Journal of Algebra 335 (2011) 96–112

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Some complete intersection symplectic quotients in positive characteristic: Invariants of a vector and a covector

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ARTICLE INFO

Article history:
Received 7 July 2010
Available online 6 April 2011
Communicated by Martin Liebeck

Keywords:
Invariant theory
Positive characteristic
Triangular matrices
Complete intersection
Symplectic quotient

ABSTRACT

Given a linear action of a group G on a K-vector space V, we consider the invariant ring $K[V \oplus V^*]^G$, where V^* is the dual space. We are particularly interested in the case where $V = \mathbb{F}_q^n$ and G is the group U_n of all upper unipotent matrices or the group B_n of all upper triangular matrices in $\mathrm{GL}_n(\mathbb{F}_q)$.
In fact, we determine $\mathbb{F}_q[V \oplus V^*]^G$ for $G = U_n$ and $G = B_n$. The result is a complete intersection for all values of n and q. We present explicit lists of generating invariants and their relations. This makes an addition to the rather short list of "doubly parametrized" series of group actions whose invariant rings are known to have a uniform description.

Introduction

Many interesting subgroups of $\mathrm{GL}_n(\mathbb{F}_q)$ come in doubly parametrized series, where one parameter is linked to n and the other to q. Important examples are the finite classical groups, the groups B_n and U_n of upper triangular matrices and unipotent upper triangular matrices in $\mathrm{GL}_n(\mathbb{F}_q)$, and the cyclic p-groups acting indecomposably. In the context of invariant theory, not only the natural actions but also others, including decomposable ones, are interesting. For the following series of groups with their natural actions, the invariant rings have been determined: the general and special linear groups (this goes back to L. Dickson, see for instance Smith [18, Chapter 8.1] or Wilkerson [19]), the groups B_n and U_n (see Neusel and Smith [17, Section 4.5, Example 2] or Smith [18, Proposition 5.5.6]), the finite symplectic groups (this goes back to D. Carlisle and P. Kropholler, see Benson [3, Chapter 8.3]),

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[1] The author is partly supported by the ANR (Project No. JC07-192339).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.03.007

and the finite unitary groups (Chu and Jow [5]). For $\mathrm{GL}_n(\mathbb{F}_q)$, $\mathrm{SL}_n(\mathbb{F}_q)$, U_n, and B_n, the invariant rings are isomorphic to polynomials rings, and their determination is fairly easy. For the finite symplectic and unitary groups, the invariant rings are complete intersections, and the same is expected for the finite orthogonal groups (see [5]). To the best of our knowledge, no results have appeared so far about the invariant rings of a doubly-parametrized series of groups with a non-trivial decomposable action.

In this paper we study the invariant rings of the type $K[V \oplus V^*]^G$, where G is a finite group acting on a finite-dimensional K-vector space V and V^* is the dual space. In the language of classical invariant theory, the elements of $K[V \oplus V^*]^G$ are called *invariants of a vector and a covector*. In the case that K has characteristic zero and G is generated by reflections, $K[V \oplus V^*]^G$ has been studied intensively in the last fifteen years, in relation with the representation theory of Cherednik algebras and the geometry of Hilbert schemes and Calogero–Moser spaces: see the pioneering work of Haiman on the symmetric group case [11–13] and, for instance, Etingof and Ginzburg [7], Ginzburg and Kaledin [8], Gordon [10], and Bellamy [2]. The ring $K[V \oplus V^*]^G$ is also important for the computation of invariants in Weyl algebras (see Kemper and Quiring [15]). Here we consider the case that $K = \mathbb{F}_q$ is a finite field and G is one of the groups B_n or U_n, and calculate the invariant ring $\mathbb{F}_q[V \oplus V^*]^G$. The result is surprisingly simple. In fact, writing $\mathbb{F}_q[V \oplus V^*] = \mathbb{F}_q[x_1, \dots, x_n, y_1, \dots, y_n]$ (where $x_1, \dots, x_n$ is the standard basis of V and $y_1, \dots, y_n$ is the dual basis) and setting

$$f_i := \prod_{h \in U_n \cdot x_i} h, \qquad f_i^* := \prod_{h \in U_n \cdot y_{n+1-i}} h \quad (1 \leqslant i \leqslant n),$$

$$\widetilde{f}_i := f_i^{q-1}, \qquad \widetilde{f}_i^* := f_i^{*q-1} \quad (1 \leqslant i \leqslant n),$$

$$u_j := \sum_{k=1}^n x_k^{q^j} y_k, \quad \text{and} \quad u_{-j} := \sum_{k=1}^n x_k y_k^{q^j} \quad (j \geqslant 0),$$

we prove:

(a) *If $n \geqslant 2$, then $\mathbb{F}_q[V \oplus V^*]^{U_n} = \mathbb{F}_q[f_1, \dots, f_n, f_1^*, \dots, f_n^*, u_{2-n}, \dots, u_{n-2}]$ is generated by $4n-3$ invariants subject to $2n-3$ relations.*
(b) *If $n \geqslant 1$, then $\mathbb{F}_q[V \oplus V^*]^{B_n} = \mathbb{F}_q[\widetilde{f}_1, \dots, \widetilde{f}_n, \widetilde{f}_1^*, \dots, \widetilde{f}_n^*, u_{1-n}, \dots, u_{n-1}]$ is generated by $4n-1$ invariants subject to $2n-1$ relations.*

The relations are given explicitly in Theorem 2.4. In particular, both $\mathbb{F}_q$-algebras $\mathbb{F}_q[V \oplus V^*]^{U_n}$ and $\mathbb{F}_q[V \oplus V^*]^{B_n}$ are complete intersections.

The special case $n = 2$ and q a prime of (a) is included in Neusel [16]. Observe that the number of generators and the number of relations are independent of q.

Notice that by a result of Kac and Watanabe [14] and Gordeev [9], the invariant ring $K[V \oplus V^*]^G$ can only be a complete intersection if G is generated by pseudo-reflections. However, even when G is generated by pseudo-reflections, it seems to be rare that $K[V \oplus V^*]^G$ is a complete intersection. A counterexample, possibly the smallest, is given by the symmetric group S_3 acting irreducibly on $V = \mathbb{C}^2$. We checked that by using the computer algebra system MAGMA (see [4]). See also Alev and Foissy [1].

Another indication that the invariant rings $\mathbb{F}_q[V \oplus V^*]^{U_n}$ and $\mathbb{F}_q[V \oplus V^*]^{B_n}$ are "lucky" cases comes from comparing them to $\mathbb{F}_q[V \oplus V]^{U_n}$ and $\mathbb{F}_q[V \oplus V]^{B_n}$. Using MAGMA, we find that for $n = 3$ and $q = 2$ or 3, $\mathbb{F}_q[V \oplus V]^{U_3}$ requires a minimum of 12 or 16 generators, respectively, and fails to be Cohen–Macaulay for $q = 3$.

The paper is organized as follows: in the first section we start by determining the invariant field $K(V \oplus V^*)^G$ for all finite groups $G \leqslant \mathrm{GL}(V)$ for which $K[V]^G$ and $K[V^*]^G$ is known. Then we prove a lemma (see 1.4) which gives a sufficient condition for a K-algebra to admit a particular presentation by generators and relations. This lemma will be used for all results in this paper. The main part of the paper is the second section, where we produce relations between our claimed generators, and show that they satisfy the hypotheses of Lemma 1.4. This leads to the main result, Theorem 2.4. In the final section we study the invariant ring $\mathbb{F}_q[V \oplus V^*]^G$ for $G = \mathrm{SL}_n(\mathbb{F}_q)$ or $\mathrm{GL}_n(\mathbb{F}_q)$. We make a conjecture about $\mathbb{F}_q[V \oplus V^*]^{\mathrm{GL}_n(\mathbb{F}_q)}$ (see 3.1).

