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Controllability of Impulsive Differential Equations

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1. INTRODUCTION

Many evolution processes are subject to short term perturbations which act instantaneously in the form of impulses. For example, biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, and frequency modulated systems do exhibit impulse effects. Thus impulsive differential equations provide a natural description of observed evolution processes of several real world problems. Moreover, the theory of impulsive differential equations is much richer than the corresponding theory of ordinary differential equations. See [1].

Control theory is the area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. To control an object means to influence its behavior so as to achieve a desired result. For basic results of what constitutes the common core of control theory, see [2].

In this paper, we initiate the study of impulsive controls relative to impulsive differential equations and obtain some simple results to demonstrate the importance of employing impulsive controls.

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2. Complete Controllability

Consider the impulsive control system

$$\begin{cases} x' = Ax + Bu, & t \neq t_k,\ t \in [0, T], \\ x(t_k^+) = [I + D^k u(t_k)]\, x(t_k), & t = t_k, \\ x(0) = v_0, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, A, B are $n \times n$, $n \times r$ matrices,

$$0 < t_1 < t_2 < \cdots < t_\rho < T,$$

for each $k = 1, 2, ..., \rho$, $D^k u(t_k)$ is a diagonal matrix such that $D^k u(t_k) = \sum_{i=1}^r d_i^k u_i(t_k) I$ and I is the identity matrix.

The control $u = u(t)$ is said to be impulsive control if at $t = t_k$, the pulses are regulated and in the rest of the given domain of definition, $u(t)$ is chosen arbitrarily.

The solution of (2.1) is given by (see [1]),

$$x(t) = e^{At} \prod_{0 < t_k < t} [I + D^k u(t_k)]\, x_0 + \int_0^t \prod_{s < t_k < t} [I + D^k u(t_k)]\, e^{A(t-s)} Bu(s)\, ds, \tag{2.2}$$

for $0 \leqslant t \leqslant T$.

To examine conditions of complete controllability of (2.1), we suppose, without loss of generality, that the final state is the origin in $\mathbb{R}^n$, and the final time is $T > 0$. Then complete controllability implies that

$$0 = x(T) = \prod_{0 < t_k < T} [I + D^k u(t_k)]\, x_0 + \int_0^T \prod_{s < t_k < T} [I + D^k u(t_k)]\, e^{-As} Bu(s)\, ds. \tag{2.3}$$

Since e^{-As} can be written as

$$e^{-As} = \sum_{i=0}^{n-1} \alpha_i(s) A^i,$$

setting

$$\beta_i = \int_0^T \prod_{s < t_k < T} [I + D^k u(t_k)]\, \alpha_i(s)\, u(s)\, ds, \tag{2.4}$$

the relation (2.3) reduces to

$$\prod_{0<t_k\leq T} [I+D^k u(t_k)]x_0 \tag{2.5}$$

$$= -\sum_{i=0}^{n-1} A^i B\beta_i = -[B \mid AB \mid \cdots \mid A^{n-1}B]\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1} \end{bmatrix}.$$

If the system (2.1) is controllable, then given any initial value $x(0)=x_0$, the relation (2.5) must be satisfied. This implies examining the following cases.

Case 1. Suppose that the $n\times nr$ matrix $[B \mid AB \mid \cdots \mid A^{n-1}B]$ is of rank n. First prescribe $u(t)$ at $t=t_k$ such that $E(j, \rho)\neq 0$, where

$$E(j, \rho)=\prod_{k=j}^{\rho} [I+D^k u(t_k)] \qquad \text{for} \quad 1\leq j\leq \rho. \tag{2.6}$$

It is not difficult to see that β_i defined by (2.4) is of the form

$$\beta_i=\int_0^{t_1} \prod_{k=1}^{\rho} [I+D^k u(t_k)]\, \alpha_i(s)\, u(s)\, ds+\int_{t_1}^{t_2} \prod_{k=2}^{\rho} [I+D^k u(t_k)]\, \alpha_i(s)\, u(s)\, ds$$

$$+\cdots+\int_{t_{\rho-1}}^{t_\rho} [I+D^\rho u(t_\rho)]\, \alpha_i(s)\, u(s)\, ds+\int_{t_\rho}^{T} \alpha_i(s)\, u(s)\, ds, \tag{2.7}$$

and consequently, the definition of $E(j, \rho)$ implies that

$$\beta_i=\beta_i(1, \rho)+\beta_i(2, \rho)+\cdots+\beta_i(\rho, \rho)+\beta_i(0, 0), \tag{2.8}$$

where

$$\beta_i(j, \rho)=\int_{t_{j-1}}^{t_j} E(j, \rho)\, \alpha_i(s)\, u(s)\, ds \qquad \text{and} \qquad \beta_i(0, 0)=\int_{t_\rho}^{T} \alpha_i(s)\, u(s)\, ds.$$

