of a groupoid accessible category \(E\) is groupoid accessible. On the other hand, any coslice category \(Y/E\) of a groupoid accessible category \(E\) is groupoid accessible. Therefore any fibre of the fibration of points \(\mathcal{Y}_E : PtE \rightarrow E\) is groupoid accessible as soon as so is \(E\).
Proof. The first point is a straightforward consequence of Proposition 1.4 applied to the forgetful functor: $\mathbb{E}/Y \to \mathbb{E}$. Suppose given now a groupoid $T_1$ in $\mathbb{E}/Y$, i.e. a groupoid $T_1$ in $\mathbb{E}$ augmented with a map $h : T_0 \to Y$. If $\phi_1 : T_1 \to \mathbb{X}_1$ is an index for $T_1$ in $\mathbb{E}$, then the following diagram provides an index for $T_1$ in $\mathbb{E}/Y$:

$$
\begin{array}{cccc}
T_1 & \xrightarrow{(\phi_1, h, d_0)} & X_1 \times Y \\
\downarrow d_0 & & \downarrow d_0 \times Y \\
T_0 & \xrightarrow{(\phi_0, h)} & X_0 \times Y \\
\downarrow h & & \downarrow p_Y \\
Y & & Y
\end{array}
$$

The third point is straightforward considering the forgetful functor: $Y \setminus \mathbb{E} \to \mathbb{E}$. Finally, the fibre $P_{tY} \mathbb{E}$ can be described as the coslice category of the slice category $\mathbb{E}/Y$ below its object $1_Y : Y \to Y$. $\square$

These slice and coslice categories give us other non-pointed examples.

2. Connected relations

We shall recall here some basic facts about the intrinsic commutator theory.

2.1. General setting

Consider $R$ and $S$ two equivalence relations on an object $X$ in any finitely complete category $\mathbb{E}$. Let us recall the following definition from [9], see also [13,17,15].

Definition 2.1. A connector on the pair $(R, S)$ is a morphism:

$$p : S \times_X R \to X, \quad (xSyRz) \mapsto p(x, y, z)$$

which satisfies the identities:

1) $xRp(x, y, z)$, 1’) $zSp(x, y, z)$,
2) $p(x, y, y) = x$, 2’) $p(y, y, z) = z$,
3) $p(x, y, p(y, u, v)) = p(x, u, v)$, 3’) $p(p(x, y, u), u, v) = p(x, y, v)$.

In set theoretical terms, condition 1) means that with any triple $xSyRz$ we can associate a square:

$$
\begin{array}{c}
X \\
\downarrow R \\
p(x, y, z) \\
\downarrow S \\
S \\
\downarrow \psi \\
\downarrow S \\
Y \\
\downarrow R \\
z
\end{array}
$$

More acutely, any connected pair produces from the following diagram in $\mathbb{E}$:
which, read from the left-hand side, produces an equivalence relation $\Sigma_1 \Rightarrow R_1$ in $GrdE$ on the equivalence relation $R$ whose two legs are discrete fibrations, namely an equivalence relation in $DiF$. It is called the centralizing double relation associated with the connector. It is clear that, conversely, any equivalence relation $\Sigma_1 \Rightarrow R_1$ in $DiF$ on the equivalence relation $R$ on the object $X$ determines a connector between $R$ and the image by the functor $(\iota_0 : GrdE \rightarrow E$ of this equivalence relation $\Sigma_1 \Rightarrow R_1$.

**Examples.** 1. An emblematical example is produced by a given discrete fibration $f_1 : R_1 \rightarrow Z_1$ whose domain $R$ is an equivalence relation. For that consider the following diagram:

$$
\begin{array}{c}
R[f_1] & \xleftarrow{p_1} & R & \xrightarrow{f_1} & Z_1 \\
R(d_0) & \downarrow & R(d_1) & \downarrow & d_1 \\
R[f_0] & \xleftarrow{p_0} & X & \xrightarrow{f_0} & Z_0 \\
\end{array}
$$

It is clear that $R[f_1]$ is isomorphic to $R[f_0] \times X R$ and that the map

$$
p : R[f_1] \xrightarrow{p_0} R d_1 X
$$

determines a connector on the pair $(R, R[f_0])$.

2. For any groupoid $X_1$, we have such a discrete fibration $e_1 X_1 : Dec(X_1) \rightarrow X_1$:

$$
\begin{array}{c}
R[d_0] & \xrightarrow{d_2} & X_1 \\
p_1 & \downarrow & \downarrow \\
X_1 & \xrightarrow{d_1} & X_0
\end{array}
$$

which implies the existence of a connector on the pair $(R[d_0], R[d_1])$. The converse is true as well, see [13] and [9]; given a reflexive graph:

$$
\begin{array}{c}
Z_1 & \xrightarrow{d_0} & Z_0 \\
\end{array}
$$

any connector on the pair $(R[d_0], R[d_1])$ determines a groupoid structure on this graph.

3. For any pair $(X, Y)$ of objects, the pair $(R[p_X], R[p_Y])$ of effective equivalence relations is canonically connected.
2.2. Connected relations in the Mal’cev context

Let \( D \) be now a Mal’cev category, i.e. a category in which any reflexive relation is an equivalence relation [12,13]. In a Mal’cev category, the previous conditions 2) and 2’) imply the other ones, and moreover a connector is necessarily unique when it exists, and thus the existence of a connector becomes a property; we then write \([R,S] = 0\) when this property holds.

The notion of Mal’cev category comes from the notion of Mal’cev variety (i.e. a variety of algebras whose theory contains a Mal’cev ternary operation \( p \), with \( p(x,y,y) = x \) and \( p(x,x,y) = y \)). From the original work [27] on commutor theory in the context of Mal’cev varieties and from the pioneering work of [24] in the Mal’cev categories, it can be showed that two congruences \( R \) and \( S \) in a Mal’cev variety have a trivial commutator in the sense of [27] if and only if they are connected according to the previous definition. From [9] recall that in a Mal’cev category:

1) \( R \land S = \Delta_X \) implies \( [R,S] = 0 \);
2) \( T \subset S \) and \( [R,S] = 0 \) imply \( [R,T] = 0 \).
3) \( [R,S] = 0 \) and \( [R',S'] = 0 \) imply \( [R \times R', S \times S'] = 0 \).

When \( D \) is a regular Mal’cev category, the direct image of an equivalence relation along a regular epimorphism is still an equivalence relation. In this case, we get moreover:

4) if \( f : X \rightarrow Y \) is a regular epimorphism, \( [R,S] = 0 \) implies \( [f(R), f(S)] = 0 \);
5) \( [R,S_1] = 0 \) and \( [R,S_2] = 0 \) imply \( [R,S_1 \vee S_2] = 0 \).

As usual, an equivalence relation \( R \) is called abelian when we have \([R,R] = 0\), and central when we have \([R,\nabla_X] = 0\). An object \( X \) in \( D \) is called commutative when \([\nabla_X, \nabla_X] = 0\).

Recall also that, in the regular Mal’cev context, any decomposition in \( GrD \) of a discrete fibration \( f_1 : X_1 \rightarrow Y_1 \) through a regular epic functor:

\[
\begin{array}{c}
X_1 \\
\downarrow q_1 \quad \downarrow f_1 \\
Q_1 \\
\downarrow \quad \downarrow \bar{f}_1 \\
Y_1
\end{array}
\]

is necessarily made of discrete fibrations.

2.3. Connected relations and faithful groupoids

Let us come back to any finitely complete category \( E \). Here is our leading observation:

**Proposition 2.1.** Given a faithful groupoid \( X_1 \) and an equivalence relation \( R \) on an object \( X \) in a category \( E \), any discrete fibration \( j_1 : R_1 \rightarrow X_1 \) is such that \( R[j_0] \) is connected with \( R \) and contains any equivalence relation \( S \) on \( X \) which is connected with \( R \).

**Proof.** We already noticed that, when \( j_1 \) is a discrete fibration, the equivalence relations \( R[j_0] \) and \( R \) are connected. Suppose moreover the groupoid \( X_1 \) is faithful, and suppose given an equivalence relation \( S \) on \( X \) such that \( S \) and \( R \) are connected. Consider the following diagram where \( \Sigma \) is the associated centralizing double relation:
This produces a reflexive relation in $\mathcal{D}$. According to Proposition 1.1, since $j_1$ is in $\mathcal{D}$, this reflexive relation is a subobject of $R_1[j_1]$ which implies in particular $S \subset R[j_0]$. 

Notice however that, if we suppose $S \subset R[j_0]$, there is no reason why, in general, the relation $S$ would be connected with $R$.

3. Mal’cev setting

This restriction no longer holds in the Mal’cev setting:

**Proposition 3.1.** Given a Mal’cev category $\mathcal{D}$, a faithful groupoid $X_1$ and an equivalence relation $R$ on an object $X$, then any discrete fibration $j_1 : R_1 \to X_1$ is such that $R[j_0]$ is the greatest equivalence relation $S$ on $X$ such that $[R, S] = 0$, namely its centralizer $Z(R)$.

**Proof.** It is just Proposition 2.1, once we know that, in the Mal’cev context, from $[R, R[j_0]] = 0$ and $S \subset R[j_0]$ we get $[R, S] = 0$. 

**Corollary 3.1.** Given a Mal’cev category $\mathcal{D}$, when the groupoid $X_1$ is faithful, we have $R[d_1] = Z(R[d_0])$.