We should mention the role of experimental work in the genesis of this paper. The starting point was the explicit computation of $\mathbb{F}_q[V \oplus V^*]^{U_n}$ for $n = 3$ and $q = 2, 3$ (and its approximate computation for $q = 4, 5$) by using MAGMA. This prompted us to guess the generators of $\mathbb{F}_q[V \oplus V^*]^{U_n}$ for $n = 3$. By obtaining the relations appearing in Example 2.5(U_3) and using Lemma 1.4, we were able to prove the case $n = 3$ of Theorem 2.4(a). Turning to the case $n = 4$, we used MAGMA again to produce some relations between our conjectured generators for several q. From these, we guessed (and verified) the relations for general q appearing in Example 2.5(U_4). We observed that these relations again satisfy the hypotheses of Lemma 1.4. We then pushed this up to $n = 5$ and 6. Only then were we able to conjecture the general relations given in Theorem 2.4(a) and to observe that they can be interpreted as special cases of the determinant identity from Lemma 2.1. This led to the (computer-free) proof of part (a) of Theorem 2.4, and part (b) was then deduced quite easily. So it is justified to say that this paper owes its existence to MAGMA.

1. Preliminaries

Let K be a field, n a positive integer, and $V = K^n$. The general linear group $\mathrm{GL}_n(K)$ acts naturally on V. It also acts on the dual space V^* by $\sigma \cdot \lambda := \lambda \circ \sigma^{-1}$ for $\sigma \in \mathrm{GL}_n(K)$ and $\lambda \in V^*$. This induces an action on the polynomial ring $K[V \oplus V^*]$, which by convention we take to be the symmetric algebra of $V \oplus V^*$. (Since $V \oplus V^*$ is self-dual, the more standard convention of taking the symmetric algebra of the dual yields the same result.) We can write

$$K\bigl[V \oplus V^*\bigr] = K[x_1, \dots, x_n, y_1, \dots, y_n],$$

where $x_1, \dots, x_n$ is the standard basis of $V = K^n$ and $y_1, \dots, y_n$ is the dual basis.

The natural pairing

$$V \otimes V^* \to K, \quad v \otimes \lambda \mapsto \lambda(v)$$

is clearly invariant under the action of $\mathrm{GL}_n(K)$. Since $V \otimes V^*$ is embedded into $K[V \oplus V^*]$, this gives rise to an invariant u_0. Explicitly, we obtain

$$u_0 = \sum_{j=1}^{n} x_j y_j \in K\bigl[V \oplus V^*\bigr]^{\mathrm{GL}_n(K)}.$$

We start by looking at the invariant field $K(V \oplus V^*)^G$. Recall that for some important finite subgroups $G \subseteq \mathrm{GL}_n(K)$, generators of the invariant ring $K[V]^G$ are known. If K is finite, these subgroups include U_n, B_n, $\mathrm{SL}_n(K)$, and $\mathrm{GL}_n(K)$ (see Smith [18, Proposition 5.5.6 and Theorems 8.1.5 and 8.1.8]).

Proposition 1.1. *Let $G \subseteq \mathrm{GL}_n(K)$ be a finite subgroup. Then $K(V \oplus V^*)^G$ is generated, as a field extension of K, by $K[V]^G$, $K[V^*]^G$, and u_0.*

Proof. Let $f_1, \dots, f_l$ (respectively $g_1, \dots, g_m$) be generators of the K-algebra $K[V]^G$ (respectively $K[V^*]^G$). The group $G \times G$ acts in the obvious way on $V \oplus V^*$, and it follows that

$$K\bigl[V \oplus V^*\bigr]^{G \times G} = K[f_1, \dots, f_l, g_1, \dots, g_m].$$

So $K(V \oplus V^*)$ is Galois as a field extension of $K(f_1, \dots, f_l, g_1, \dots, g_m)$ with group $G \times G$. It follows that it is also Galois as a field extension of $L := K(f_1, \dots, f_l, g_1, \dots, g_m, u_0)$. Clearly $L \subseteq K(V \oplus V^*)^G$, so if we can show that the Galois group $\mathrm{Gal}(K(V \oplus V^*)/L)$ is contained in G embedded diagonally, then Galois theory yields $K(V \oplus V^*)^G = L$.

So take an arbitrary element from this Galois group $\mathrm{Gal}(K(V \oplus V^*)/L)$, which we can write as $(\sigma, \tau) \in G \times G$. We need to show that $\sigma = \tau$. We have

$$\bigl(\sigma\tau^{-1}, \mathrm{id}\bigr)(u_0) = \bigl(\sigma\tau^{-1}, \mathrm{id}\bigr)\bigl((\tau, \tau)(u_0)\bigr) = (\sigma, \tau)(u_0) = u_0.$$

Since the y_i are algebraically independent over $K[x_1, \dots, x_n]$, this shows that $(\sigma\tau^{-1})(x_j) = x_j$ for all j, so $\sigma\tau^{-1} = \mathrm{id}$. This concludes the proof. □

We have an involution

$$* : K\bigl[V \oplus V^*\bigr] \to K\bigl[V \oplus V^*\bigr], \quad x_i \mapsto y_{n+1-i}, \quad y_i \mapsto x_{n+1-i}.$$

For $\sigma \in \mathrm{GL}_n(K)$ we set

$$\sigma^* := \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix} \cdot \bigl(\sigma^{-1}\bigr)^T \cdot \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \iddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.$$

It is easy to verify that for $\sigma \in \mathrm{GL}_n(K)$ and $f \in K[V \oplus V^*]$, the rule

$$(\sigma \cdot f)^* = \sigma^* \cdot f^*$$

holds. So if $G \subseteq \mathrm{GL}_n(K)$ is stable under the automorphism $*$ of $\mathrm{GL}_n(K)$, then $*$ induces an automorphism of the invariant ring $K[V \oplus V^*]^G$, and this automorphism restricts to an isomorphism between $K[V]^G$ and $K[V^*]^G$.

Example 1.2. The groups U_n, B_n, $\mathrm{SL}_n(K)$ and $\mathrm{GL}_n(K)$ are $*$-stable.

We obtain the following corollary from Proposition 1.1.

Corollary 1.3. *Let $G \subseteq \mathrm{GL}_n(K)$ be a $*$-stable finite subgroup. Assume that $K[V]^G$ is generated by the invariants $f_1, \dots, f_m$ (as a K-algebra). Then $K(V \oplus V^*)^G$ is generated (as a field extension of K) by f_i, f_i^* $(i = 1, \dots, m)$, and u_0.*

For the proof of our main results we will use the following lemma. It gives a sufficient condition for a K-algebra to admit a particular presentation by generators and relations.

Lemma 1.4. *Let A be a graded algebra over K. Suppose that A is an integral domain. Let $f_1, \dots, f_n, g_1, \dots, g_m, h_1, \dots, h_l \in A$ be homogeneous elements of positive degree such that*

(a) *$f_1, \dots, f_n, g_1, \dots, g_m$ form a homogeneous system of parameters of A (i.e., they are algebraically independent and A is an integral extension of the subalgebra formed by them), and*
(b) *for the field of fractions we have*

$$\mathrm{Quot}(A) = K(f_1, \dots, f_n, g_1, \dots, g_m, h_1, \dots, h_l).$$

Moreover, let $R_1, \dots, R_l$ be homogeneous elements of the kernel of the homomorphism

$$\varphi : P := K[X_1, \dots, X_n, Y_1, \dots, Y_m, Z_1, \dots, Z_l] \to A, \quad X_i \mapsto f_i, \quad Y_i \mapsto g_i, \quad Z_i \mapsto h_i,$$

were P is a polynomial ring graded in such a way that φ is degree-preserving. Suppose that

(c) *$X_1, \dots, X_n, Y_1, \dots, Y_m, R_1, \dots, R_l$ form a homogeneous system of parameters of P, and*
(d) *for*

$$x := \overline{X}_1 \cdots \overline{X}_n \in B := P/(R_1, \dots, R_l)$$

(*where $\overline{X}_i$ denotes the class in B of X_i*), *the localization $B_x := B[x^{-1}]$ is generated by x^{-1}, $\overline{X}_1, \dots, \overline{X}_n$, and m further elements. Moreover, for $y := \overline{Y}_1 \cdots \overline{Y}_m$, B_y is generated by y^{-1}, $\overline{Y}_1, \dots, \overline{Y}_m$, and n further elements.* (*Loosely speaking, this means that after localizing by x or y, the relations allow us to eliminate l of the generators.*)

Then

$$A = K[f_1, \dots, f_n, g_1, \dots, g_m, h_1, \dots, h_l].$$

Moreover, A is a complete intersection, and the kernel of φ is generated by $R_1, \dots, R_l$.

