The Symmetry of the Modular Burnside Ring

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Let $b(G)$ be the Burnside ring of a finite group $G$ and let $k$ be a field of prime characteristic. It is the purpose of this paper to give a characterization of whether a block of $k \otimes_k b(G)$ is a symmetric $k$-algebra. This proves a blockwise version of a corresponding result about $k \otimes_k b(G)$ by W. Gustafson (1977, Comm. Algebra 5, 1–15).

1. INTRODUCTION

Let $G$ be a finite group and let $b(G)$ be the Burnside ring of $G$, i.e., the Grothendieck ring of the category of finite $G$-sets where addition and multiplication are given by direct sums and cartesian products (see [1, Sect. 80] for an introduction to Burnside rings). For any commutative ring $k$ we set $B_k(G) := k \otimes_k b(G)$. If $k$ is a field of characteristic 0 then $B_k(G)$ is a semi-simple split $k$-algebra and therefore isomorphic to a direct product of copies of $k$ which was first proved by Solomon [5]. This is no longer true in general for a field $k$ of characteristic $p \neq 0$. In this case it follows from results of Dress [3] that $B_k(G)$ has a block decomposition which is parametrized by the conjugacy classes of $p$-perfect subgroups of $G$ (see Section 2 for an exact description of this).

For any $p$-perfect subgroup $H$ of $G$ let $B_H$ denote the corresponding block of $B_k(G)$. We look at the question of whether $B_H$ is a symmetric $k$-algebra. In general a $k$-algebra $A$ is called symmetric if there exists a $k$-linear map $\lambda: A \rightarrow k$ with the following properties:

(i) $\lambda(ab) = \lambda(ba)$ for any $a, b \in A$.

(ii) $I \not\subseteq \text{Ker} \lambda$ for any nontrivial ideal $I$ in $A$. 

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Theorem 1. Let $G$ be a finite group, let $k$ be a field of characteristic $p$, and let $H$ be a $p$-perfect subgroup of $G$. Then the block $B_H$ of $B_k(G)$ corresponding to $H$ is a symmetric $k$-algebra if and only if $|N_G(H) : H|$ is not divisible by $p^2$.

As an immediate consequence we have

Corollary 2 (Gustafson [4]). For any finite group $G$ and any field $k$ of characteristic $p$ the $k$-algebra $B_k(G)$ is symmetric if and only if $|G|$ is not divisible by $p^2$.

2. THE MODULAR BURNSIDE RING

For any subgroup $H$ of $G$ let $G/H$ be the set of left cosets of $G$ with respect to $H$ which becomes a $G$-set by left multiplication. We use the symbols $\sim$ and $\leq$ for conjugation and subconjugation in $G$. Let us choose a transversal $\mathcal{S}$ of the conjugacy classes of subgroups of $G$. Then $b(G)$ is free as a $\mathbb{Z}$-module with basis $[G/H] (H \in \mathcal{S})$ where $[X]$ denotes the isomorphism class of a $G$-set $X$. The species of $b(G)$, i.e., the nontrivial ring homomorphisms $b(G) \to \mathbb{Z}$, play an important role within the treatment of Burnside rings. Any species of $b(G)$ is of the form $\phi_H': b(G) \to \mathbb{Z}, [X] \to |X^H|$ for a subgroup $H$ of $G$; here $X^H$ denotes the set of $H$-fixed points of $X$. These maps are called mark homomorphisms and for subgroups $H, I$ of $G$ we have $\phi_H = \phi_I$ if and only if $H \sim I$. Moreover, the product

$$\phi := \prod_{H \in \mathcal{S}} \phi_H: b(G) \to \mathbb{Z}^{[\mathcal{S}]}$$

is known to be a ring monomorphism.

Let $\mathcal{O}$ be a Dedekind ring with quotient field $K$ of characteristic 0, let $p$ be a prime, let $\mathfrak{p}$ be a maximal ideal in $\mathcal{O}$ with $p \in \mathfrak{p}$, and let $k := \mathcal{O}/\mathfrak{p}$ be the corresponding residue field of characteristic $p$. Any species of $\phi_H$ of $b(G)$ clearly induces maps $B_{\mathcal{O}}(G) \to \mathcal{O}$ and $B_k(G) \to K$, which we also denote by $\phi_H$. The above product map then becomes an isomorphism $\phi: B_k(G) \to K^{[\mathcal{S}]}$ of $K$-algebras. As a consequence we get the primitive idempotents $e_H$ ($H \in \mathcal{S}$) of $B_k(G)$ as the preimages of the canonical primitive idempotents of $K^{[\mathcal{S}]}$; i.e., $e_H$ is characterized by $\phi_H(e_H) = \delta_H$ for for $I \in \mathcal{S}$ where $\delta_H$ is the Kronecker delta.