**Proof.** Consider the discrete fibration $\epsilon_1 X_1 : \text{Dec}_1 X_1 \to X_1$:

$$
\begin{array}{ccc}
R[d_0] & \xrightarrow{p_2} & X_1 \\
p_0 & \downarrow & \downarrow \ \\
X_1 & \xrightarrow{d_1} & X_0
\end{array}
$$

and apply the previous proposition. 

3.1. Properties of centralizers

In this section we shall suppose $\mathcal{D}$ is any Mal’cev category and collect the elementary properties of the centralizer without any assumption apart from its simple existence.

**Definition 3.1.** When $R$ is an equivalence relation on an object $X$, we define $Z(R)$, and call centralizer of $R$, the greatest equivalence relation on $X$ connected with $R$.

If $Z(R)$ does exist, we have then:

$$[S, R] = 0 \iff S \subset Z(R)$$

It is clear that if $Z(R)$ and $Z^2(R)$ do exist, we have $R \subset Z^2(R)$.

**Proposition 3.2.** Suppose $Z(R)$, $Z(S)$ and $Z^2(S)$ do exist. Then:

$$R \subset S \implies Z(S) \subset Z(R)$$

Accordingly, if $Z(R)$, $Z^2(R)$ and $Z^3(R)$ do exist, we have $Z(R) = Z^3(R)$. 
Proof. If we have \( R \subseteq S \), we get \( R \subseteq Z^2(S) \) and \([R, Z(S)] = 0\). Accordingly we have \( Z(S) \subseteq Z(R)\).

We have naturally \( Z(R) \subseteq Z^2(R) \). And, according to our first point, we have \( Z^3(R) \subseteq Z(R) \) since we have \( R \subseteq Z^2(R) \). \( \square \)

**Corollary 3.2.** We have \( R = Z^2(R) \) if and only if there is some equivalence relation \( S \) such that \( R = Z(S) \).

**Proposition 3.3.** Suppose \( \mathcal{D} \) is a regular Mal’cev category and \( f : X \rightarrow Y \) a regular epimorphism in \( \mathcal{D} \). Let \( R \) be an equivalence relation on \( X \). Suppose that \( Z(R) \) and \( Z(f(R)) \) do exist. Then we have \( f(Z(R)) \subseteq Z(f(R)) \) and a factorization \( Z(f) : Z(R) \rightarrow Z(f(R)) \). Let \( S \) be an equivalence relation on \( Y \), then we get \( Z(f^{-1}(S)) \subseteq f^{-1}(Z(S)) \).

**Proof.** Since the map \( f \) is a regular epimorphism, from \([R, Z(R)] = 0\), we get \([f(R), f(Z(R))] = 0\). Whence \( f(Z(R)) \subseteq Z(f(R)) \) and a factorization we shall denote by \( Z(f) \):

\[
\begin{array}{ccc}
Z(R) & Z(f(R)) \\
p_0 & p_1 & p_0 \downarrow & p_1 \downarrow \\
X & f & Y
\end{array}
\]

Suppose \( R = f^{-1}(S) \). Then from \( f(Z(f^{-1}(S))) \subseteq Z(f(f^{-1}(S))) = Z(S) \), we get \( Z(f^{-1}(S)) \subseteq f^{-1}(Z(S)) \). \( \square \)

**Proposition 3.4.** Suppose \( \mathcal{D} \) is a regular Mal’cev category. Let \( R \) (resp. \( S \)) be an equivalence relation on \( X \) (resp. \( Y \)). Suppose that \( ZR, ZS \) and \( Z(R \times S) \) do exist. Then we have \( Z(R \times S) = ZR \times ZS \) provided that \( X \) and \( Y \) have global support (i.e. are such that their terminal map \( X \rightarrow 1 \) is a regular epimorphism). When \( \mathcal{D} \) is pointed, we have this in any case.

**Proof.** We have \([R, ZR] = 0 \) and \([S, ZS] = 0 \). So we have \([R \times S, ZR \times ZS] = 0 \), and consequently \( ZR \times ZS \subseteq Z(R \times S) \). When \( X \) has global support, the projection \( p_Y : X \times Y \rightarrow Y \) is a regular epimorphism. Moreover the object \( R \) has global support, and \( p_S : R \times S \rightarrow S \) is a regular epimorphism. Consequently \( S \) is the direct image of \( R \times S \) along \( p_Y \). From the previous proposition we get a map \( Z(p_Y) \):

\[
\begin{array}{ccc}
Z(R \times S) & ZS \\
p_0 & p_1 \downarrow & p_0 \downarrow & p_1 \downarrow \\
X \times Y & Y
\end{array}
\]

The same holds for \( R \) when \( Y \) has global support. Finally we get a factorization \((Z(p_X), Z(p_Y)) \) which means that \( Z(R \times S) \subseteq ZR \times ZS \):

\[
\begin{array}{ccc}
Z(R \times S) & ZR \times ZS \\
p_0 \downarrow & p_0 \downarrow & p_1 \times p_0 \downarrow & p_1 \times p_1 \downarrow \\
X \times Y & X \times Y \quad \square
\end{array}
\]
In conclusion, when \( \mathcal{D} \) is a Mal’cev category in which there are centralizers, the double centralization \( Z^2(R) = \overline{R} \) is a closure operator on equivalence relations which satisfies:

\[
R \subset \overline{R}; \quad \overline{R} = \overline{\overline{R}} \quad \text{and} \quad R \subset S \implies \overline{R} \subset \overline{S}
\]

In the regular Mal’cev context, we can easily add: \( Z(R \vee S) = Z(R) \land Z(S) \) and \( Z(R) \lor Z(S) \subset Z(R \land S) \).

### 3.2. Eccentral groupoids

In this section we shall prepare an intrinsic characterization of the faithful groupoids in the context of groupoid accessible Mal’cev categories. We recalled above that an internal reflexive graph \( (d_0, d_1): X_1 \Rightarrow X_0 \) in a Mal’cev category \( \mathcal{D} \) is a groupoid if and only if we have \([R[d_0], R[d_1]] = 0\), namely \( R[d_1] \subset Z(R[d_0]) \). Of course, there is an extremal situation:

**Definition 3.2.** Let \( \mathcal{D} \) be a Mal’cev category. A groupoid \( X_1 \) in \( \mathcal{D} \) is said to be eccentral when we have \( Z(R[d_0]) = R[d_1] \).

In other word a groupoid \( X_1 \) in \( \mathcal{D} \) is eccentral if and only if any reflexive relation \( \Sigma_1 \) on \( \text{Dec}_1 X_1 \) in \( \text{DiF} \) has its legs coequalized by \( \xi_1 X_1 \):

\[
\Sigma_1 \xleftarrow{p_0} \text{Dec}_1 X_1 \xrightarrow{\xi_1 X_1} X_1
\]

We already noticed that any faithful groupoid \( X_1 \) in a Mal’cev category \( \mathcal{D} \) is necessarily eccentral by Corollary 3.1.

**Proposition 3.5.** If the groupoid \( X_1 \) in \( \mathcal{D} \) is eccentral, then any discrete fibration with domain \( X_1 \) is monomorphic. When \( \mathcal{D} \) is exact, this last property is characteristic.

**Proof.** Compose this discrete fibration \( f_1: X_1 \to Y_1 \) with \( \xi_1 X_1 \) and complete the diagram with the kernel equivalence relation of the discrete fibration \( f_1, \xi_1 X_1 \). Then certainly we have \([R[d_0], R[f_0, d_1]] = 0\). Whence \( R[f_0, d_1] \subset Z(R[d_0]) = R[d_1] \), and consequently \( R[f_0, d_1] = R[d_1] \). Since \( d_1 \) is a split epimorphism, then \( f_0 \) is a monomorphism.

Now suppose that \( \mathcal{D} \) is exact [1] and \( S \) is an equivalence relation such that \([S, R[d_0]] = 0\). Then consider the equivalence relation \( \Sigma_1 \) on the object \( S \) determined by the double centralizing relation \( \Sigma' \):

\[
\Sigma_1 = S \times_X R[d_0] \xleftarrow{p_1} R[d_0] \xrightarrow{\text{d_1}} X_1
\]

and take its direct image along the functor \( \xi_1 X_1 \); this gives us an equivalence relation \( T_1 \) on \( X_1 \) in \( \text{DiF} \):
In $\mathbb{D}$, this gives us the following diagram where any commutative square is a pullback:

$$
\begin{array}{c}
\Sigma_1 \overset{\gamma_1}{\longrightarrow} T_1 \\
p_0 \downarrow \quad \downarrow p_1 \quad \downarrow p_0 \downarrow p_1 \\
Dec X_1 \overset{\varepsilon_1 X_1}{\longrightarrow} X_1
\end{array}
$$

Since $\mathbb{D}$ is exact, we can complete the diagram by the quotient functor $q_1$ which lies inside the discrete fibrations. According to our assumption on $X_1$, it is a monomorphism, and the two legs of the pair $T_1 \rightrightarrows X_1$ are the same. Accordingly we get that $\varepsilon_1 X_1.p_0 = \varepsilon_1 X_1.p_1$ which implies $S \subset R[d_1]$. Consequently we have $R[d_1] = Z(R[d_0])$. □

**Proposition 3.6.** If the equivalence relation $R$ is eccentric, then $Z(R) = \Delta X$. When $\mathbb{D}$ is regular, this last property is characteristic.

**Proof.** Let $R$ be eccentric and $S$ such that $[R, S] = 0$. Let us denote by $\Sigma$ the associated centralizing double relation, and let us consider the following diagram:

The upper part of the diagram shows that $[\Sigma, R[p_0]] = 0$; whence $\Sigma \subset R[p_1]$. The commutations of the following diagram:

implies that the two projections $S \rightrightarrows X$ are equal, so that we have $S = \Delta X$. Whence $Z(R) = \Delta X$.