Proof. The first goal is to show that B is an integral domain. We conclude from (c) that

$$\dim\bigl(P/(R_1, \dots, R_l, X_1, \dots, X_n, Y_1, \dots, Y_m)\bigr) = 0. \tag{1.1}$$

Therefore B is a complete intersection of dimension $n+m$. In particular, B is Cohen–Macaulay (see Eisenbud [6, Proposition 18.13]). It follows from (1.1) that for every $i \in \{1, \dots, n\}$, $\overline{X}_i$ lies in no minimal prime ideal of B. By the unmixedness theorem (see [6, Corollary 18.14]), all associated prime ideals of (0) are minimal, so it follows that $\overline{X}_i$ is a non-zero-divisor. Since this holds for all i, also x is a non-zero-divisor. Therefore B embeds into B_x. In particular, B_x has transcendence degree at least $n+m$. So it follows from (d) that B_x is a localized polynomial ring and in particular an integral domain. This implies that B is also an integral domain. Similarly, B_y is a localized polynomial ring. This will be used in a moment.

Now we show that B is normal. Let $\mathfrak{p} \in \operatorname{Spec}(B)$ be a prime ideal of height one. It follows from (1.1) that for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, m\}$ the ideal $(\overline{X}_i, \overline{Y}_j) \subseteq B$ has height 2. Therefore $\mathfrak{p}$ cannot contain both x and y, so $B_\mathfrak{p}$ is a localization of B_x or of B_y and therefore normal. This shows that B satisfies Serre's condition R1 (see [6, Theorem 11.5]). Moreover, applying the unmixedness theorem again, we see that B also satisfies the condition S2 (see [6, Theorem 11.5]). By Serre's criterion [6, Theorem 11.5], B is normal.

Consider the epimorphism

$$\psi : B \to A' := K[f_1, \dots, f_n, g_1, \dots, g_m, h_1, \dots, h_l] \subseteq A$$

induced from φ. It follows from (a) that A' has dimension $n+m$, the same as B. Since B is an integral domain, it follows that $\ker(\psi) = \{0\}$, so ψ is an isomorphism. In particular, A' is normal. Applying (a) again, we see that A is integral over A'. But by (b), $A \subseteq \operatorname{Quot}(A')$, so the normality of A' implies $A = A'$. We have already seen that $A' \cong B$ is a complete intersection. The injectivity of ψ means that the kernel of φ is generated by the R_i. So the proof is complete. □

Readers may find it helpful to take a look at Example 2.5 already now. There, Lemma 1.4 is applied several times, so the example serves to illustrate the less intuitive hypotheses (c) and (d) of the lemma.

2. The invariant ring of U_n and B_n

From now on, we assume that $K = \mathbb{F}_q$ is a finite field with q elements.

Some invariants. The homomorphisms

$$F : \mathbb{F}_q\big[V \oplus V^*\big] \to \mathbb{F}_q\big[V \oplus V^*\big], \quad x_i \mapsto x_i^q, \quad y_i \mapsto y_i,$$

and

$$F^* : \mathbb{F}_q\big[V \oplus V^*\big] \to \mathbb{F}_q\big[V \oplus V^*\big], \quad x_i \mapsto x_i, \quad y_i \mapsto y_i^q, \tag{2.1}$$

commute with the action of $\mathrm{GL}_n(\mathbb{F}_q)$. Therefore we get further invariants in $\mathbb{F}_q[V \oplus V^*]^{\mathrm{GL}_n(\mathbb{F}_q)}$ by setting, for $i \geqslant 0$,

$$u_i := F^i(u_0) = \sum_{j=1}^n x_j^{q^i} y_j \quad \text{and} \quad u_{-i} := \big(F^*\big)^i(u_0) = \sum_{j=1}^n x_j y_j^{q^i}.$$

Notice that $u_{-i} = u_i^*$ for all $i \in \mathbb{Z}$.

Now we turn our attention to the case where $G \in \{U_n, B_n\}$. Apart from the invariants u_i defined above, we get obvious invariants by taking the orbit-products (for $1 \leqslant i \leqslant n$)

$$f_i := \prod_{h \in U_n \cdot x_i} h = \prod_{\alpha_1,\dots,\alpha_{i-1} \in \mathbb{F}_q} \left(x_i + \sum_{j=1}^{i-1} \alpha_j x_j\right).$$

Then

$$f_i^* = \prod_{h \in U_n \cdot y_{n+1-i}} h = \prod_{\alpha_1,\dots,\alpha_{i-1} \in \mathbb{F}_q} \left(y_{n+1-i} + \sum_{j=1}^{i-1} \alpha_j y_{n+1-j}\right).$$

The f_i and f_i^* are homogeneous of degrees $\deg(f_i) = \deg(f_i^*) = q^{i-1}$. Similarly, we set (for $1 \leqslant i \leqslant n$)

$$\widetilde{f}_i := f_i^{q-1} = -\prod_{h \in B_n \cdot x_i} h,$$

so that

$$\widetilde{f}_i^* = \big(f_i^*\big)^{q-1} = -\prod_{h \in B_n \cdot y_{n+1-i}} h.$$

The minus sign comes from the fact that $\prod_{\xi \in \mathbb{F}_q^\times} \xi = -1$. It is well known (see Neusel and Smith [17, Section 4.5, Example 2] or Smith [18, Proposition 5.5.6]) that

$$\mathbb{F}_q[V]^{U_n} = \mathbb{F}_q[f_1, \dots, f_n] \quad \text{and} \quad \mathbb{F}_q[V]^{B_n} = \mathbb{F}_q[\widetilde{f}_1, \dots, \widetilde{f}_n]. \tag{2.2}$$

So if we want to use Lemma 1.4 for showing that the f_i, f_i^* (respectively, $\widetilde{f}_i$ and $\widetilde{f}_i^*$) together with some u_i generate the invariant ring, the hypotheses (a) and (b) are already satisfied. So everything hinges on our ability to find some suitable relations between the generators.

Some relations. The following identity provides the source of our relations.

Lemma 2.1. *Let R be a commutative ring with identity element, n a positive integer, and $a_{i,j}, b_{i,j} \in R$ $(i, j = 1, \ldots, n)$. Then for $1 \leqslant k \leqslant n$ we have*

$$\sum_{i=1}^{k} \sum_{j=1}^{n+1-k} \sum_{l=1}^{n} (-1)^{i+j+n+1} a_{i,l} b_{j,n+1-l} \cdot \det \begin{pmatrix} a_{1,1} & \cdots & a_{1,k-1} \\ \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} \\ a_{i+1,1} & \cdots & a_{i+1,k-1} \\ \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,k-1} \end{pmatrix}$$
$$\cdot \det \begin{pmatrix} b_{1,1} & \cdots & b_{1,n-k} \\ \vdots & & \vdots \\ b_{j-1,1} & \cdots & b_{j-1,n-k} \\ b_{j+1,1} & \cdots & b_{j+1,n-k} \\ \vdots & & \vdots \\ b_{n+1-k,1} & \cdots & b_{n+1-k,n-k} \end{pmatrix}$$
$$= \det \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,k} \end{pmatrix} \cdot \det \begin{pmatrix} b_{1,1} & \cdots & b_{1,n+1-k} \\ \vdots & & \vdots \\ b_{n+1-k,1} & \cdots & b_{n+1-k,n+1-k} \end{pmatrix}. \tag{2.3}$$

In the case $k = 1$, the first determinant in the left-hand side of (2.3) *is to be understood as* 1*, and in the case $k = n$, the second determinant is to be understood as* 1*.*

Proof. First, observe that, for $1 \leqslant l \leqslant n$,

$$\sum_{i=1}^{k} (-1)^i a_{i,l} \cdot \det \begin{pmatrix} a_{1,1} & \cdots & a_{1,k-1} \\ \vdots & & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,k-1} \\ a_{i+1,1} & \cdots & a_{i+1,k-1} \\ \vdots & & \vdots \\ a_{k,1} & \cdots & a_{k,k-1} \end{pmatrix} = (-1)^k \det \begin{pmatrix} a_{1,1} & \cdots & a_{1,k-1} & a_{1,l} \\ \vdots & & \vdots & \vdots \\ a_{k,1} & \cdots & a_{k,k-1} & a_{k,l} \end{pmatrix} \tag{2.4}$$

and

$$\sum_{j=1}^{n+1-k} (-1)^j b_{j,n+1-l} \cdot \det \begin{pmatrix} b_{1,1} & \cdots & b_{1,n-k} \\ \vdots & & \vdots \\ b_{j-1,1} & \cdots & b_{j-1,n-k} \\ b_{j+1,1} & \cdots & b_{j+1,n-k} \\ \vdots & & \vdots \\ b_{n+1-k,1} & \cdots & b_{n+1-k,n-k} \end{pmatrix}$$
$$= (-1)^{n+1-k} \det \begin{pmatrix} b_{1,1} & \cdots & b_{1,n-k} & b_{1,n+1-l} \\ \vdots & & \vdots & \vdots \\ b_{n+1-k,1} & \cdots & b_{n+1-k,n-k} & b_{n+1-k,n+1-l} \end{pmatrix}. \tag{2.5}$$