The species of $B_k(G)$ are obtained as reductions $\bar{\phi}_H$ of the maps $\phi_H: B_{\mathcal{O}}(G) \to \mathcal{O}$ modulo $\mathfrak{p}$. Similarly we write $\bar{x}$ for the reduction of an element $x \in B_{\mathcal{O}}(G)$ modulo $\mathfrak{p}$. For any subgroup $H$ of $G$ let $O^\mathfrak{p}(H)$
denote the (unique) smallest normal subgroup $N$ of $H$ such that $H/N$ is a $p$-group. We call $H$ $p$-perfect if $O^p(H) = H$. Let $\mathcal{P}$ be a transversal of the conjugacy classes of $p$-perfect subgroups of $G$. For each $H \in \mathcal{P}$ we set $\mathcal{H}_H = \{I \in \mathcal{P} : O^p(I) \sim H\}$. We choose a preimage $S_H$ of a Sylow $p$-subgroup of $N_G(H)/H$ which is unique up to conjugation in $G$, and we assume $S_H \in \mathcal{H}_H$. Indeed $S_H$ is the greatest element in $\mathcal{H}_H$ with respect to subconjugation.

**Lemma 3** (Dress [3], Yoshida [6]). (i) For subgroups $H, I$ of $G$ we have $
abla = \Delta$ iff $O^p(H) \sim O^p(I)$.

(ii) For any $H \in \mathcal{P}$ let $f_H := \sum_{I \in \mathcal{H}_H} e_I$. Then the primitive idempotents of $B_k(G)$ are given by $f_H$ ($H \in \mathcal{P}$); in particular we have $\dim_k B_H = |\mathcal{H}_H|$.

Of course any primitive idempotent of $B_k(G)$ is a block idempotent since $B_k(G)$ is commutative. Thus the sum $1 = \sum_{H \in \mathcal{P}} f_H$ gives rise to the block decomposition $B_k(G) = \bigoplus_{H \in \mathcal{P}} B_H$ where $B_H$ denotes the block $B_k(G)_{/H}$ for $H \in \mathcal{P}$.

Every $G$-set $X$ can be considered as an $H$-set for any subgroup $H$ of $G$ by restriction of the operation. This induces a restriction map $\text{Res}^G_H : b(G) \rightarrow b(H)$, which clearly is a ring homomorphism.

In the following lemma we state some elementary properties of idempotents without proof:

**Lemma 4.** Suppose $H \in \mathcal{P}$ and $I \leq G$.

(i) For $x \in B_k(G)$ we have $x \cdot e_I = \phi_H(x) e_H$.

(ii) If $[G/I] e_I \neq 0$ then $H \leq I$.

(iii) If $H \leq I$ then

$$\text{Res}^G_H (\overline{f_H}) = \sum_{L \in \mathcal{P}(I)} \overline{f_{L/L}},$$

where $\mathcal{P}(I)$ is a transversal of $p$-perfect subgroups of $I$ and $\overline{f_{L/L}}$ denotes the primitive idempotent of $B_k(I)$ corresponding to $L$.

For the rest of this section fix a $p$-perfect subgroup $H$ of $G$. Since the multiplication in $b(G)$ is explicitly given by

$$[G/I] \cdot [G/L] = \sum_{g \in L \cap G/L} [G/I \cap gLg^{-1}].$$

for $I, L \leq G$ it is not difficult to see that

$$b'(G, H) := \bigoplus_{I \in \mathcal{P}} \mathbb{Z}[G/I]$$

for $H \leq I$.
is an ideal in \( b(G) \). We also define 
\[ B'_K(G, H) := K \otimes_{\mathbb{Z}} b'(G, H) \]
and 
\[ B'_K(G, H) := k \otimes_{\mathbb{Z}} b'(G, H). \]

**Lemma 5.**
(i) For \( I \in \mathcal{I}_H \) we have \( [G/I] \equiv [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} (mod B'_K(G, H)). \)
(ii) The elements \( [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} (I \in \mathcal{I}_H) \) form a \( k \)-basis of \( B_H \).
(iii) Suppose \( H := O^p(G) \). Then the \( k \)-linear map \( \alpha: B'_K(G/H) \to B'_K(G/H)_H \) defined by \( [(G/H)/(I/H)] \to [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} \) is an isomorphism of \( k \)-algebras.