Conversely suppose $R$ is such that $Z(R) = \Delta X$. Let $\Sigma$ be an equivalence relation on the object $R$ such that $[\Sigma, R[p_0]] = 0$. Since $p_1 : R \to X$ is a regular epimorphism, we get $[p_1(\Sigma), p_1(R[p_0])] = 0$, with $p_1(R[p_0]) = R$. Whence $p_1(\Sigma) \subset Z(R) = \Delta X$. Accordingly we get $\Sigma \subset R[p_1]$, and $R[p_1] = Z(R[p_0])$. □
We get also the stability of eccentral groupoids under products:

**Proposition 3.7.** Suppose the Mal'cev category $\mathbb{D}$ is regular. Then the eccentral groupoids are stable under products, provided that these groupoids have global support. When $\mathbb{D}$ is pointed, we have this in any case.

**Proof.** It is straightforward from Proposition 3.4 when the groupoids have global support. □

In the regular context, one important point is that eccentral groupoids are also related to the existence of centralizers:

**Proposition 3.8.** Suppose the Mal'cev category $\mathbb{D}$ is regular, $X_1$ an eccentral groupoid and $R$ an equivalence relation on $X$. Then any regular epimorphic discrete fibration $j_1 : R_1 \to X_1$ is such that $R[j_0]$ is the centralizer $Z(R)$.

**Proof.** Consider the following diagram in $Grd\mathbb{D}$ where $\Sigma_1$ is the double centralizing relation associated with a connected pair $[R, S] = 0$. We shall show that $\Sigma_1$ factorizes through $R[j_1]$ or, equivalently, that $j_1$ coequalizes $p_0$ and $p_1$. For that take the direct image along the regular epimorphic discrete fibration $Dec_1j_1$ of the equivalence relation $Dec_1\Sigma_1$:

The maps $\pi_0$ and $\pi_1$ are discrete fibrations since they come from a decomposition of a discrete fibration through a regular epic functor. Accordingly $Dec_1j_1(\Sigma_1)$ is a reflexive relation in $DiF\mathbb{D}$, which factorizes through $Dec_1^2X_1$ since $X_1$ is an eccentral groupoid. So $\xi_1X_1, Dec_1j_1$ coequalizes $Dec_1p_0$ and $Dec_1p_1$, and since $\xi_1\Sigma_1$ is a regular epic functor, the functor $j_1$ coequalizes $p_0$ and $p_1$. □

3.3. Groupoid accessible Mal'cev category

We are now in position to characterize the faithful groupoids inside the groupoid accessible Mal'cev categories:

**Proposition 3.9.** Suppose $\mathbb{D}$ is a groupoid accessible Mal'cev category. A groupoid is faithful if and only if it is eccentral.

**Proof.** Suppose the groupoid $T_1$ is eccentral. Take an index towards a faithful groupoid $X_1$ and compose with $\xi_1T_1$:
Then we have \( R[\phi_0,d_1] = ZR[d_0] = R[d_1] \) since \( T_1 \) is eccentric. Since \( d_1 \) is a split epimorphism, the map \( \phi_0 \) is a monomorphism (Barr–Kock theorem for the split epimorphisms) and \( T_1 \) is a fibrant subobject of the faithful groupoid \( X_1 \). Accordingly \( T_1 \) is faithful. \( \square \)

From this and Proposition 3.7 we get immediately:

**Proposition 3.10.** When \( \mathbb{D} \) is a regular groupoid accessible Mal’cev category, the faithful groupoids are stable under products, provided that these groupoids have global support. When \( \mathbb{D} \) is pointed, we have this in any case.

### 3.3.1. Existence of centralizers

Evidently, by Proposition 3.1, we get the existence of centralizers in any groupoid accessible Mal’cev category:

**Theorem 3.1.** Suppose \( \mathbb{D} \) is a groupoid accessible Mal’cev category. Any equivalence relation \( R \) admits a centralizer.

We have also the following property:

**Proposition 3.11.** Suppose the Mal’cev category \( \mathbb{D} \) is regular and groupoid accessible. Let \( R \) and \( S \) be equivalence relations on \( X \) and \( Y \). Suppose moreover that \( f : X \to Y \) is a map in \( \mathbb{D} \) which produces a discrete fibration, denoted \( f_1 : R_1 \to S_1 \), between them. Then \( Z(R) \) factorizes through \( Z(S) \) in such a way that we have \( Z(R) = f^{-1}(Z(S)) \). Whenever this map \( f \) is a regular epimorphism, we have moreover \( Z(S) = f(Z(R)) \).

**Proof.** According to Proposition 1.3, the discrete fibration \( f_1 : R_1 \to S_1 \) determines a monomorphic discrete fibration \( \bar{f}_1 : X_1 \to Y_1 \) between the indexes of \( R \) and \( S \). Complete the following diagram in \( Grd\mathbb{D} \) with the kernel equivalence relations:

\[
\begin{array}{ccc}
R_1[\phi_1] & \xrightarrow{d_1} & R_1 \\
\downarrow{R_1(f_1)} & & \downarrow{f_1} \\
R_1[\phi_1'] & \xrightarrow{d_0} & S_1 \\
\end{array}
\]

The image of this diagram by the functor \( ()_0 : Grd\mathbb{D} \to \mathbb{D} \) gives rise to the following diagram in \( \mathbb{D} \):

\[
\begin{array}{ccc}
Z(R) = R[\phi_0] & \xrightarrow{d_1} & X \\
\downarrow{R(f)} & & \downarrow{f} \\
Z(S) = R[\phi_0'] & \xrightarrow{d_0} & Y \\
\end{array}
\]

The map \( R(f) \) is the desired factorization \( Z(f) \). Since \( f_0 \) is a monomorphism, we have \( Z(R) = f^{-1}(Z(S)) \). If, moreover \( f \) is a regular epimorphism, we get \( f(Z(R)) = f(f^{-1}(Z(S))) = Z(S) \). \( \square \)

So, in those circumstances, any fibrant monomorphism \( m_1 : R_1 \to S_1 \) between two equivalence relations is such that \( Z(R) = m^{-1}(Z(S)) \).
3.3.2. Characterization of the abelian equivalence relations

The nature of the index will allow us to characterize the abelian equivalence relations.

**Proposition 3.12.** Suppose \( \mathcal{D} \) is a groupoid accessible Mal’cev category. When an index of an equivalence relation \( R \) on an object \( Y \) is totally disconnected, then \( R \) is an abelian equivalence relation. Conversely, when \( \mathcal{D} \) is regular, if \( R \) is abelian, its index is totally disconnected. In particular an object \( Y \) in \( \mathcal{D} \) is commutative if and only the index \( X_1 \) of \( \nabla_Y \) is such that its object of objects \( X_0 \) is a subobject of 1. Accordingly, when \( \mathcal{D} \) is pointed, an object \( Y \) in \( \mathcal{D} \) is commutative if and only the index of \( \nabla_Y \) is an internal group.

**Proof.** Consider the following index:

\[
\begin{array}{ccc}
R & \xrightarrow{\phi_1} & X_1 \\
p_0 & \downarrow & d \\
Y & \phi_0 & X_0
\end{array}
\]

We get \( \phi_0 \cdot p_0 = d \cdot \phi_1 = \phi_0 \cdot p_1 \), and consequently \( R \subseteq R[\phi_0] = ZR \). Accordingly \( [R, R] = 0 \), and \( R \) is abelian.

Suppose \( R \) abelian and consider its index:

\[
\begin{array}{ccc}
R & \xrightarrow{\phi_1} & X_1 \\
p_0 & \downarrow & d_0 \\
Y & \phi_0 & X_0
\end{array}
\]

We have \( R \subseteq ZR = R[\phi_0] \). Accordingly \( \phi_0 \cdot p_0 = \phi_0 \cdot p_1 \) and thus \( d_0 \cdot \phi_1 = d_1 \cdot \phi_1 \). Since \( \phi_1 \) is a regular epimorphism, we get \( d_0 = d_1 \), and \( X_1 \) is totally disconnected.

Suppose \( R = \nabla_Y \). Then \( Y \) is commutative if and only if \( R[\phi_0] = Z \nabla_Y = \nabla_Y \). If \( X_0 \) is a subobject of 1, it is straightforward. Conversely, since \( \phi_0 \) is a regular epimorphism, the equality \( R[\phi_0] = \nabla_Y \) makes \( X_0 \) a subobject of 1 (which implies that \( X_1 \) is totally disconnected). \( \square \)

3.4. Exact Mal’cev setting

Eventually, in the exact Mal’cev context, we shall produce a characterization of the existence of centralizers by the existence of enough eccentral groupoids.

**Proposition 3.13.** Let \( \mathcal{D} \) be an exact Mal’cev category with centralizers. Given any equivalence relation \( R \), there is a, unique up to isomorphism, regular epic discrete fibration \( j_1 : R_1 \rightarrow X_1 \) to an eccentral groupoid \( X_1 \).