Moreover, the right-hand side of (2.4) (respectively (2.5)) is zero if $l \leqslant k-1$ (respectively $l \geqslant k+1$). So by multiplying (2.4) and (2.5) and summing over $l = 1, \dots, n$, we obtain (2.3). Notice that the special cases $k = 1$ and $k = n$ pose no problems in the proof. $\square$

We apply Lemma 2.1 to $R = \mathbb{F}_q[V \oplus V^*]$, $a_{i,j} = x_j^{q^{i-1}}$, and $b_{i,j} = y_{n+1-j}^{q^{i-1}} = (x_j^{q^{i-1}})^*$. We wish to express the relations obtained in this way in terms of the invariants u_i, f_i, $\widetilde{f}_i$, f_i^*, and $\widetilde{f}_i^*$. First, note that the sums $\sum_{l=1}^n a_{i,l} b_{j,n+1-l}$ in (2.3) specialize to

$$\sum_{l=1}^{n} x_l^{q^{i-1}} y_l^{q^{j-1}} = u_{i-j}^{q^{\min\{i-1,j-1\}}}.$$

Therefore, setting

$$d_{k,i} := \det \begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ x_1^q & x_2^q & \cdots & x_k^q \\ \vdots & \vdots & & \vdots \\ x_1^{q^{i-1}} & x_2^{q^{i-1}} & \cdots & x_k^{q^{i-1}} \\ x_1^{q^{i+1}} & x_2^{q^{i+1}} & \cdots & x_k^{q^{i+1}} \\ \vdots & \vdots & & \vdots \\ x_1^{q^k} & x_2^{q^k} & \cdots & x_k^{q^k} \end{pmatrix}$$

and shifting the summations indices i and j in (2.3) down by 1, we obtain

$$\sum_{i=0}^{k-1} \sum_{j=0}^{n-k} (-1)^{i+j+n+1} u_{i-j}^{q^{\min\{i,j\}}} \cdot d_{k-1,i} \cdot d_{n-k,j}^* = d_{k,k} \cdot d_{n+1-k,n+1-k}^* \tag{2.6}$$

for $1 \leqslant k \leqslant n$. The following lemma expresses the determinants $d_{k,i}$ in terms of our invariants.

Lemma 2.2. *For $1 \leqslant k \leqslant n$ and $0 \leqslant i \leqslant k$ we have*

$$d_{k,i} = \prod_{j=1}^{k} f_j \cdot \left(\sum_{1 \leqslant j_1 < j_2 < \cdots < j_{k-i} \leqslant k} \prod_{l=1}^{k-i} \widetilde{f}_{j_l}^{q^{i+l-j_l}} \right).$$

For $i = k$, the sum on the right-hand side should be interpreted as 1.

Proof. Most of the ideas in the proof are taken from Wilkerson [19]. We first treat the case $i = k$ using induction on k. We have

$$d_{1,1} = x_1 = f_1.$$

Now we go from k to $k+1$. Substituting $x_{k+1} = \sum_{j=1}^k \alpha_j x_j$ with $\alpha_j \in \mathbb{F}_q$ into $d_{k+1,k+1}$ yields 0. Since the x_{k+1}-degree of $d_{k+1,k+1}$ is q^k, we conclude that as polynomials in x_{k+1}, both $d_{k+1,k+1}$ and f_{k+1} have the same roots. So they are equal up to a factor in $\mathbb{F}_q(x_1, \dots, x_k)$. By comparing leading coefficients, we see that

$$d_{k+1,k+1} = d_{k,k} \cdot f_{k+1}. \tag{2.7}$$

(This equation even holds for $k = n$ if we define $f_{n+1} := \prod_{\alpha_1,\dots,\alpha_n\in\mathbb{F}_q}(x_{n+1} + \sum_{j=1}^{n}\alpha_j x_j)$ with an additional indeterminate x_{n+1}.) From (2.7), we obtain the desired result for $d_{k+1,k+1}$ by induction.

Expanding the determinant $d_{k+1,k+1}$ along the last column gives

$$d_{k+1,k+1} = \sum_{i=0}^{k}(-1)^{k+i} d_{k,i} x_{k+1}^{q^i}.$$

So by (2.7) we can write

$$f_{k+1} = \sum_{i=0}^{k}(-1)^{k+i} c_{k,i} x_{k+1}^{q^i}$$

with $c_{k,i} := d_{k,i}/d_{k,k} \in \mathbb{F}_q[x_1,\dots,x_k]$. So we need to show that

$$c_{k,i} = \sum_{1\leqslant j_1<j_2<\cdots<j_{k-i}\leqslant k} \prod_{l=1}^{k-i} \widetilde{f}_{j_l}^{q^{i+l-j_l}}. \tag{2.8}$$

Again we use induction on k, this time starting with $k = 0$. We have $f_1 = x_1$, so $c_{0,0} = 1$ as claimed. For $0 < k \leqslant n$ we have

$$\begin{aligned}
f_{k+1} &= \prod_{\alpha_1,\dots,\alpha_k\in\mathbb{F}_q}\left(x_{k+1} + \sum_{j=1}^{k}\alpha_j x_j\right) = \prod_{\alpha_k\in\mathbb{F}_q} f_k(x_1,\dots,x_{k-1},x_{k+1}+\alpha_k x_k) \\
&= \prod_{\alpha_k\in\mathbb{F}_q}\bigl(f_k(x_1,\dots,x_{k-1},x_{k+1}) + \alpha_k f_k(x_1,\dots,x_{k-1},x_k)\bigr) \\
&= f_k(x_1,\dots,x_{k-1},x_{k+1})^q - f_k(x_1,\dots,x_{k-1},x_{k+1})\cdot\widetilde{f}_k \\
&= \sum_{i=0}^{k-1}(-1)^{k+i+1}\bigl(c_{k-1,i}^q\cdot x_{k+1}^{q^{i+1}} - \widetilde{f}_k c_{k-1,i}\cdot x_{k+1}^{q^i}\bigr).
\end{aligned}$$

This yields the recursive formula

$$c_{k,i} = c_{k-1,i-1}^q + \widetilde{f}_k c_{k-1,i},$$

where we set $c_{k-1,-1} = c_{k-1,k} := 0$. For $i = k$ we have $c_{k,i} = 1$, satisfying (2.8) by convention. For $0 < i < k$ we use induction and obtain

$$\begin{aligned}
c_{k,i} &= \sum_{1\leqslant j_1<j_2<\cdots<j_{k-i}\leqslant k-1}\prod_{l=1}^{k-i}\bigl(\widetilde{f}_{j_l}^{q^{i+l-j_l-1}}\bigr)^q + \sum_{1\leqslant j_1<j_2<\cdots<j_{k-i-1}\leqslant k-1}\prod_{l=1}^{k-i-1}\widetilde{f}_{j_l}^{q^{i+l-j_l}}\widetilde{f}_k \\
&= \sum_{1\leqslant j_1<j_2<\cdots<j_{k-i}\leqslant k}\prod_{l=1}^{k-i}\widetilde{f}_{j_l}^{q^{i+l-j_l}}
\end{aligned}$$

as desired. (No problem arises in the special case $i = k - 1$.) For $i = 0$ we obtain

$$c_{k,i} = \widetilde{f}_k c_{k-1,0} = \widetilde{f}_k \cdot \prod_{l=1}^{k-1} \widetilde{f}_l = \prod_{l=1}^{k} \widetilde{f}_l.$$