**Proof.**
(i) For \( I \in \mathcal{I}_H \) we have
\[
[G/I] = [G/I] \cdot 1 = [G/I] \sum_{H' \in \mathcal{I}_H} \bar{f}_{H'} = \sum_{H' \in \mathcal{I}_H} [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H}.
\]
In the case \( H' \not\leq I \) we have \( L \not\leq I \) for any \( L \in \mathcal{I}_H \) and therefore by Lemma 4(ii)
\[
[G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} = \sum_{L \in \mathcal{I}_H} [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} \cdot e_L = 0.
\]
It follows that \( [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} = 0 \) and
\[
[G/I] = \sum_{H' \in \mathcal{I}_H} [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H},
\]
Now let \( H' \in \mathcal{I} \setminus \{H\} \) and \( H' \not\leq I \). Then \( H' = O^p(H') \not\leq O^p(I) \sim H \), so for any \( L \in \mathcal{I}_H \) we have \( H \not\leq L \) since otherwise we would have \( H' \leq H = O^p(H) \not\leq O^p(L) = H' \) and \( H' \sim H \). Hence
\[
[G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} = \sum_{L \in \mathcal{I}_H} [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} \cdot e_L = \sum_{L \in \mathcal{I}_H} \phi_L([G/I])_{\mathcal{F}_H}^{\mathcal{F}_H}
\]
\[ \in \sum_{L \in \mathcal{I}_H} \left( \sum_{L' \leq L} K[G/L'] \right) \subseteq B'_K(G, H) \]
and \( [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} \subseteq B'_K(G, H) \). Finally we have
\[
[G/I] = [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} \quad (mod B'_K(G, H)).
\]
(ii) By (i) the elements \( [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} + B'_K(G, H) (I \in \mathcal{I}_H) \) are linearly independent in \( B'_K(G/H) \). Thus \( [G/I]_{\mathcal{F}_H}^{\mathcal{F}_H} (I \in \mathcal{I}_H) \) are linearly independent in \( B_H \) and a \( k \)-basis in \( B_H \) since \( |\mathcal{I}_H| = \dim_k B_H \).
(iii) Clearly \( \alpha \) is a homomorphism of \( k \)-algebras. By (ii) \( \alpha \) is surjective and thus an isomorphism since \( \dim_k B'_K(G/H) = |\mathcal{I}_H| = \dim_k B_H \).
**Proposition 6.** The Jacobson radical of $B_H$ is

$$JB_H = \bigoplus_{I \in \mathcal{I}_H \setminus \{S_H\}} k[G/I]_{H}.$$  

**Proof.** For any $I \in \mathcal{I}_H \setminus \{S_H\}$ and $L \leq G$ we have

$$[G/I]_{H} \cdot [G/L]_{H} = \sum_{l \in L \cap G/L} [G/I \cap gLg^{-1}]_{H},$$

In the case $H \leq I \cap gLg^{-1}$ we have $[G/I \cap gLg^{-1}]_{H} = 0$ as in the proof of Lemma 5(i). Hence

$$[G/I]_{H} \cdot [G/L]_{H} \in \bigoplus_{M \in \mathcal{I}_H \setminus \{S_H\}} \bigoplus_{M \leq l} k[G/M]_{H}$$

showing that $J := \bigoplus_{I \in \mathcal{I}_H \setminus \{S_H\}} k[G/I]_{H}$ is an ideal in $B_H$. Since the elements $[G/I]_{H} (I \in \mathcal{I}_H)$ are linearly independent $J$ has codimension 1 and thus is the Jacobson radical of the local $k$-algebra $B_H$.  

Finally let us consider $G$-stable elements. Suppose $S$ is a subgroup of $G$ and $g \in G$. An element $x \in B_k(S)$ is called $g$-stable if

$$\text{Res}^S_{S \cap gS^{-1}}(x) = \text{Res}^S_{S \cap gS^{-1}}(gx),$$

where $gx$ is defined by $^g[S/I] = [gSg^{-1}/gIg^{-1}]$ for $I \leq S$. The element $x$ is called $G$-stable if $x$ is $g$-stable for any $g \in G$. By $B_k(S)^G$ we denote the set of $G$-stable elements of $B_k(S)$. Now we can formulate the following transfer theorem:

**Proposition 7 (Yoshida).** (i) The map

$$B_k(G)_{H} \rightarrow B_k(S_H)^{G}_{S_H,H}, \quad x \mapsto \text{Res}^G_{S_H}(x)$$

is an isomorphism of $k$-algebras.

(ii) $B_k(S_H)^{G}_{S_H,H} = B_k(S_H)^{N_G(H)}_{S_H,H}$.  