**Proof.** Let \( \Sigma_1 \) be the double centralizing relation associated with the connected pair \( [R, Z(R)] = 0 \). Since \( \mathcal{D} \) is exact, we can take a levelwise quotient of this double relation which produces a regular epic discrete fibration:

\[
\begin{array}{ccc}
\Sigma_1 & \xrightarrow{\pi_0} & R_1 \\
\quad & \downarrow & \quad \\
\quad & \pi_1 \\
\end{array}
\]
We have to show now that the groupoid $X_1$ is eccentric. Suppose given an equivalence relation $\Lambda_1$ on $\text{Dec}_1 X_1$ which is in $\text{DiF}$. Then consider the inverse image of $\Lambda_1$ along the regular epic discrete fibration $\text{Dec}_1 j_1$ in the following diagram:

![Diagram](attachment:diagram.png)

Its direct image $\Gamma_1$ along the regular epic discrete fibration $\epsilon_1 R_1$ is an equivalence relation in $\text{DiF}$ which is a double centralizing relation associated with $R$. Accordingly this direct image factorizes through $\Sigma_1$ according to Proposition 3.8, and produces the left-hand side vertical dotted factorization. Accordingly the pair $(\bar{\pi}_0, \bar{\pi}_1)$ is coequalized by $\epsilon_1 X_1$. And since $h_1$ is an epimorphism, the pair $(\pi_0, \pi_1)$ is coequalized by $\epsilon_1 X_1$. Accordingly $\Lambda_1$ factorizes through $\text{Dec}_2 X_1$, and $X_1$ is eccentric.

Suppose now there are two regular epic discrete fibrations $j_1$ and $j'_1$ to eccentric groupoids $X_1$ and $X'_1$. Then $R[j_0] = Z(R) = R[j'_0]$. Accordingly $X_0$ is isomorphic to $X'_0$. Since $j_1$ and $j'_1$ are discrete fibrations, the two equivalence relations $R[j_1]$ and $R[j'_1]$ are part of the double centralizing relation associated with the pair $(R, Z(R))$. Since $\mathcal{D}$ is a Mal'cev category, this double centralizing relation is unique (up to isomorphism), and consequently we get $R[j_1] = R[j'_1]$; so that $X_1$ is isomorphic to $X'_1$. □

**Theorem 3.2.** Let $\mathcal{D}$ be an exact Mal’cev category. Then $\mathcal{D}$ has centralizers if and only if $\mathcal{D}$ has “enough” eccentric groupoids with respect to $\text{DiF}$: namely, from any groupoid $T_1$ there is a regular epic discrete fibration $\phi_1 : T_1 \rightarrow X_1$ with $X_1$ an eccentric groupoid. In this case the eccentric groupoid is unique up to isomorphism.

**Proof.** If $\mathcal{D}$ has enough eccentric groupoids, $\mathcal{D}$ has centralizers according to Proposition 3.8. Conversely suppose $\mathcal{D}$ has centralizers. Let $j_1 : \text{Dec}_1 T_1 \rightarrow X_1$ be the regular epic discrete fibration, with $X_1$ eccentric, given by the previous proposition. Since $X_1$ is eccentric, the functor $j_1$ trivializes the equivalence relation $\text{Dec}_2 T_1$ since it is in $\text{DiF}$, according to Proposition 3.8.

![Diagram](attachment:diagram2.png)

Accordingly there is a factorization $\phi_1$ which is regular epic and a discrete fibration, since so is $j_1$. The eccentric codomain $X_1$ of this factorization $\phi_1$ is unique up to isomorphism since, by the previous proposition, it was already the case for the codomain of $j_1$. □

4. Protomodular setting

In this section we are going to give a specific characterization of the faithful groupoids and of the groupoid accessible categories in the protomodular context, dealing only with the split epimorphisms, i.e. with the fibration of points. This will give us the opportunity to develop another type of example
of $\Theta$-faithful objects. Recall that a category $\mathcal{D}$ is said to be protomodular [2] when the change of base functors with respect to the fibrations of points $\mathcal{D} : \mathcal{P} \mathcal{D} \to D$ are conservative (i.e. reflect the isomorphisms), and that any protomodular category is a Mal’cev category. Let us begin by a remark:

**Lemma 4.1.** Let $\mathcal{D}$ be a protomodular category. Then any internal group is faithful.

**Proof.** Let $A$ be a (necessarily abelian) a group in $\mathcal{D}$. Consider the following diagram in $\mathcal{D}$, where the internal functors $u_1, h_1$ and $\bar{h}_1$ are discrete fibrations:

$$
\begin{array}{c}
T'_1 \\ u_1 \\
\downarrow \\
T_1 \\
\downarrow \\
A
\end{array}
\quad
\begin{array}{c}
h_1 \\
\downarrow \\
\bar{h}_1
\end{array}

\begin{array}{c}
d_0 \\
\downarrow \\
d_1
\end{array}
\quad
\begin{array}{c}
d_0 \\
\downarrow \\
d_1
\end{array}

\begin{array}{c}
d_0 \\
\downarrow \\
o
\end{array}

Suppose we have $h_1 = \bar{h}_1$. Since $\mathcal{D}$ is protomodular and any of the left-hand side squares is a pullback, the pair $(u_1, s_0 : T_0 \to T_1)$ is jointly strongly epic. Now, the pair $(h_1, \bar{h}_1)$ is clearly equalized by $s_0$; if, by assumption, it is equalized by $u_1$, we get $h_1 = h_1$ and $\bar{h}_1 = \bar{h}_1$. 

From that we get immediately:

**Proposition 4.1.** Any additive category $\mathcal{A}$ is groupoid accessible.

**Proof.** Any additive category is protomodular. The previous lemma shows that, for any object $A$, the terminal split epimorphism $A \rightrightarrows 1$ produces a faithful internal groupoid. Moreover, any internal groupoid in $\mathcal{A}$ with object of objects $T$ is determined by a map $\gamma : C \to T$ and consequently produces a discrete fibration to a faithful groupoid:

$$
\begin{array}{c}
T \oplus C \\ p_C \\
\downarrow \\
C
\end{array}
\quad
\begin{array}{c}
T \\
\downarrow \\
1
\end{array}

4.1. Action accessible categories

4.1.1. Non-pointed case

Given any fibration $U : \mathcal{E} \to \mathcal{E}'$, the class of $U$-cartesian maps is a proper class in $\mathcal{E}$. We shall call $U$-faithful any object in $\mathcal{E}$ which is faithful relatively to this class, and we shall call $U$-accessible a category $\mathcal{E}$ with enough $U$-faithful objects. More particularly:

**Definition 4.1.** Given a finitely complete category $\mathcal{E}$, a split epimorphism in $\mathcal{E}$ will be called faithful, when it is $\mathcal{E}$-faithful. A category $\mathcal{E}$ will be said to be action accessible when it is $\mathcal{E}$-accessible.

**Proposition 4.2.** When an internal groupoid $X_1$, in any category $\mathcal{E}$, is such that its underlying split epimorphism $(d_0, s_0) : X_1 \rightrightarrows X_0$ is faithful, then it is a faithful groupoid.

**Proof.** This comes from Proposition 1.4 applied to the forgetful faithful functor: $Y_\mathcal{E} : \text{Grd}\mathcal{E} \to \mathcal{P}t\mathcal{E}$. 


First we shall show that any groupoid accessible protomodular category is action accessible, and then, that, in the regular context, the two notions coincide. This was already observed in the pointed case in [11]. Let us begin by a remark:

**Lemma 4.2.** Let $\mathcal{D}$ be a protomodular category. Then any split terminal map is faithful.

**Proof.** The proof can be copied from the proof of Lemma 4.1. $\square$

In the protomodular context, the conditions of the previous Proposition 4.2 become a characterization:

**Proposition 4.3.** Suppose the category $\mathcal{D}$ is protomodular. A groupoid $X_1$ is faithful if and only if the split epimorphism $(d_0, s_0) : X_1 \rightrightarrows X_0$ is faithful. Accordingly if the category $\mathcal{D}$ is groupoid accessible, it is action accessible.

**Proof.** Let be given a faithful groupoid $X_1$ and a pair of parallel maps in $\text{Pt}\mathcal{D}$ with codomain $(d_0, s_0) : X_1 \rightrightarrows X_0$ which are equalized by a monomorphic $\mathcal{D}$-cartesian map $(v, u)$. This gives us the lower part of the following diagram, where any commutative square is a pullback:

Once it is completed by the kernel equivalence relations (upper part of the diagram), any of the upper functors are discrete fibrations. From $\text{id}_X \cdot \cdot \cdot = \text{id}_X$, we get $d_1, h = d_1, \text{id}_X$ since the groupoid $X_1$ is faithful. Whence $f = d_1, s_0, f = d_1, h \cdot s_0 = d_1, \text{id}_X \cdot s_0 = d_1, \text{id}_X \cdot f = d_1, \text{id}_X$. Since $\mathcal{D}$ is protomodular, the pair $(u, t)$ is jointly strongly epic. By assumption we have $h, u = h, u$. From $h, t = s_0, f = s_0, f = h, t$, we get $h = h$. Accordingly the split epimorphism $(d_0, s_0)$ is faithful.

Suppose now $\mathcal{D}$ is groupoid accessible, and start with a split epimorphism $(f, s) : X \rightrightarrows Y$. Then its kernel equivalence relation $R[f]$ has an index $\phi_1 : R[f] \rightarrow X_1$ which produces the following right-hand side pullbacks:

Since $f$ is split, we have also the left-hand side pullback, and since the groupoid $X_1$ is faithful, the split epimorphism $(d_0, s_0) : X_1 \rightrightarrows X_0$ is faithful. $\square$

We can now give an intrinsic characterization of the faithful split epimorphisms:

**Corollary 4.1.** Let $\mathcal{D}$ be a protomodular groupoid accessible category. A split epimorphism $(f, s) : X \rightrightarrows Y$ is faithful if and only if $s^{-1}(\Delta_Y) = \Delta_Y$. 