This completes the proof. $\square$

It follows from Lemma 2.2 that both sides of (2.6) are divisible by $\prod_{j=1}^{k-1} f_j \cdot \prod_{j=1}^{n-k} f_j^*$. Therefore, setting

$$c_{s,t} := \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{s-t} \leqslant s} \prod_{l=1}^{s-t} \widetilde{f}_{j_l}^{q^{t+l-j_l}} = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_{s-t} \leqslant s} \prod_{l=1}^{s-t} f_{j_l}^{q^{t+l-j_l}(q-1)} \tag{2.9}$$

for $0 \leqslant t < s \leqslant n$ and $c_{s,s} := 1$ for $0 \leqslant s \leqslant n$, we obtain the relation

$$\sum_{i=0}^{k-1} \sum_{j=0}^{n-k} (-1)^{i+j+n+1} c_{k-1,i} \cdot c^*_{n-k,j} \cdot u_{i-j}^{q^{\min\{i,j\}}} - f_k \cdot f^*_{n+1-k} = 0 \tag{R_k}$$

for $1 \leqslant k \leqslant n$. We deduce some further relations from (R_k) by applying the homomorphisms F and F^* (see (2.1)). This yields

$$\sum_{i=0}^{k-1} \sum_{j=0}^{n-k} (-1)^{i+j+n+1} c^q_{k-1,i} \cdot c^*_{n-k,j} \cdot u_{i-j+1}^{q^{\min\{i+1,j\}}} - f^q_k \cdot f^*_{n+1-k} = 0 \tag{R_k^+}$$

and

$$\sum_{i=0}^{k-1} \sum_{j=0}^{n-k} (-1)^{i+j+n+1} c_{k-1,i} \cdot c^{*q}_{n-k,j} \cdot u_{i-j-1}^{q^{\min\{i,j+1\}}} - f_k \cdot f^{*q}_{n+1-k} = 0. \tag{R_k^-}$$

The relations produced so far involve the U_n-invariants f_i, f_i^*, and u_i. In order to obtain relations between the B_n-invariants $\widetilde{f}_i$, $\widetilde{f}_i^*$, and u_i, we raise $f_k \cdot f^*_{n+1-k}$ and the remaining sum in (R_k) to the $(q-1)$st power. This yields

$$\left(\sum_{i=0}^{k-1} \sum_{j=0}^{n-k} (-1)^{i+j} c_{k-1,i} \cdot c^*_{n-k,j} \cdot u_{i-j}^{q^{\min\{i,j\}}} \right)^{q-1} - \widetilde{f}_k \cdot \widetilde{f}^*_{n+1-k} = 0. \tag{$\widetilde{R}_k$}$$

Furthermore, by subtracting the $\widetilde{f}_k$-fold of (R_k) from (R_k^+), we obtain

$$\sum_{i=0}^{k-1} \sum_{j=0}^{n-k} (-1)^{i+j} \left(c^q_{k-1,i} \cdot c^*_{n-k,j} \cdot u_{i-j+1}^{q^{\min\{i+1,j\}}} - \widetilde{f}_k \cdot c_{k-1,i} \cdot c^*_{n-k,j} \cdot u_{i-j}^{q^{\min\{i,j\}}} \right) = 0. \tag{$\widetilde{R}_k^+$}$$

Remark 2.3.

(a) It may be of interest that $c_{s,t}$ is the t-th Dickson invariant in $x_1, \ldots, x_s$ (see Smith [18, Section 8.1] or Wilkerson [19]). This follows from the proof of Lemma 2.2.

(b) It is easy to see that the relations (R_k), (R_k^+), (R_k^-), $(\widetilde{R}_k)$, and $(\widetilde{R}_k^+)$ are homogeneous. (Their degrees are listed on p. 110.)

Main result. We are now ready to prove the main result of this paper.

Theorem 2.4. *With the above notation, we have*:

(a) *If* $n \geqslant 2$, *then*

$$\mathbb{F}_q\big[V \oplus V^*\big]^{U_n} = \mathbb{F}_q\big[f_1, \dots, f_n, f_1^*, \dots, f_n^*, u_{2-n}, \dots, u_{n-2}\big]$$

is generated by $4n-3$ *invariants. If* $n \geqslant 3$, *the ideal of relations has the following* $2n-3$ *generators*:

$$R_1^+, R_2, R_3^-, R_3, R_4^-, R_4, R_5^-, \dots, R_{n-2}, R_{n-1}^-, R_{n-1}, R_n^-. \tag{2.10}$$

If $n = 2$, *then the ideal of relations is generated by*

$$u_0^q - \big(f_1 f_1^*\big)^{q-1} u_0 - f_1^q f_2^* - f_1^{*q} f_2 = 0. \tag{2.11}$$

(b) *If* $n \geqslant 1$, *then*

$$\mathbb{F}_q\big[V \oplus V^*\big]^{B_n} = \mathbb{F}_q\big[\widetilde{f}_1, \dots, \widetilde{f}_n, \widetilde{f}_1^*, \dots, \widetilde{f}_n^*, u_{1-n}, \dots, u_{n-1}\big]$$

is generated by $4n-1$ *elements, and the ideal of relations has the following* $2n-1$ *generators*:

$$\widetilde{R}_1, \widetilde{R}_1^+, \widetilde{R}_2, \widetilde{R}_2^+, \dots, \widetilde{R}_{n-1}, \widetilde{R}_{n-1}^+, \widetilde{R}_n. \tag{2.12}$$

In particular, both $\mathbb{F}_q$*-algebras* $\mathbb{F}_q[V \oplus V^*]^{U_n}$ *and* $\mathbb{F}_q[V \oplus V^*]^{B_n}$ *are complete intersections. The generating invariants given in* (a) *and* (b) *are minimal, except in the case* $q = 2$ *of* (b) (*in which* $B_n = U_n$).

Before proving Theorem 2.4, we shall provide examples in the case where $n \in \{1, 2, 3, 4\}$.

Example 2.5.

(U_1) If $n = 1$, then $U_n = U_1 = \{1\}$ and $\mathbb{F}_q[V \oplus V^*]^{U_1} = \mathbb{F}_q[V \oplus V^*] = \mathbb{F}_q[x_1, y_1]$. This case is not covered by the uniform description of Theorem 2.4(a).

(U_2) If $n = 2$, we have

$$f_1 = x_1, \qquad f_1^* = y_2, \qquad f_2 = x_2^q - x_2 x_1^{q-1},$$
$$f_2^* = y_1^q - y_1 y_2^{q-1}, \quad \text{and} \quad u_0 = x_1 y_1 + x_2 y_2.$$

The relation (2.11) can be verified by direct computation:

$$\begin{aligned}&(x_1 y_1 + x_2 y_2)^q - (x_1 y_2)^{q-1}(x_1 y_1 + x_2 y_2) - x_1^q\big(y_1^q - y_1 y_2^{q-1}\big) - y_2^q\big(x_2^q - x_2 x_1^{q-1}\big)\\ &\quad = x_1^q y_1^q + x_2^q y_2^q - x_1^q y_1 y_2^{q-1} - x_1^{q-1} x_2 y_2^q - x_1^q y_1^q + x_1^q y_1 y_2^{q-1} - x_2^q y_2^q + x_1^{q-1} x_2 y_2^q = 0.\end{aligned}$$

We have already seen that f_1, f_1^*, f_2, f_2^*, and u_0 satisfy the hypotheses (a) and (b) from Lemma 1.4 (see after (2.2)). The relation (2.11) also satisfies (c) from Lemma 1.4. Indeed, if we treat u_0 and the f_i and f_i^* as indeterminates for a moment, it is clear that the relation together with f_1, f_2, f_1^*, and f_2^* forms a homogeneous system of parameters. Moreover, if we localize by f_1, the relation can be used to eliminate f_2^* as a generator; and localizing by f_1^* eliminates the generator f_2. So (d) is also satisfied, and applying Lemma 1.4 proves Theorem 2.4(a) for $n=2$. Why are the relations for $n=2$ not given by (2.10)? Notice that the relations R_1, R_2^- read

$$-f_1^{*q-1}u_0 + u_{-1} - f_1 f_2^* = 0 \quad \text{and} \quad -f_1^{q-1}u_{-1} + u_0^q - f_2 f_1^{*q} = 0,$$

so they involve u_{-1}, which is not included in the list of generators. But (2.11) can be obtained by adding the f_1^{q-1}-fold of R_1 to R_2^-.