Since $u(t)$ is already chosen such that $E(1, \rho)\neq 0$ at $t=t_k$, $k=1, 2, ..., \rho$, setting

$$\beta_i^0=E^{-1}(1, \rho)[\beta_i(1, \rho)+\beta_i(2, \rho)+\cdots+\beta_i(\rho, \rho)+\beta_i(0, 0)], \tag{2.9}$$

we see that (2.5) reduces to

$$x_0 = -\sum_{i=0}^{n-1} A^i B \beta_i^0 = -[B \mid AB \mid \cdots \mid A^{n-1}B] \begin{bmatrix} \beta_0^0 \\ \beta_1^0 \\ \vdots \\ \beta_{n-1}^0 \end{bmatrix}. \tag{2.10}$$

Now $u(t)$ can be chosen uniquely and hence the impulsive system (2.1) is completely controllable in Case 1.

Case 2. Suppose that the rank of $[B \mid AB \mid \cdots \mid A^{n-1}B]$ is d, $0 < d < n$. Then there exists an invertible $n \times n$ matrix S such that the matrices $\tilde{A} = S^{-1}AS$, $\tilde{B} = S^{-1}B$ have the block structure given by

$$\tilde{A} = \begin{bmatrix} A_1, & A_2 \\ 0, & A_3 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{2.11}$$

where A_1, A_2, A_3, and B_1 are, respectively, $d \times d$, $d \times (n-d)$, $(n-d) \times (n-d)$, and $d \times r$ matrices. See [2]. Thus with the change of variable $x = Sy$, the system (2.1) is transformed into

$$\left| \begin{array}{ll} y_1' = A_1 y_1 + A_2 y_2 + B_1 u, & t \neq t_k, \\ y_2' = A_3 y_2, \quad t \neq t_k, & \\ y_1(t_k^+) = N_k^{11} y_1(t_k), & t = t_k \\ y_2(t_k^+) = N_k^{22} y_2(t_k), & t = t_k, \\ y_1(0) = y_{10}, \quad y_2(0) = y_{20}, & \end{array} \right. \tag{2.12}$$

where N_k^{11}, N_k^{22} are, respectively, $d \times d$ and $(n-d) \times (n-d)$ diagonal matrices such that

$$S^{-1} N_k S = \begin{bmatrix} N_k^{11} & 0 \\ 0 & N_k^{22} \end{bmatrix}$$

and $N_k = N_k u(t_k) = [I + D^k u(t_k)]$. Then we obtain from (2.12) the relations

$$\begin{cases} y_2(t) = e^{A_3 t} \prod_{0 < t_k < t} N_k^{22} y_{20}, \\ y_1(t) = e^{A_1 t} \prod_{0 < t_k < t} N_k^{11} y_{10} + \int_0^t \prod_{s < t_k < t} N_k^{11} e^{A_1(t-s)} [B_1 u(s) + A_2 y_2(s)] \, ds \end{cases} \tag{2.13}$$

for $0 \leqslant t \leqslant T$. Complete controllability implies

$$\begin{cases} \text{(a)} \quad 0 = \prod_{0 < t_k < T} N_k^{22} y_{20}, \\ \text{(b)} \quad 0 = \prod_{0 < t_k < T} N_k^{11} y_{10} + \int_0^T \prod_{s < t_k < T} N_k^{11} e^{-A_1 s}[B_1 u(s) + A_2 y_2(s)] \, ds. \end{cases} \tag{2.14}$$

We choose next $u(t_k)$ such that $\prod_{0 < t_k < T} N_k^{22} = 0$ so that (2.14) (a) is satisfied. We then get from (2.14) (b)

$$0 = \int_{t_\rho}^T B_1 u(s) \, ds$$

Now, $u(t)$ can be chosen such that $u(t) = 0$, $t_\rho < t \leqslant T$ and hence the system (2.1) is completely controllable in Case 2 also.

Case 3. If the matrix B in (2.1) is a null matrix so that the rank of $[B \mid AB \mid \cdots \mid A^{n-1}B]$ is zero, then the relations (2.5) are satisfied by choosing impulsive control $u(t)$ such that $\prod [I + D^k u(t_k)] = 0$. For example, a simple choice is $I + D^k u(t_k) \neq 0$ for $k = 1, 2, ..., \rho - 1$, and $I + D^\rho u(t_\rho) = 0$. Hence the system (2.1) is controllable completely in this case as well.

The foregoing considerations prove the following.