Proof. Let \((f, s) : X \rightrightarrows Y\) be a split epimorphism, and let us consider the last diagram above. Since the groupoid \(X_1\) is faithful, we have \(Z(R[f]) = R[\phi_0]\). On the other hand, the whole rectangle of this diagram determines a \(\mathcal{D}\)-cartesian map \((f, s) \to (d_0, s_0)\) in \(Pt\mathcal{D}\). If the split epimorphism \((f, s)\) is faithful, this map is a monomorphism according to Proposition 1.1. So that we get \(s^{-1}(Z(R[f])) = s^{-1}(R[\phi_0]) = R[\phi_0]\). On the other hand, the whole rectangle of this diagram determines a \(\mathcal{D}\)-cartesian map \((f, s) \to (d_0, s_0)\) in \(Pt\mathcal{D}\). If the split epimorphism \((f, s)\) is faithful, this map is a monomorphism according to Proposition 1.1. So that we get \(s^{-1}(Z(R[f])) = s^{-1}(R[\phi_0]) = R[\phi_0]\). Conversely suppose we have \(s^{-1}(Z(R[f])) = \Delta Y\), then \(R[\phi_0] = \Delta Y\) and the map \(\phi_0.s\) is a monomorphism; accordingly \((f, s) \to (d_0, s_0)\) is a monomorphic \(\mathcal{D}\)-cartesian map having a faithful codomain \((d_0, s_0)\), so that \((f, s)\) is itself faithful. □

Actually we shall get the converse to Proposition 4.3 (namely: action accessibility implies groupoid accessibility) in the regular context by taking a way back to the Mal'cev setting. First step:

Proposition 4.4. Suppose \(\mathcal{D}\) is a regular Mal'cev category, and consider the forgetful functor \(\Upsilon_{\mathcal{D}} : Grd\mathcal{D} \to Pt\mathcal{D}\). Then any regular epimorphic \(\mathcal{D}\)-cartesian map in \(Pt\mathcal{E}\) produces a \(\Upsilon_{\mathcal{D}}\)-cocartesian map above it.

Proof. Suppose given an internal groupoid \(X_1\) and a pullback in \(\mathcal{D}\):

we have first to complete the right-hand side split epimorphism into an internal groupoid. For that let us consider the following diagram:

Since any of the commutative left-hand side squares are pullback and \(\mathcal{D}\) is a Mal'cev category, then the pair \((R(s_0), s_0) : X_1 \to R(q_1)\) is jointly strongly epic. So, we can check that the map \(q_0.d_1 : X_1 \to Z_0\) trivializes the equivalence relation \(R[q_1]\) by composition with this pair, which is straightforward. Whence a unique map \(d_1 : Z_1 \to Z_0\) which produces an internal reflexive graph. Since \(X_1\) is an internal groupoid and \(q_1\) a regular epimorphism, the reflexive graph \(Z_1\) is underlying a groupoid.

Suppose moreover there is a functor \(g_1 : X_1 \to \mathcal{T}_1\) whose image by \(\Upsilon_{\mathcal{D}}\) factorizes through our initial pullback:
This means that the pair \((h_0, h_1)\) commutes only with the \(d_0\). We have to check that it commutes also with the \(d_1\). This is straightforward since \(q_1\) is a regular epimorphism, and \(g_1\) a functor. \(\square\)

Second step:

**Proposition 4.5.** Suppose \(\mathbb{D}\) is a regular Mal'cev category. If it is action accessible, it is groupoid accessible.

**Proof.** Start with a groupoid \(X_1\) and take a \(\mathfrak{g}_\mathbb{D}\)-index (given by the pair \((Z_0, Z_1)\)) for the split epimorphism \((d_0, s_0) : X_1 \rightrightarrows X_0\). Then consider its epimorphic part (given by the pair \((Z'_0, Z'_1)\)):

This gives rise to two pullbacks. Since the maps \(m_0\) and \(m_1\) are monic, the split epimorphism \((d_0, s_0) : Z'_1 \rightrightarrows Z'_0\) is still faithful. Since \(q_1\) is a regular epimorphism, we can complete this split epimorphism into a groupoid, which is faithful by Proposition 4.2. \(\square\)

Finally:

**Corollary 4.2.** Suppose \(\mathbb{D}\) is a regular protomodular category. It is action accessible if and only if it is groupoid accessible.

**Proof.** It is a straightforward consequence of the previous proposition and Proposition 4.3. \(\square\)

### 4.1.2. Pointed case

In this section we shall show that, when the category \(\mathbb{D}\) is pointed protomodular, action accessibility in the sense of Definition 4.1 coincides with action accessibility in the sense of [11]. Let \(\mathbb{D}\) be pointed protomodular category. Recall that a *split extension with kernel* \(K*\):
was called *faithful* in [11], when, from any other split extension with kernel $K$, there is at most one morphism of split extensions into it:

![Diagram of split extensions](image)

**Lemma 4.3.** Given any pointed protomodular category $\mathcal{D}$, a split epimorphism $(f, s) : H \rightarrowtail G$ is faithful according to Definition 4.1 if and only if its associated split extension:

![Diagram of split epimorphism](image)

is faithful in the sense of [11]. Accordingly the respectively associated notion of accessible groupoids coincide.

**Proof.** Suppose that the previous split extension is faithful according to [11]. Consider the following situation, where the right-hand side commutative squares are pullbacks, and the two vertical edges are equal:

![Diagram of split extensions](image)

and complete it with the extensions to kernels. Since the upper right-hand side square is a pullback, the factorization $K(\nu)$ is an isomorphism. Accordingly the isomorphisms $K(\chi)$ and $K(\chi')$ are equal since they are equalized by $K(\nu)$. Since the lower split extension is faithful, and modulo the inverse of the isomorphism $K(\chi)$, we get $\phi = \phi'$ and $\psi = \psi'$.

Conversely suppose that the split epimorphism $(f, s)$ is faithful according to Definition 4.1. Moreover suppose you have the following lower commutative parallel diagram, which implies that the two lower right-hand side squares are pullbacks, namely $\mathcal{D}$-cartesian:

![Diagram of split extensions](image)
Then extend it with the equalizers $\mu$ of the pair $(\phi, \phi')$ and $\nu$ of the pair $(\psi, \psi')$. By commutation of limits, you get the upper horizontal split extension. According to the upper left and side vertical equality, the upper right-hand side is a pullback, namely $\mathcal{F}_{\mathcal{D}}$-cartesian. Since the split epimorphism $(f, s)$ is faithful, then we get $\phi = \phi'$ and $\psi = \psi'$. □

In [11], a pointed protomodular category $\mathcal{D}$ was called action accessible when there is “enough” faithful split extensions. Accordingly we get the following:

**Corollary 4.3.** Any pointed protomodular category is action accessible in the sense of [11] if and only if it is action accessible according to Definition 4.1. In the same way, any pointed protomodular category is groupoid accessible category in the sense of [11] if and only if its is groupoid accessible in our sense.

### 4.2. Reflection of commutative objects in $\mathcal{D}/Y$

One of the main consequences of the groupoid accessibility in the protomodular context is the reflection under pullback of the abelian object in the slice categories, which implies the reflection under pullback of the extensions with abelian kernel relations.

**Theorem 4.1.** Suppose $\mathcal{D}$ is a groupoid accessible regular protomodular category. The pullback functors reflect the commutative objects of the slice categories.

**Proof.** The map $f : X \to Y$ is commutative in $\mathcal{D}/Y$ if and only if $R[f]$ is an abelian equivalence relation. Consider the following diagram where the lower square is pullback, and $X_1$ is the index of $R[f]$:

$$
\begin{array}{ccccc}
R[f'] & \xrightarrow{R(g)} & R[f] & \xrightarrow{\phi_1} & X_1 \\
p_0 & \downarrow & p_1 & & \\
X' & \xrightarrow{g} & X & \xrightarrow{\phi_0} & X_0 \\
f' & \downarrow & f & & \\
Y' & \xrightarrow{h} & Y & & \\
\end{array}
$$

Then the two upper squares give rise to a discrete fibration $R_1[f'] \to X_1$ which becomes an index of $R[f']$. Now consider the regular epimorphic part of this fibration which gives us the index of $R[f']$: 

$$
\begin{array}{ccccc}
R[f'] & \xrightarrow{R(g)} & R[f] & \xrightarrow{\phi_1} & X_1 \\
p_0 & \downarrow & p_1 & & \\
X' & \xrightarrow{g} & X & \xrightarrow{\phi_0} & X_0 \\
f' & \downarrow & f & & \\
Y' & \xrightarrow{h} & Y & & \\
\end{array}
$$
In this diagram any square is a pullback. Since $\mathbb{D}$ is supposed to be protomodular, the pair $(l_1, s_0 : X_0 \rightarrow X_1)$ is jointly strongly epic. We have $d_0' = d_1'$ since $R[f']$ is abelian by Proposition 3.12. Accordingly $d_0 l_1 = d_1 l_1$. Since we know that $d_0 s_0 = 1_{X_0} = d_1 s_0$, we get $d_0 = d_1$, and $R[f]$ is abelian. 

4.3. Action representative categories

An action representative category $[4,5,3]$ is a pointed protomodular category which admits, for any object $K$ a split extension classifier, i.e. a split extension:

$$1 \rightarrow K \xrightarrow{\gamma} D_1 K \xrightarrow{d_0} DK \rightarrow 1$$

which is universal, as explained in the introduction. Consequently, any split extension classifier is faithful, and any action representative category is action accessible. As recalled in the introduction, the main examples of action representative categories are the categories $Gp$ of groups and the category $K$-$Lie$ of Lie-algebras. Obtaining the following result (which is well known in the category $Gp$ of groups where $D(X) = \text{Aut} X$) was one of the starting point of this work:

**Theorem 4.2.** Suppose that the category $\mathbb{D}$ is an action presentative homological (= pointed + protomodular + regular) category. Then the canonical comparison map $D(X) \times D(Y) \rightarrow D(X \times Y)$ is a monomorphism, and the canonical comparison functor $D_1(X) \times D_1(Y) \rightarrow D_1(X \times Y)$ is $(0)_0$-cartesian (i.e. fully faithful).