(U_3) For $n=3$, the relations are

$$u_{-1}^q - \left(f_1^{*q(q-1)} + f_2^{*q-1}\right)u_0^q + \left(f_1^* f_2^*\right)^{q-1}u_1 - f_1^q f_3^* = 0, \tag{R_1^+}$$

$$u_0^q - f_1^{*q-1}u_1 - f_1^{q-1}u_{-1} + \left(f_1 f_1^*\right)^{q-1}u_0 - f_2 f_2^* = 0, \tag{R_2}$$

$$u_1^q - \left(f_1^{q(q-1)} + f_2^{q-1}\right)u_0^q + (f_1 f_2)^{q-1}u_{-1} - f_3 f_1^{*q} = 0. \tag{R_3^-}$$

It is clear that the relations satisfy (c) from Lemma 1.4. Moreover, if we localize the algebra B defined by the relations by $f_1^* f_2^*$, we obtain an algebra that is generated by $f_1^*, f_2^*, f_3^*, f_1, u_{-1}, u_0$, and $(f_1^* f_2^*)^{-1}$. (By abuse of notation, we write f_i^* for the element corresponding to f_i^* in B and so on.) In fact, we can use (R_1^+) to eliminate the generator u_1 of $B_{f_1^* f_2^*}$, then (R_2) to eliminate f_2 and, finally, (R_3^-) to eliminate f_3. We can also localize by $f_1 f_2$. Then we use (R_3^-) to eliminate u_{-1}, then (R_2) to eliminate f_2^*, and, finally, (R_1^+) to eliminate f_3^*. So we are left with the generators $f_1, f_2, f_3, f_1^*, u_0, u_1$, and $(f_1 f_2)^{-1}$.

There is a total of nine relations of the type $(R_k^{(\pm)})$, but as it happens, just the above three serve for the proof of Theorem 2.4(a) in the case $n=3$. Besides, some of the nine relations involve invariants other than $f_1, f_2, f_3, f_1^*, f_2^*, f_3^*, u_{-1}, u_0, u_1$. For example, (R_1) reads

$$u_{-2} - \left(f_1^{*q(q-1)} + f_2^{*q-1}\right)u_{-1} + \left(f_1^* f_2^*\right)^{q-1}u_0 - f_1 f_3^* = 0, \tag{R_1}$$

so it serves to express u_{-2} in terms of the above nine invariants.

(U_4) For $n=4$, the relations from Theorem 2.4(a) read

$$u_{-2}^q - \left(f_1^{*q^2(q-1)} + f_2^{*q(q-1)} + f_3^{*q-1}\right)u_{-1}^q + \left(\left(f_1^* f_2^*\right)^{q(q-1)} + \left(f_1^{*q} f_3^*\right)^{q-1} + \left(f_2^* f_3^*\right)^{q-1}\right)u_0^q$$
$$- \left(f_1^* f_2^* f_3^*\right)^{q-1}u_1 - f_1^q f_4^* = 0, \tag{R_1^+}$$

$$u_{-1}^q - f_1^{q-1}u_{-2} - \left(f_1^{*q(q-1)} + f_2^{*q-1}\right)u_0^q + f_1^{q-1}\left(f_1^{*q(q-1)} + f_2^{*q-1}\right)u_{-1} + \left(f_1^* f_2^*\right)^{q-1}u_1$$
$$- \left(f_1 f_1^* f_2^*\right)^{q-1}u_0 - f_2 f_3^* = 0, \tag{R_2}$$

$$u_0^{q^2} - f_1^{*q(q-1)}u_1^q - \left(f_1^{q(q-1)} + f_2^{q-1}\right)u_{-1}^q + \left(f_1^{q(q-1)} + f_2^{q-1}\right)f_1^{*q(q-1)}u_0^q + (f_1 f_2)^{q-1}u_{-2}$$
$$- \left(f_1 f_2 f_1^{*q}\right)^{q-1}u_{-1} - f_3 f_2^{*q} = 0, \tag{R_3^-}$$

$$u_1^q - f_1^{*q-1}u_2 - \left(f_1^{q(q-1)} + f_2^{q-1}\right)u_0^q + \left(f_1^{q(q-1)} + f_2^{q-1}\right)f_1^{*q-1}u_1 + (f_1 f_2)^{q-1}u_{-1}$$
$$- \left(f_1 f_2 f_1^*\right)^{q-1}u_0 - f_3 f_2^* = 0, \tag{R_3}$$

$$u_2^q - \left(f_1^{q^2(q-1)} + f_2^{q(q-1)} + f_3^{q-1}\right)u_1^q + \left((f_1 f_2)^{q(q-1)} + \left(f_1^q f_3\right)^{q-1} + (f_2 f_3)^{q-1}\right)u_0^q$$
$$- (f_1 f_2 f_3)^{q-1}u_{-1} - f_4 f_1^{*q} = 0. \tag{R_4^-}$$

With these relations, we can make an argument analogous to the above for U_3, showing that Lemma 1.4 is applicable. We will do this in general in the forthcoming proof of Theorem 2.4(a).

Notice that applying the involution $*$ transforms (R_1^+) into (R_4^-) and (R_2) into (R_3); but (R_3^-) is not invariant under $*$. This "violation of symmetry" can be fixed by adding the (f_2^{q-1})-fold of (R_2) to (R_3^-). The result is

$$u_0^{q^2} - f_1^{*q(q-1)}u_1^q - f_1^{q(q-1)}u_{-1}^q + \bigl((f_1f_1^*)^{q(q-1)} - (f_2f_2^*)^{q-1}\bigr)u_0^q + \bigl(f_2f_1^*f_2^*\bigr)^{q-1}u_1 + \bigl(f_1f_2f_2^*\bigr)^{q-1}u_{-1} - \bigl(f_1f_2f_1^*f_2^*\bigr)^{q-1}u_0 - f_3f_2^{*q} - f_2^qf_3^* = 0, \qquad (R_3')$$

which is $*$-invariant and can substitute the relation (R_3^-). This also demonstrates that there is some arbitrariness in our choice of generating relations.

(B_1) If $n=1$, then $\widetilde{f}_1 = x_1^{q-1}$, $\widetilde{f}_1^* = y_1^{q-1}$, and $u_0 = x_1y_1$. Theorem 2.4(b) asserts that $\mathbb{F}_q[V \oplus V^*]^{B_1}$ is generated by $\widetilde{f}_1$, $\widetilde{f}_1^*$, and u_0, subject to the relation

$$u_0^{q-1} - \widetilde{f}_1\widetilde{f}_1^* = 0. \qquad (\widetilde{R}_1)$$

This can easily be verified by hand.

(B_2) If $n=2$, one gets the following three relations between the B_2-invariants:

$$\bigl(u_{-1} - \widetilde{f}_1^*u_0\bigr)^{q-1} - \widetilde{f}_1\widetilde{f}_2^* = 0, \qquad (\widetilde{R}_1)$$

$$u_0^q - \widetilde{f}_1u_{-1} - \widetilde{f}_1^*u_1 + \widetilde{f}_1\widetilde{f}_1^*u_0 = 0, \qquad (\widetilde{R}_1^+)$$

$$(u_1 - \widetilde{f}_1u_0)^{q-1} - \widetilde{f}_2\widetilde{f}_1^* = 0. \qquad (\widetilde{R}_2)$$

This looks nicely symmetric, in the sense that the set of relations is stable under the involution $*$. But it is clear that the symmetry will be lost when n becomes bigger. In fact, our choice of generating relations of the B_n-invariants is arbitrary, just as in the case of U_n-invariants.

Proof of Theorem 2.4. The proofs of (a) and (b) are very similar and both rely on the use of Lemma 1.4.

• Let us first prove (a). Since Example 2.5(U_2) deals with the case where $n=2$, we may assume that $n>2$. We want to apply Lemma 1.4 to $A = \mathbb{F}_q[V \oplus V^*]^{U_n}$, $m=n$, $l=2n-3$, $g_i = f_i^*$, $(h_1,\dots,h_l) = (u_{2-n},\dots,u_{-1},u_0,u_1,\dots,u_{n-2})$, and $R_1,\dots,R_l$ being replaced by $R_1^+, R_2, R_3^-, R_3, R_4^-, R_4, R_5^-,\dots, R_{n-2}, R_{n-1}^-, R_{n-1}, R_n^-$.