THEOREM 2.1. *The impulsive system* (2.1) *is always completely controllable.*

Remark 2.1. The simple choice of $D^k u(t_k)$ has become necessary to obtain the expression (2.2) which has helped our investigation. Any generalization of $D^k u(t_k)$ leads to difficulties in obtaining (2.2) and analyzing further until restrictive assumptions are imposed. At this time, this problem is open.

3. OPTIMAL CONTROL

We next consider the optimal control problem given by the system of impulsive control system

$$\begin{aligned} &x' = Ax + Bu, \qquad t \neq t_k, \ t \in \mathbb{R}_+, \\ &x(t_k^+) = x(t_k) + D_k u(t_k), \qquad t = t_k, \\ &x(0) = x_0, \end{aligned} \tag{3.1}$$

to determine the matrix F of the optimal control vector

$$u(t) = -Fx(t), \qquad t \neq t_k, \tag{3.2}$$

so as to minimize the performance index

$$J(x_0, u) = \int_0^\infty (x^T Q x + u^T R u)\, ds. \tag{3.3}$$

Here A, B, D_k, F, Q, and R are $n \times n$, $n \times r$, $n \times r$, $r \times n$, $n \times n$, and $r \times r$ matrices, respectively, Q, R are symmetric, positive definite matrices, and $0 < t_1 < t_2 < \cdots t_k$ such that $t_k \to \infty$ as $k \to \infty$. Substituting (3.2) in (3.1) and (3.3), we arrive at

$$\begin{cases} x' = (A - BF)x, & t \neq t_k \\ x(t_k^+) = [I - D_k F]\, x(t_k), & t = t_k \\ x(0) = x_0, \end{cases} \tag{3.4}$$

and

$$J(x_0, u) = \int_0^\infty x^T (Q + F^T R F)\, x ds. \tag{3.5}$$

Choosing the Lyapunov function $V(x) = x^T P x$ where P is an $n \times n$ positive definite matrix, we find that

$$\begin{cases} V'(x) = x^T[(A - BF)^T P + P(A - BF)]\, x = -x^T[Q + F^T R F]\, x, & t \neq t_k, \\ V(x^+) = x^T[P - PG_k - G_k^T P + G_k^T P G_k]\, x. & \\ \qquad = x^T P x - x^T \Omega_k x, \quad t = t_k, \end{cases} \tag{3.6}$$

where $G_k = D_k F$, Ω_k is an $n \times n$ matrix and the relations

$$\begin{cases} (A - BF)^T P + P(A - BF) + Q + F^T R F = 0, \\ -PG_k - G_k^T P + G_k^T P G_k + \Omega_k = 0, \quad \text{for each } k, \end{cases} \tag{3.7}$$

are satisfied. Suppose that for some λ, η_k, we have

$$x^T[Q + F^T R F] \geqslant -\lambda V(x), \quad \text{and} \quad x^T \Omega_k x \geqslant -\eta_k V(x), \tag{3.8}$$

then it follows that

$$\begin{aligned} V'(x) &\leqslant \lambda V(x), \qquad t \neq t_k, \\ V(x(t_k^+)) &\leqslant (1 + \eta_k)\, V(x(t_k)), \qquad t = t_k. \end{aligned} \tag{3.9}$$

Consequently, we conclude (see Corollary 3.2.1 in [1]) that the trivial solution of (3.4) is asymptotically stable if

$$0 \leqslant \alpha(1+\eta_k) \leqslant \exp[-\lambda(t_{k+1}-t_k)], \tag{3.10}$$

for some $\alpha > 1$ and for each k. This implies, because of (3.5) that we obtain the relation

$$0 = \lim_{t\to\infty} V(x(t)) = V(x_0) - \left[\sum_{k=1}^{\infty} M_k x(t_k) + \int_0^{\infty} x^T(s)\,[Q + F^T R F]\, x(s)\, ds\right],$$

which, in turn, leads to

$$J(x_0, u) = V(x_0) - \sum_{k=1}^{\infty} M_k x(t_k), \tag{3.11}$$

where $M_k x = x^T \Omega_k x$. Thus

$$0 \leqslant \sum_{k=1}^{\infty} M_k x(t_k) \leqslant V(x_0).$$

The minimum of $J(x_0, u)$ will occur if F can be chosen such that

$$x_0^T P x_0 = \sum_{k=1}^{\infty} M_k x(t_k). \tag{3.12}$$

Hence we have the following result.

THEOREM 3.1. *The impulsive control system* (3.1) *and* (3.2) *with the performance index* (3.3) *is optimal if* (3.7), (3.8), (3.10), *and* (3.12) *hold.*

Remark. If $\lambda < 0$ and $\eta_k = 0$ for each k in (3.9), then Theorem 3.1 reduces to the corresponding result for ODE without impulses. As it is, condition (3.10) offers other possibilities for λ and η_k showing the advantage of impulsive controls.

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