**Proof.** The upper split exact sequence produces a unique factorization $\gamma_0$:

$$X \times Y \xrightarrow{j_X \times j_Y} D_1(X) \times D_1(Y) \xrightarrow{d_1 \times d_1} D(X) \times (Y)$$

$$X \times Y \xrightarrow{j_X \times j_Y} D_1(X \times Y) \xrightarrow{d_0 \times d_0} D(X \times Y)$$

which commutes also with the $d_1$ (see [3]) and consequently produces the discrete fibration $\gamma_1$. Since the groupoids $D_1(X)$ and $D_1(Y)$ are faithful, such is their product $D_1(X) \times D_1(Y)$ by Proposition 3.10; so that the factorization $\gamma_1$ is necessarily a monomorphism, as any discrete fibration with domain a faithful groupoid. According to Lemma 1.1, the functor $\gamma_1$ is $(0)_0$-cartesian. 

5. Further examples of groupoid accessible categories

This section will be devoted to produce new examples.
5.1. Rings with unit and associative $R$-algebras with unit

Let us denote by $Rg_*$ the (non-pointed) category of rings with unit and $Rg$ the (pointed) category of rings. We shall denote by $\mathbb{K} : Pt(Rg_*) \to Rg$ the functor associating with any split epimorphism its kernel. It is straightforward that a split epimorphism $(f, s) : A \twoheadrightarrow B$ in $Rg_*$ gives $\mathbb{K}(f, s) = \text{Ker } f$ a structure of $B$-algebra (which is a certain kind of action of $B$). Conversely any $B$-algebra $K$ allows us to define a split epimorphism $B \times K \twoheadrightarrow B$ in $Rg_*$. In this section we shall show that the category $Rg_*$ is groupoid accessible. Since $Rg_*$ is regular and protomodular, it is enough to show it is action accessible.

**Lemma 5.1.** A morphism in $Pt_{Rg_*}$ is $\mathbb{K}_{Rg_*}$-cartesian if and only if its image by $\mathbb{K}$ is an isomorphism. The pair $(\mathbb{K}_{Rg_*}, \mathbb{K})$ of functors is jointly faithful.

**Proof.** The first point is well known, and the second one a direct consequence of the fact that $Rg$ is pointed protomodular. □

Let $(f, s) : A \twoheadrightarrow B$ be a split epimorphism in $Rg_*$. In this category, we do know that centralizers exist. So according to the characterization given by Corollary 4.1, we are immediately interested in the ideal $s^{-1}(Z(\text{Ker } f))$, the inverse image of the centralizer (i.e. annihilator) of $\text{Ker } f$, which we shall denote by:

$$I_s = \{ b \in B \mid \forall k \in \text{Ker } f, s(b).k = 0 = k.s(b) \}$$

We shall show that this ideal exactly measures the obstruction to the faithfulness of the split epimorphism $(f, s) : A \twoheadrightarrow B$.

**Lemma 5.2.** $I_s$ is an ideal of $B$, and its direct image $s(I_s)$ is an ideal of $A$.

**Proof.** The first point is straightforward. As for the second one, it is a direct consequence of the fact that $I_s$ is an ideal of $B$, once we are aware that we have $a = (a - s.f(a)) + s.f(a)$ for any $a \in A$. □

**Lemma 5.3.** When $I_s$ is trivial, the split epimorphism $(f, s) : A \twoheadrightarrow B$ is faithful.

**Proof.** Let us consider the following diagram where any commutative square is a pullback and $g.u = \tilde{g}.u$, $h.v = \tilde{h}.v$:

Since the left-hand side square is a pullback, its image by $\mathbb{K} : Pt(Rg_*) \to Rg$ is an isomorphism. Accordingly we get $\mathbb{K}(g, h) = \mathbb{K}(\tilde{g}, \tilde{h})$ and consequently $\text{Ker } f' \subset A''$. We have also $\text{Ker } f' \simeq \text{Ker } f$. We are now going to show that $g(\beta) - \tilde{g}(\beta)$ is in $I_s$ for any $\beta \in B'$, which will imply $g(\beta) = \tilde{g}(\beta)$ when $I_s$ is supposed to be trivial. So, let $k$ be in $\text{Ker } f$. There is a $\kappa$ in $\text{Ker } f'$ such that $h(\kappa) = k = \tilde{h}(\kappa)$. Then we get:

$$k.s(g(\beta)) = h(\kappa).h(s'(\beta)) = h(\kappa.s'(\beta)) = \tilde{h}(\kappa.s'(\beta)) = \tilde{h}(\kappa).\tilde{h}(s'(\beta)) = k.s(\tilde{g}(\beta))$$
the middle equality coming from the fact that \( \kappa . s'(\beta) \) is in \( \text{Ker} f' \) and thus in \( A'' \). The same thing holds for \( s(g(\beta)).\kappa \). Accordingly \( g(\beta) - \bar{g}(\beta) \) is in \( I_s \), and \( g = \bar{g} \). Since the pair \((\mathfrak{f}_{R*}, K)\) is jointly faithful, we get also \( h = \bar{h} \). □

Whence the following:

**Theorem 5.1.** The category \( R^*_g \) is action accessible, and thus groupoid accessible. A split epimorphism \((f, s) : A \rightrightarrows B\) is faithful if and only if \( I_s \) is trivial.

**Proof.** Starting from any split epimorphism \((f, s) : A \rightrightarrows B\), we shall construct a cartesian map to a faithful split epimorphism in the following way:

The squares are pullbacks since the kernels of \( q_B \) and \( q_A \) are isomorphic; accordingly \( \text{Ker} f \simeq \text{Ker} \phi \).

We have to show that the right-hand side split epimorphism is faithful. For that it is enough to show that the ideal \( I_{\sigma} \) is trivial. So let the class \( \bar{b} \) be in \( I_{\sigma} \). This means that, for any \( k \in \text{Ker} f \), \( k.\sigma(\bar{b}) = k.s(\bar{b}) = 0 \), in other words \( k.s(\bar{b}) \) is in \( s(I_s) \) (and the same thing for \( \sigma(\bar{b}), k \)); i.e. there is a \( \beta \in I_s \) such that \( k.s(\bar{b}) = s(\beta) \). The image by \( f \) of this equality implies that \( \beta = 0 \), so that \( k.s(\bar{b}) = 0 \) (and the same thing holds when we have \( \sigma(\bar{b}), k = 0 \)). Accordingly \( b \) is in \( I_s \) and \( \bar{b} = 0 \). The last point of the theorem is just a translation of Corollary 4.1. □

Given any commutative ring \( R \), exactly the same scheme of proof holds to show that the category \( R\text{-Ass} \) of unitary associative \( R \)-algebras is groupoid accessible.

5.2. Birkhoff subcategories

There is a very general result which extends the one known in the pointed case, see [11]:

**Proposition 5.1.** Let \( C \) be a full replete subcategory, stable under regular epimorphism, of a regular groupoid accessible category \( D \). Then \( C \) is groupoid accessible. In particular, any Birkhoff subcategory of a groupoid accessible Mal’cev category is groupoid accessible. When \( D \) is a regular and groupoid accessible Mal’cev category, then any fibre \( \text{Grd}_X D \) is groupoid accessible.

**Proof.** Let \( T_1 \) be a groupoid in \( C \). Take an index \( X_1 \) in \( D \) and its regular epimorphic part:

Then \( X'_1 \) is faithful as a fibrant subobject of a faithful groupoid. Moreover it is in \( C \) since the category \( C \) is stable under regular epimorphisms.

The fibre \( \text{Gph}_X D \) of internal reflexive graphs with object of objects \( X \) in a groupoid accessible category \( D \) is groupoid accessible as being the coslice category below the object \( \Delta_X \) in the slice
category $\mathcal{D}/(X \times X)$. Since $\mathcal{D}$ is a regular Mal’cev category, the fibre $\text{Grd}_X \mathcal{D}$ is a full subcategory of $\text{Gph}_X \mathcal{D}$ which is stable under regular epimorphisms. □

As a quick application, the categories $BRg_*$ of unitary boolean rings and $VNRg_*$ of unitary von Neumann regular [26] rings are groupoid accessible.

5.3. Topological Mal’cev algebras

Let $\mathbb{T}$ a Mal’cev theory. Suppose that the corresponding variety $\mathbb{V}(\mathbb{T})$ of $\mathbb{T}$-algebras is action accessible (and thus, according to Proposition 4.5, groupoid accessible, since any variety of algebras is exact and consequently regular). We are going to show that the category $\text{Top}(\mathbb{T})$ of topological $\mathbb{T}$-algebras is action accessible, and thus groupoid accessible, since it is necessarily a regular Mal’cev category according to [16] (the regular epimorphisms being the open surjective maps). From this, the category $\text{GpTop}$ of topological groups and $Rg_*\text{Top}$ of topological rings with unit will be groupoid accessible. Let us begin by the following:

Lemma 5.4. Let $\mathbb{T}$ be a Mal’cev theory. Then the forgetful left exact functor $U: \text{Top}(\mathbb{T}) \to \mathbb{V}(\mathbb{T})$ reflects the pullback of split epimorphisms along regular epimorphisms.