From (2.2) we deduce that $f_1,\dots,f_n$, $f_1^*,\dots,f_n^*$ satisfy the hypothesis (a) from Lemma 1.4. From Corollary 1.3 and again (2.2), it follows that

$$\mathbb{F}_q\bigl(V \oplus V^*\bigr)^{U_n} = \mathbb{F}_q\bigl(f_1,\dots,f_n,f_1^*,\dots,f_n^*,u_0\bigr),$$

so the hypothesis (b) of Lemma 1.4 is also satisfied.

In order to establish the hypotheses (c) and (d) of Lemma 1.4, we analyze the relations $(R_k^{(\pm)})$.

We will say that one of the relations is a relation *for* a u_i if the relation equates a power of u_i to a polynomial in our claimed generators, and each monomial of this polynomial involves at least one of the f_i or f_i^*. We will say that one of the relations f*-eliminates* a (claimed) generator g if this relation, viewed as a polynomial in g, has degree 1 and leading coefficient a product of powers of the f_i. In the same way, we speak of relations that f^**-eliminate* generators. Notice that $c_{s,0}$, as defined in (2.9), is a product of powers of the f_i, and $c_{s,0}^*$ is a product of powers of the f_i^*. Using this terminology, our analysis of the relations can be summarized in the following table:

Relation	Involves	Relation for	f-Eliminates	f^*-Eliminates	Range
R_k^+	$f_1, \dots, f_k$, $f_1^*, \dots, f_{n+1-k}^*$, $u_{k-n+1}, \dots, u_k$	u_{2k-n}	f_{n+1-k}^*	u_k	$k = 1$
R_k	$f_1, \dots, f_k$, $f_1^*, \dots, f_{n+1-k}^*$, $u_{k-n}, \dots, u_{k-1}$	u_{2k-n-1}	f_{n+1-k}^*, u_{k-n}	f_k, u_{k-1}	$2 \leqslant k \leqslant n-1$
R_k^-	$f_1, \dots, f_k$, $f_1^*, \dots, f_{n+1-k}^*$, $u_{k-n-1}, \dots, u_{k-2}$	u_{2k-n-2}	u_{k-n-1}	f_k	$3 \leqslant k \leqslant n$

The last column of the table indicates the range of k specified in (2.10). We make several observations.

First, since $n > 2$, the relations in (2.10) involve the invariants $f_1, \dots, f_n$, $f_1^*, \dots, f_n^*$, and $u_{2-n}, \dots, u_{n-2}$, which are exactly the generators claimed in Theorem 2.4(a).

Second, in (2.10) we have one relation for every u_i (with $2 - n \leqslant i \leqslant n-2$). If we regard the f_i, f_i^*, and u_i as indeterminates for a moment, it follows that the affine variety in $\overline{\mathbb{F}_q}^{4n-3}$ given by the equations $f_i = 0$, $f_i^* = 0$ and the relations in (2.10) consists of only one point, the origin. It follows that the hypothesis (c) of Lemma 1.4 is satisfied.

It remains to show that (d) is also satisfied. By another abuse of notation, we will now regard the f_i, f_i^*, and u_i as elements of the algebra B defined by the relations in (2.10). We can use the relations

$$\begin{array}{ll} R_1^+,\ R_2,\ R_3^-,\ R_3,\ R_4^-,\ R_4,\ R_5^-, \dots,\ R_{n-2},\ R_{n-1}^-,\ R_{n-1},\ R_n^- & \text{(in this order) to show that} \\ u_1,\ \ f_2,\ \ f_3,\ \ u_2,\ \ f_4,\ \ u_3,\ \ f_5, \dots,\ \ u_{n-3},\ \ f_{n-1},\ \ u_{n-2},\ \ f_n & \text{(also in this order)} \end{array}$$

lie in $\mathbb{F}_q[(f_1^* \cdots f_n^*)^{-1}, f_1^*, \dots, f_n^*, f_1, u_{2-n}, \dots, u_0]$. So this algebra is equal to $B[(f_1^* \cdots f_n^*)^{-1}]$. We can also use

$$\begin{array}{ll} R_n^-,\ R_{n-1}, R_{n-1}^-, R_{n-2}, \dots, R_5^-,\ \ R_4,\ \ R_4^-,\ \ R_3,\ \ R_3^-,\ \ R_2,\ \ R_1^+ & \text{(in this order) to show that} \\ u_{-1}, f_2^*,\ \ u_{-2},\ \ f_3^*, \dots,\ \ u_{4-n}, f_{n-3}^*, u_{3-n}, f_{n-2}^*, u_{2-n}, f_{n-1}^*, f_n^* & \text{(also in this order)} \end{array}$$

lie in $\mathbb{F}_q[(f_1 \cdots f_n)^{-1}, f_1, \dots, f_n, f_1^*, u_0, \dots, u_{n-2}]$. So this algebra is equal to $B[(f_1 \cdots f_n)^{-1}]$. We have shown that the hypothesis (d) in Lemma 1.4 is satisfied, so Theorem 2.4(a) follows.

• Let us now prove (b). From (2.2) and Corollary 1.3, we get that $\mathbb{F}_q(V \oplus V^*)^{B_n} = \mathbb{F}_q(\widetilde{f}_1, \dots, \widetilde{f}_n, \widetilde{f}_1^*, \dots, \widetilde{f}_n^*, u_0)$, so the hypotheses (a) and (b) of Lemma 1.4 follow.

Now we analyze the relations (2.12) in the same manner as in the proof of (a). This results in the following table:

Relation	Involves	Relation for	$\widetilde{f}$-Eliminates	$\widetilde{f}^*$-Eliminates	Range
$\widetilde{R}_k$	$\widetilde{f}_1, \dots, \widetilde{f}_k$, $\widetilde{f}_1^*, \dots, \widetilde{f}_{n+1-k}^*$, $u_{k-n}, \dots, u_{k-1}$	u_{2k-n-1}	$\widetilde{f}_{n+1-k}^*$	$\widetilde{f}_k$	$1 \leqslant k \leqslant n$
$\widetilde{R}_k^+$	$\widetilde{f}_1, \dots, \widetilde{f}_k$, $\widetilde{f}_1^*, \dots, \widetilde{f}_{n-k}^*$, $u_{k-n}, \dots, u_k$	u_{2k-n}	u_{k-n}	u_k	$1 \leqslant k \leqslant n-1$

We first observe that the relations (2.12) involve only the claimed generators. Secondly, there is one relation for each u_i, so the hypothesis (c) of Lemma 1.4 is satisfied. Finally, to see that (d) is also satisfied, we use the relations

$$\begin{array}{ll} \widetilde{R}_1,\ \widetilde{R}_1^+,\ \widetilde{R}_2,\ \widetilde{R}_2^+,\ \widetilde{R}_3,\ \widetilde{R}_3^+,\ \ldots,\ \widetilde{R}_{n-2},\ \widetilde{R}_{n-2}^+,\ \widetilde{R}_{n-1},\ \widetilde{R}_{n-1}^+\ \widetilde{R}_n & \text{(in this order) to show that} \\ \widetilde{f}_1,\ u_1,\ \widetilde{f}_2,\ u_2,\ \widetilde{f}_3,\ u_3,\ldots,\ \widetilde{f}_{n-2},\ u_{n-2},\ \widetilde{f}_{n-1},\ u_{n-1},\ \widetilde{f}_n & \text{(also in this order)} \end{array}$$

lie in $\mathbb{F}_q[(\widetilde{f}_1^* \cdots \widetilde{f}_n^*)^{-1}, \widetilde{f}_1^*, \ldots, \widetilde{f}_n^*, u_{1-n}, \ldots, u_0]$. We can also use

$$\begin{array}{ll} \widetilde{R}_n, \widetilde{R}_{n-1}^+, \widetilde{R}_{n-1}, \widetilde{R}_{n-2}^+, \widetilde{R}_{n-2}, \ldots, \widetilde{R}_3^+,\ \widetilde{R}_3,\ \widetilde{R}_2^+,\ \widetilde{R}_2,\ \widetilde{R}_1^+,\ \widetilde{R}_1 & \text{(in this order) to show that} \\ \widetilde{f}_1^*, u_{-1},\ \widetilde{f}_2^*,\ u_{-2},\ \widetilde{f}_3^*, \ldots,\ u_{3-n}, \widetilde{f}_{n-2}^*, u_{2-n}, \widetilde{f}_{n-1}^*, u_{1-n}, \widetilde{f}_n^* & \text{(also in this order)} \end{array}$$

lie in $\mathbb{F}_q[(\widetilde{f}_1 \cdots \widetilde{f}_n)^{-1}, \widetilde{f}_1, \ldots, \widetilde{f}_n, u_0, \ldots, u_{n-1}]$. This shows that (d) of Lemma 1.4 is also satisfied, so applying the lemma yields the desired result.