**Proof.** See [7, Lemma 4.1, Corollary 4.2].  

3. **Socle Elements**

By Soc $A$ we denote the socle of an algebra $A$.

**Lemma 8.** (i) $[G/H]_{H} \in \text{Soc} B_H$.

(ii) Let $S$ be a $p$-group and $V < U < S$. Then the number of subgroups $I$ of $S$ with $UI = S$ and $U \cap I = V$ is divisible by $p$. 


(iii) Let $S$ be a $p$-group of order $p'$ and let $i \in \{1, \ldots, r - 1\}$. Assume that all subgroups of order $p'$ are normal in $S$. Then

$$s_i := \sum_{I \leq S} [S/I] \in \text{Soc } B_k(S).$$

Proof. (i) For $I \in \mathcal{S}_H$ the following holds:

$$[G/H]_{I_H} \cdot [G/I]_{I_H} = \sum_{Hg \in H \setminus G/I} [G/H \cap gl^{-1}]_{I_H}.$$  

In the case $H \leq gl^{-1}$ we have $[G/H \cap gl^{-1}]_{I_H} = [G/H]_{I_H}$; otherwise we have $H \cap gl^{-1} < H$ and $[G/H \cap gl^{-1}]_{I_H} = 0$ by Lemma 4(ii). This implies that $k[G/H]_{I_H}$ is a one-dimensional ideal in $B_H$ and therefore $k[G/H]_{I_H} \subseteq \text{Soc } B_H$.

(ii) This is an easy exercise. A proof which is based on a generalization of Sylow’s theorem can be found in [2].

(iii) It suffices to show that $J B_k(S)$ is annihilated by $s_i$. By Proposition 6 we have $J B_k(S) = \sum_{L \leq S} k[L]$. Let $L \leq S$ and let $I$ be a subgroup of $S$ of order $p'$. Because $I$ is normal in $S$ the following holds:

$$[S/L] \cdot [S/I] = \sum_{Ls \in L \setminus S/I} [S/L \cap sIs^{-1}] = \sum_{Li \in L \setminus S} [S/L \cap I]$$

$$= |S:LI|[S/L \cap I] = \begin{cases} [S/L \cap I], & \text{if } LI = S, \\ 0, & \text{if } LI < S. \end{cases}$$

Thus we have

$$[S/L] \cdot s_i = \sum_{I \leq S, |I| = p'} [S/L \cap I] = 0$$

by (ii) and therefore $J B_k(S) \cdot s_i = 0$. \(\square\)

The proof of Theorem 1 is based on the following easy lemma:

**Lemma 9.** Suppose $A$ is a finite-dimensional commutative $k$-algebra and $\dim_k A/IA = 1$. Then $A$ is symmetric iff the socle $\text{Soc } A$ of $A$ is one-dimensional.

Proof. If $A$ is symmetric we have $A/IA \cong \text{Soc } A$ (for example, see [1, Proposition 9.12]).

On the other hand let $\dim_k \text{Soc } A = 1$. We choose a $k$-basis $b_1, \ldots, b_n$ of $A$ such that $\text{Soc } A = k \cdot b_1$, and we define the linear map

$$\lambda: A \rightarrow k, \quad \sum_{i=1}^n \alpha_i b_i \mapsto \alpha_1.$$
Since any nontrivial ideal \( I \) in \( A \) contains a simple \( A \)-module \( S \) and \( \text{Soc} \ A \) is one-dimensional we have \( S = \text{Soc} \ A \not\subseteq \text{Ker} \lambda \) and therefore \( I \not\subseteq \text{Ker} \lambda \), so \( A \) together with \( \lambda \) is a symmetric \( k \)-algebra.

4. PROOF OF THEOREM 1

We are now ready to prove our main result.

Proof of Theorem 1. In the case \( p^2 \nmid |S_H : H| \) we have \( \mathcal{S}_H = \{ H \} \) or \( \mathcal{S}_H = \{ S_H, H \} \); hence \( \dim_k B_H \leq 2 \). Thus \( B_H \) is either semi-simple and therefore symmetric or \( \dim_k \text{Soc} \ B_H = 1 \) and by Lemma 9 \( B_H \) is symmetric, too.

Now let \( p^2 | |S_H : H| \). We consider the block \( B_{S_H, H} := B_H(S_H) \) of \( B_{S_H, H} \). By Proposition 7(i) \( B_H \) is isomorphic to the set \( B_{S_H, H}^G \) of \( G \)-stable elements of \( B_{S_H, H} \) as a \( k \)-algebra. By Lemma 9 it suffices to show that \( \dim_k \text{Soc} \ B_{S_H, H}^G \geq 2 \).