Proof. This forgetful functor is well known to be left exact and faithful. First, let us begin by the reflection of pullbacks of split epimorphisms along split epimorphisms. Let us consider the following external commutative diagram of split epimorphisms in $\text{Top}(\mathbb{T})$:

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & Y \\
\downarrow{\sigma} & & \downarrow{\psi} \\
P & \xrightarrow{f} & X' \\
\end{array}
\]

Let the internal diagram be a pullback in $\text{Top}(\mathbb{T})$, and $\kappa: X \to P$ the induced factorization. It is a regular epimorphism since $\text{Top}(\mathbb{T})$ is a Mal’cev category. Now, suppose the external diagram in question is mapped by $U$ onto a pullback in $\mathbb{V}(\mathbb{T})$. Then $U(\kappa)$ is an isomorphism, and consequently $\kappa$ is a monomorphism. Accordingly $\kappa$ is an isomorphism.

Dealing with a commutative diagram with only regular epimorphic horizontal arrows, we complete it by the kernel equivalence relations:

\[
\begin{array}{ccc}
R[f] & \xrightarrow{R[\phi]} & X \\
R(f) & \xrightarrow{R(s)} & Y \\
\end{array}
\]

Suppose the right-hand side square mapped by $U$ onto a pullback. The left-hand side squares are also mapped by $U$ onto pullbacks. As pullbacks of split epimorphisms, these left-hand side squares are themselves pullbacks. Since $\text{Top}(\mathbb{T})$ is a regular category, our initial square is a pullback by the Barr–Kock theorem. □
Proposition 5.2. Let $\mathbb{T}$ a Mal’cev theory such that $\forall(\mathbb{T})$ is action accessible (= groupoid accessible). Then the category $\text{Top}(\mathbb{T})$ of topological $\mathbb{T}$-algebras is action accessible (= groupoid accessible).

Proof. Start with a split epimorphism $(f, s): X \twoheadrightarrow Y$ in $\text{Top}(\mathbb{T})$. Then take a regular epic index of its image by $U$ to a faithful split epimorphism in $\forall(\mathbb{T})$:

\[
\begin{array}{c}
U(X) \overset{q_X}{\rightarrow} X' \\
U(f) \quad U(s) \\
U(Y) \overset{q_Y}{\rightarrow} Y'
\end{array}
\]

Endow $X'$ and $Y'$ with the quotient topologies. This make a commutative diagram in $\text{Top}(\mathbb{T})$ above the previous one:

\[
\begin{array}{c}
X \overset{\bar{q}_X}{\rightarrow} \overline{X}' \\
f \quad s \\
Y \overset{\bar{q}_Y}{\rightarrow} \overline{Y}'
\end{array}
\]

It is a pullback according the previous lemma. Its codomain is a faithful split epimorphism in $\text{Top}(\mathbb{T})$ according to Proposition 1.4, since the functor $U$ is left exact and faithful. □

5.4. The extremal Mal’cev cases

There are two extremal Mal’cev cases, when any pair $(R, S)$ of equivalence relations is connected and when the only ones are the ones such that $R \wedge S = \Delta_X$. We are now going to investigate their eccentral and faithful groupoids.

5.4.1. The naturally Mal’cev context

A naturally Mal’cev category $\mathbb{D}$ in the sense of [14] is a category in which any object is endowed with a natural Mal’cev operation. It is a Mal’cev category where any object is commutative, or, equivalently, any internal graph is an internal groupoid, or, again equivalently, where any pair $(R, S)$ of equivalence relations on the same object $X$ is connected. Accordingly, they are exactly those Mal’cev categories with centralizers which are such that, given any equivalence relation $R$ on an object $X$, we have $Z(R) = \nabla_X$.

Recall (see [2]) that when $\mathbb{D}$ is a Mal’cev category, any fibre $\text{Grd}_X \mathbb{D}$ is a protomodular naturally Mal’cev category. A pointed category is naturally Mal’cev if and only if it is additive.

Proposition 5.3. Suppose $\mathbb{D}$ is a naturally Mal’cev category. Then a groupoid $X_1$ is eccentral if and only if $X_0$ is a subobject of $1$.

Proof. In a naturally Mal’cev context, a groupoid $X_1$ is eccentral if and only if $R[d_1] = Z[R[d_0]] = \nabla_{X_1}$. Since $d_1$ is split, it is the quotient of $R[d_1]$, so that, $R[d_1] = \nabla_{X_1}$ if and only if $X_0$ is a subobject of $1$ (Barr–Kock theorem for the split epimorphisms). It is clear that this implies $d_0 = d_1$ and that $X_1$ is totally disconnected. □

Proposition 5.4. Suppose $\mathbb{D}$ is a groupoid accessible Mal’cev category. Then it is a naturally Mal’cev category if and only if the only faithful groupoids $X_1$ are the ones where $X_0$ is a subobject of $1$. In particular a pointed
Mal’cev category $\mathbb{D}$ is additive if and only if it is a groupoid accessible category in which the only faithful groupoids are the internal groups.

**Proof.** Suppose $\mathbb{D}$ is a groupoid accessible naturally Mal’cev category. According to the previous proposition the only faithful (= eccentral) groupoids $X_1$ are the ones where $X_0$ is a subobject of 1. Conversely, suppose $\mathbb{D}$ is a groupoid accessible Mal’cev category whose only faithful groupoids $X_1$ are the ones where $X_0$ is a subobject of 1. They are totally disconnected. In particular the index of any $\nabla X$ is totally disconnected, and, according to Proposition 3.12, we have $[\nabla X, \nabla X] = 0$. So any object $X$ is commutative and $\mathbb{D}$ is naturally Mal’cev. The second point is now straightforward. \(\square\)

**Proposition 5.5.** Suppose $\mathbb{D}$ is a protomodular naturally Mal’cev category. Then a groupoid $X_1$ is faithful if and only if it is eccentral. Any exact protomodular naturally Mal’cev category $\mathbb{D}$ is groupoid accessible.

**Proof.** In the Mal’cev context, we know that any faithful groupoid $X_1$ is eccentral. Conversely, suppose $X_1$ is eccentral, then $X_0$ is a subobject of 1. Since $\mathbb{D}$ is protomodular, we can just copy the proof of Lemma 4.1 to show that this kind of groupoid is necessarily faithful. Now, since any naturally Mal’cev category admits centralizers we can apply Proposition 3.2 in the exact case. \(\square\)

Accordingly, given an exact Mal’cev category $\mathbb{D}$, any fibre $Grd_1 \mathbb{D}$ is a groupoid accessible protomodular and naturally Mal’cev category.

5.4.2. The stiffly Mal’cev context

Recall the following [2]:

**Proposition 5.6.** For a given Mal’cev category $\mathbb{D}$, the following condition are equivalent:

1) any internal groupoid is an equivalence relation;
2) any abelian equivalence relation is discrete;
3) for any pair $(R, S)$ of equivalence relations: $R \wedge S = \Delta_X \iff [R, S] = 0$.

**Definition 5.1.** We shall say that a Mal’cev category $\mathbb{D}$ is a stiffly Mal’cev category when it satisfies any of the previous conditions.

The second condition shows immediately that the categories $BRg^*_*$ of unitary boolean rings and $VNRg^*_*$ of unitary von Neumann regular rings are stiffly Mal’cev categories. On the other hand, recall that the dual $\mathcal{E}^{op}$ of any elementary topos $\mathcal{E}$ is an exact stiffly Mal’cev category, see [25] and [7].

Recall [2] that, when the Mal’cev category $\mathbb{D}$ is regular, it is a stiffly Mal’cev category if and only if it is weakly congruence distributive; namely such that, for any triple of equivalence relations:

$$T \wedge R = \Delta_X \quad \text{and} \quad T \wedge S = \Delta_X \quad \Rightarrow \quad T \wedge (R \vee S) = \Delta_X$$

When the Mal’cev category $\mathbb{D}$ is exact, it is stiffly Mal’cev if and only if it is congruence distributive [25].

The aim of this section is to investigate the eccentral and faithful groupoids in this context and, in particular, to show that the dual of any boolean topos $\mathcal{E}$ is a groupoid accessible stiffly Mal’cev category in which the only faithful groupoids are the undiscrete equivalence relations $\nabla Y$. As a collateral benefit, we shall characterize those groupoid accessible Mal’cev categories which have the undiscrete equivalence relations as only faithful groupoids.

**Proposition 5.7.** A groupoid accessible Mal’cev category $\mathbb{D}$ is a stiffly Mal’cev category if and only if the only faithful groupoids are equivalence relations.
Proof. Any groupoid in a stiffly Mal’cev category is an equivalence relation; so the part “only if” is trivial. Conversely if any groupoid $T_1$ has an equivalence relation as index, it is itself an equivalence relation as the domain of a discrete fibration with codomain an equivalence relation. □

We need now the following observation: given a pair of equivalence relations $(R, S)$ on an object $X$, we denote by $S \Box R$ the inverse image of $R \times R$ along $(p_0, p_1): S \hookrightarrow X \times X$:

$S \Box R \xrightarrow{\text{p}_0 \times \text{p}_1} R \times R \xrightarrow{p_0 \text{ and } p_1} X \times X$.

$S \Box R$ is the largest double relation on the pair $(R, S)$. In the Mal’cev context, this double relation is a centralizing double relation if and only if we have $R \wedge S = \Delta_X$, see [10]. In this case, the following diagram is an equivalence relation which is nothing but $R \vee S$:

$S \Box R \xleftarrow{d_1, p_1} X \xrightarrow{d_0, p_0}$

Lemma 5.5. Suppose $\mathbb{D}$ is a regular Mal’cev category. Suppose $(R, S)$ is pair of equivalence relations on $X$ such that $R \wedge S = \Delta_X$. Then if $T$ is another equivalence relation on $X$ such that $T \subset R$, we get:

$T \vee S = R \vee S \implies T = R$

Proof. Consider the following diagram:

$S \Box T \xrightarrow{\text{p}_0 \times \text{p}_1} S \Box R \xrightarrow{\text{p}_0 \text{ and } \text{p}_1} R$.