The statement about the minimality of generators will be proved below. $\square$

Bigrading. There is an obvious bigrading on $K[V \oplus V^*]$, given by assigning the bidegree $(1, 0)$ to every x_i and $(0, 1)$ to every y_i. This bigrading passes to $K[V \oplus V^*]^G$ for every $G \leqslant \mathrm{GL}_n(K)$. It is interesting in itself, and also provides an easy way to prove the minimality statement in Theorem 2.4. All generating invariants occurring in Theorem 2.4 are bihomogeneous, and their bidegrees are listed in the following table:

Invariant	f_i	f_i^*	$\widetilde{f}_i$	$\widetilde{f}_i^*$	u_i
Bidegree	$(q^{i-1}, 0)$	$(0, q^{i-1})$	$((q-1)q^{i-1}, 0)$	$(0, (q-1)q^{i-1})$	$(q^i, 1)$ if $i \geqslant 0$, $(1, q^{-i})$ if $i \leqslant 0$

The relations are also bihomogeneous of the following bidegrees:

Relation	R_k	R_k^+	R_k^-	$\widetilde{R}_k$	$\widetilde{R}_k^+$
Bidegree	(q^{k-1}, q^{n-k})	(q^k, q^{n-k})	(q^{k-1}, q^{n+1-k})	$(q-1) \cdot (q^{k-1}, q^{n-k})$	(q^k, q^{n-k})

Proof of Theorem 2.4 (continued). To prove the minimality of the generating invariants, we assume, by way of contradiction, that one of the given generators is unnecessary. Then there exists a relation equating this generator to a polynomial in the other generators. We may assume this relation to be bihomogeneous of the same bidegree as the unnecessary generator. This implies that one of the generating relations must have bidegree bounded above (in both components) by the bidegree of the unnecessary generator. By comparing the bidegrees of the generating invariants and the bidegrees of the relations (and keeping in mind for which ranges of k each relation appears in Theorem 2.4), we see that this only happens in one case: if $q = 2$, then $\widetilde{R}_1$ and $\widetilde{R}_n$ have bidegrees $(1, q^{n-1})$ and $(q^{n-1}, 1)$, respectively. Since this case was excluded in the minimality statement, the proof is complete. $\square$

Since $\mathbb{F}_q[V \oplus V^*]^{U_n}$ and $\mathbb{F}_q[V \oplus V^*]^{B_n}$ are complete intersections, we can also write down their bigraded Hilbert series. For a general bigraded vector space V (with finite-dimensional bihomogeneous components $V_{d,e}$), the bigraded Hilbert series is defined as

$$H(V, s, t) := \sum_{d,e=0}^{\infty} \dim_K(V_{d,e}) s^d t^e \in \mathbb{Z}[[s, t]].$$

The results are

$$H\big(\mathbb{F}_q\big[V \oplus V^*\big]^{U_n}, s, t\big) = \frac{\prod_{k=2}^{n-1}(1 - s^{q^{k-1}} t^{q^{n-k}}) \prod_{k=1}^{n-1}(1 - s^{q^k} t^{q^{n-k}})}{\prod_{i=0}^{n-1}((1 - s^{q^i})(1 - t^{q^i})) \prod_{i=0}^{n-2}(1 - s^{q^i} t) \prod_{i=1}^{n-2}(1 - s t^{q^i})}$$

and

$$H\left(\mathbb{F}_q\left[V \oplus V^*\right]^{B_n}, s, t\right) = \frac{\prod_{k=1}^{n}(1 - s^{(q-1)q^{k-1}} t^{(q-1)q^{n-k}}) \prod_{k=1}^{n-1}(1 - s^{q^k} t^{q^{n-k}})}{\prod_{i=0}^{n-1}((1 - s^{(q-1)q^i})(1 - t^{(q-1)q^i})) \prod_{i=0}^{n-1}(1 - s^{q^i} t) \prod_{i=1}^{n-1}(1 - s t^{q^i})}.$$

Notice that the Hilbert series with respect to the usual total degree can be obtained from the bigraded Hilbert series by setting $s = t$.

3. A conjecture about $\mathrm{GL}_n(\mathbb{F}_q)$

We have also considered the invariant ring $\mathbb{F}_q[V \oplus V^*]^{\mathrm{GL}_n(\mathbb{F}_q)}$ of the general linear group. It is well known that the invariant ring $\mathbb{F}_q[V]^{\mathrm{GL}_n(\mathbb{F}_q)}$ is generated by the Dickson invariants $c_{n,0}, \dots, c_{n,n-1}$ (see Wilkerson [19, Theorem 1.2] or Smith [18, Theorem 8.1.5]). The $c_{n,i}^*$ are further invariants in $\mathbb{F}_q[V \oplus V^*]^{\mathrm{GL}_n(\mathbb{F}_q)}$, and we also have the invariants u_i. Various computations in the computer algebra system MAGMA (see [4]) have prompted us to make the following conjecture.

Conjecture 3.1. *If $n \geqslant 2$, the invariant ring of the general linear group is generated by $4n - 1$ invariants as follows*

$$\mathbb{F}_q\left[V \oplus V^*\right]^{\mathrm{GL}_n(\mathbb{F}_q)} = \mathbb{F}_q\left[c_{n,0}, \dots, c_{n,n-1}, c_{n,0}^*, \dots, c_{n,n-1}^*, u_{1-n}, \dots, u_{n-1}\right].$$

*The invariant ring is Gorenstein but **not** a complete intersection.*

We have been able to verify the conjecture computationally for $(n, q) \in \{(2, 2), (2, 3), (2, 4), (3, 2)\}$. For $(n, q) \in \{(2, 5), (2, 7), (3, 3), (4, 2)\}$, we managed to gain evidence for the conjecture by checking that all invariants up to some degree (as far as the computer calculation was possible) lie in the algebra that Conjecture 3.1 claims to be the invariant ring.

Theorem 2.4 and Conjecture 3.1 (if true) tell us that for $G \in \{U_n, B_n, \mathrm{GL}_n(\mathbb{F}_q)\}$, the invariant ring $\mathbb{F}_q[V \oplus V^*]^G$ is generated by generators of $\mathbb{F}_q[V]^G$, their $*$-images, and invariants of the form u_i. How general is this phenomenon? To find out, we considered the special linear groups.

Example 3.2. For $G = \mathrm{SL}_2(\mathbb{F}_3)$ and $V = \mathbb{F}_3^2$ the natural G-module, we have

$$\mathbb{F}_3[V]^G = \mathbb{F}_3\big[\underbrace{x_1^3 x_2 - x_1 x_2^3}_{=: f_1}, \underbrace{x_1^6 + x_1^4 x_2^2 + x_1^2 x_2^4 + x_2^6}_{=: f_2}\big].$$

(In fact, the invariants of $\mathrm{SL}_n(\mathbb{F}_q)$ acting on its natural module are well known for general n and q, see Smith [18, Theorem 8.1.8].) Turning to the action on $\mathbb{F}_3[V \oplus V^*]$, we verify that the G-orbit of $h := x_1 y_2 - x_2 y_1$ has length 6 and includes $-h$. Therefore a square root of the negative of the orbit product is an invariant, which we write as $g \in \mathbb{F}_3[V \oplus V^*]^G$. The bidegree of g is $(3, 3)$.

On the other hand, the f_i and their $*$-images have bidegrees $(4, 0)$, $(6, 0)$, $(0, 4)$, and $(0, 6)$, and the u_i and u_{-i} have bidegrees $(3^i, 1)$ and $(1, 3^i)$, respectively, for i non-negative. So $g \in \mathbb{F}_3[f_1, f_2, f_1^*, f_2^*, u_0, u_1, u_{-1}, \dots]$ would imply $g = \pm u_0^3$, which is not the case. We conclude that for $G = \mathrm{SL}_2(\mathbb{F}_3)$, the invariant ring $\mathbb{F}_3[V \oplus V^*]^G$ is *not* generated by generators of $\mathbb{F}_3[V]^G$, their $*$-images, and invariants of the form u_i.

Further calculations show that this carries over to other special linear groups $\mathrm{SL}_n(\mathbb{F}_q)$.

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