By Lemma 8(i) we have an element \( x := [S_H/H] \) of \( \text{Soc} \ B_{S_H, H} \). Here we describe how we get another socle element of \( B_{S_H, H} \): By Lemma 5(ii) there is an isomorphism \( B_{S_H, H}^G \to B_{S_H, H}^G \) given by \( [S_H/H] \to [S_H/H] \). Applying Lemma 8(ii) to the \( p \)-group \( S_H \) we get an element

\[
\sum_{H \leq I \leq S_H \atop |S_H : I| = p} [(S_H/H)/(1/H)] \in \text{Soc} B_h(S/H)
\]

and therefore an element

\[
y := \sum_{H \leq I \leq S_H \atop |S_H : I| = p} [S_H/I] \in \text{Soc} B_{S_H, H}.
\]

By Lemma 5(ii) \( x \) and \( y \) are linear independent, so it remains to show that both elements are \( G \)-stable, in which case we would have

\[
x, y \in (\text{Soc} B_{S_H, H}) \cap B_{S_H, H}^G \subseteq \text{Soc} B_{S_H, H}^G
\]

and \( \dim_k \text{Soc} B_{S_H, H}^G \geq 2 \).

By Proposition 7(ii) it is enough to show that \( x \) and \( y \) are \( g \)-stable for \( g \in N_G(H) \). We fix \( g \in N_G(H) \) and set \( D := S_H \cap gS_Hg^{-1} \). By Lemma 5(iii) we have

\[
\text{Res}_H^G(\overline{f_{S_H, H}}) = \sum_{L \in \mathcal{P}(D) \atop L \sim_s S_H} \overline{f_{D, L}} = \overline{f_{D, H}}
\]
and similarly
\[ \text{Res}_{D}^{g_{H}g^{-1}}(\overline{f_{S_{H},H}}) = \overline{f_{D,H}}. \]
Therefore
\[
\text{Res}_{D}^{g_{H}g^{-1}}(x) = \text{Res}_{D}^{g_{H}g^{-1}}([S_{H}/H]) \cdot \text{Res}_{D}^{g_{H}g^{-1}}(\overline{f_{S_{H},H}})
= \sum_{D \subseteq H \subseteq S_{H}} [D/D \cap sHs^{-1}]\overline{f_{D,H}}
= \sum_{D \subseteq H \subseteq S_{H}} [D/D \cap H]\overline{f_{D,H}}
= |S_{H} : D|[D/H]\overline{f_{D,H}}
\]
and
\[
\text{Res}_{D}^{g_{H}g^{-1}}(x) = |S_{H} : D|[D/H]\overline{f_{D,H}}
\]
Since
\[
p \nmid |S_{H} : D| \iff D = S_{H} \iff S_{H} = gS_{H}g^{-1} \iff D = gS_{H}g^{-1} \iff p \nmid |gS_{H}g^{-1} : D|,
\]
in this case we get
\[
\text{Res}_{D}^{g_{H}g^{-1}}(x) = [D/H]\overline{f_{D,H}} = \text{Res}_{D}^{g_{H}g^{-1}}(\overline{x}),
\]
and otherwise
\[
\text{Res}_{D}^{g_{H}g^{-1}}(x) = 0 = \text{Res}_{D}^{g_{H}g^{-1}}(\overline{x}).
\]
Thus we have proved that \( x \) is \( g \)-stable.

As for \( y \) we have
\[
\text{Res}_{D}^{g_{H}g^{-1}}(y) = \sum_{H \leq I \leq S_{H}} \sum_{|S_{H} : I| = p} \text{Res}_{D}^{g_{H}g^{-1}}([S_{H}/I])\overline{f_{D,H}}
= \sum_{H \leq I \leq S_{H}} \sum_{D \subseteq I \subseteq S_{H} : |S_{H} : I| = p} [D/D \cap sIs^{-1}]\overline{f_{D,H}}
= \sum_{H \leq I \leq S_{H}} |S_{H} : D|[D/D \cap I]\overline{f_{D,H}}
= \sum_{H \leq I \leq S_{H}} [D/D \cap I]\overline{f_{D,H}}
\]
since any maximal subgroup $I$ of $S_H$ is normal and

$$p \nmid |S_H : DI| \iff DI = S_H \iff D \leq I.$$ 

Now Lemma 8(ii) shows that $\text{Res}^{S_H}_H(y) = 0$ and by an analogous argument we also have $\text{Res}^{S_H}_{H_g}((y)) = 0$, so $y$ is a $g$-stable element. 

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**References**