Since $\mathbb{D}$ is a regular Mal’cev category, according to example 2.4.2 in [10], this diagram is underlying a fibrant subobject (i.e any commutative square is a pullback). The upper horizontal map is an isomorphism since $T \Box S = T \vee S = R \vee S = R \Box S$, accordingly such is the lower one. □

Proposition 5.8. Suppose $\mathbb{D}$ is a regular stiffly Mal’cev category. Given any pair $(X, Y)$ of objects, we have $Z(R[p_X]) = R[p_Y]$. Accordingly any undiscrete groupoid $\nabla_Y$ is eccentric.

Proof. Let $S$ be an equivalence relation on $X \times Y$ such that $[S, R[p_X]] = 0$, i.e. $S \wedge R[p_X] = \Delta_X$. Since we have also $R[p_Y] \wedge R[p_X] = \Delta_X$, we have necessarily $(S \vee R[p_Y]) \wedge R[p_X] = \Delta_X$ by the weak congruence distributivity. On the other hand it is clear that $R[p_Y] \vee R[p_X] = \nabla_{X \times Y} = S \vee R[p_Y] \vee R[p_X]$. According to the previous lemma, from $R[p_Y] \subset S \vee R[p_Y]$, we get $R[p_Y] = S \vee R[p_Y]$, and thus $S \subset R[p_Y]$. Accordingly $Z(R[p_X]) = R[p_Y]$. When $X = Y$, we have $Z(R[p_0]) = R[p_1]$, and any undiscrete groupoid $\nabla_Y$ is eccentric. □

Corollary 5.1. Suppose $\mathbb{D}$ is a regular stiffly Mal’cev category which is groupoid accessible. Then any undiscrete groupoid $\nabla_Y$ is faithful.
We shall need now the following:

**Definition 5.2.** Suppose \( D \) is a Mal’cev category. We say that an equivalence relation \( S \) is a complement of the equivalence relation \( R \) on \( X \), when we have \( R \land S = \Delta_X \) and \( R \lor S = \nabla_X \).

Given any pair \((X, Y)\) of objects, the equivalence relation \( R[p_Y] \) on \( X \times Y \) is a complement of \( R[p_X] \).

**Lemma 5.6.** Let \( D \) be a regular stiffly Mal’cev category. Suppose the equivalence relation \( R \) on \( X \) has a complement \( S \). Then we have \( S = Z(R) \). Accordingly any complement, when it exists, is unique. We shall denote it by \( R \).

**Proof.** From \( S \land R = \Delta_X \), we get \( [R, S] = 0 \). Let \( T \) be an equivalence relation on \( X \) such that \( [T, R] = 0 \), i.e. such that \( T \land R = \Delta_X \). By the weak congruence distributivity, from \( S \land R = \Delta_X \) we get \( (T \lor S) \land R = \Delta_X \). Then, by Lemma 5.5, from \( S \subset T \lor S \) and \( S \lor R = \nabla_X \), we get \( S = T \lor S \) and \( T \subset S \). Accordingly we have \( S = Z(R) \).

**Proposition 5.9.** Let \( D \) be a regular stiffly Mal’cev category with centralizers. Consider the following conditions:

1) any equivalence relation \( R \) has a complement;
2) for any equivalence relation \( R \), we have \( Z^2(R) = R \);
3) the only eccentral equivalence relations are the undiscrete ones \( \nabla_X \).

Then we have: 1) \( \Rightarrow \) 2) \( \Rightarrow \) 3).

When moreover \( D \) is exact, then the three conditions are equivalent.

**Proof.** Suppose condition 1) holds. Then \( Z(R) \) is the complement of \( R \) and \( Z^2(R) \); accordingly we have \( R = Z^2(R) \). Suppose condition 2) holds. We know that \( R \) is eccentral if and only if \( Z(R) = \Delta_X \). Then \( R = Z^2(R) = Z(\Delta_X) = \nabla_X \).

Suppose moreover that \( D \) is exact. Suppose condition 3) holds. Since \( D \) has centralizers, its has enough eccentral groupoids. So, given any equivalence relation \( R \) on \( X \), take its eccentral index \( \nabla_U \):

![Diagram](https://example.com/diagram.png)

We have \( Z(R) = R[q_0] \), and since \( q_0 \) is a regular epimorphism, we have \( R \lor R[q_0] = q_0^{-1}(q_0(R)) = q_0^{-1}(\nabla_U) = \nabla_X \). Accordingly \( Z(R) \) is a complement of \( R \).

**Corollary 5.2.** Let \( D \) be a regular groupoid accessible stiffly Mal’cev category. Then the three conditions are equivalent:

1) any equivalence relation \( R \) has a complement;
2) for any equivalence relation \( R \), we have \( Z^2(R) = R \);
3) the only faithful equivalence relations are the undiscrete ones \( \nabla_X \).

**Proof.** According to the previous proposition we have 1) \( \Rightarrow \) 2) \( \Rightarrow \) 3). And since \( D \) is groupoid accessible, we have an index to an undiscrete groupoid, and we can prove 3) \( \Rightarrow \) 1) on the model of the end of the previous proof.
Eventually we get:

**Proposition 5.10.** Any exact stiffly Mal’cev category $\mathcal{D}$ with complements and having a terminal object without proper subobject is groupoid accessible.

**Proof.** Since $\mathcal{D}$ is stiffly Mal’cev with complement, it has centralizers. Since moreover it is exact, it has enough eccentric groupoids which, according to the previous proposition, are the undiscrete equivalence relations. So we have to show that, when the terminal subobject has no proper subobject, any undiscrete equivalence relation is faithful. This is the object of the next proposition. □

**Proposition 5.11.** Suppose the regular category $\mathcal{C}$ has a terminal object $1$ without any proper subobject. Then any fibrant subobject $m_1 : R_1 \to \nabla_1 Y$ is an isomorphism and any discrete fibration $f_1 : S_1 \to \nabla_1 Y$ is a regular epic functor. When moreover $\mathcal{D}$ is a stiffly Mal’cev category, then any undiscrete groupoid $\nabla_Y$ is faithful.

**Proof.** Since we have a fibrant subobject, then we have $R = m_0^{-1}(\nabla_X)$, according to Lemma 1.1. Accordingly $R$ is an effective equivalence relation. Consider the following diagram, and complete it with the quotient of $R$:

\[
\begin{array}{ccc}
R & \xrightarrow{m_1} & X \times X \\
\downarrow{p_0} & \downarrow{p_1} & \downarrow{p_1} \\
U & \xrightarrow{m_0} & X \\
\downarrow{q} & & \downarrow{q} \\
Q & \xrightarrow{1} & 1
\end{array}
\]

The factorization $Q \to 1$ is necessarily a monomorphism. According to our assumption about the terminal object $1$, it is an isomorphism. The lower square is a pullback by the Barr–Kock theorem, and $m_0$ is itself an isomorphism. As for the second point, take the canonical decomposition of $f_1 : S_1 \to \nabla_1 Y$.

Suppose now $\mathcal{D}$ is a stiffly Mal’cev category. By Proposition 5.8 we know that any undiscrete groupoid $\nabla_Y$ is eccentric. We have to show it is faithful. For that, consider a diagram in $Di\mathcal{D}$ in which $g_1 \cdot m_1 = h_1 = g'_1 \cdot m_1$:

\[
\begin{array}{ccc}
S & \xrightarrow{m_1} & R \\
\downarrow{h_1} & & \downarrow{g_1} \\
\nabla_Y & \xrightarrow{g'_1} & \nabla_Y
\end{array}
\]

According to our first point, the functors $g_1$ and $g'_1$ are regular epimorphisms. On the other hand, according to Proposition 3.8, they have same kernel equivalence relations, since $\nabla_Y$ is eccentric. Accordingly there is an isomorphism $\phi_1 : \nabla_Y \to \nabla_Y$, such that $\phi_1 \cdot g_1 = g'_1$. Whence: $\phi_1 \cdot h_1 = \phi_1 \cdot g_1 \cdot m_1 = g'_1 \cdot m_1 = h_1$. Now $h_1$ is also a regular epimorphism, so that $\phi_1 = 1_{\nabla_Y}$, and $g_1 = g'_1$. □

We are now in position to assert what we had in mind:

**Corollary 5.3.** Suppose the elementary topos $\mathcal{E}$ is boolean. Then its dual $\mathcal{E}^{op}$ is an exact groupoid accessible stiffly Mal’cev category in which the only faithful groupoids are the undiscrete groupoids $\nabla_Y$. 
Proof. We already noticed that, when $\mathbf{E}$ is a boolean topos, the category $\mathbf{E}^{\text{op}}$ is an exact stiffly Mal’cev category. Its terminal object has no proper subobject, since, in any topos, the initial object is strict. The equivalence relations on an object $X$ in the dual $\mathbf{E}^{\text{op}}$ of any topos $\mathbf{E}$ are in bijection ($\mathbf{E}^{\text{op}}$ being exact) with the subobjects of $X$ in $\mathbf{E}$. Accordingly, when the topos $\mathbf{E}$ is boolean, the stiffly Mal’cev category $\mathbf{E}^{\text{op}}$ has complements. □

References