2D backward stochastic Navier–Stokes equations with nonlinear forcing✩

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Abstract

The paper is concerned with the existence and uniqueness of a strong solution to a two-dimensional backward stochastic Navier–Stokes equation with nonlinear forcing, driven by a Brownian motion. We use the spectral approximation and the truncation and variational techniques. The methodology features an interactive analysis on the basis of the regularity of the deterministic Navier–Stokes dynamics and the stochastic properties of the Itô-type diffusion processes.

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1. Introduction

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) be a complete, filtered probability space, on which defined is a standard one-dimensional Brownian motion \({W_t}_{t \geq 0}\), whose natural augmented filtration is denoted by \({\mathcal{F}_t}, t \in [0, T]\) with the finite positive time \(T\) fixed throughout this work. We denote by \(P\) the \(\sigma\)-Algebra of the predictable sets on \(\Omega \times [0, T]\) associated with \({\mathcal{F}_t}\) \(t \geq 0\). The expectation will be exclusively denoted by \(E\) and the conditional expectation on \(\mathcal{F}_s\) will be denoted by \(E_{\mathcal{F}_s}\). We use \(a.s.\) to denote that an equality or inequality holds almost surely with respect to the probability measure \(P\).

The theory of backward stochastic differential equations (BSDEs) has received an extensive study in the past two decades in connection with a wide range of applications such as in stochastic control theory, econometrics, mathematical finance, and nonlinear partial differential equations. See [8,3,9,10,14,17,28] for details. In the past few years, a rather complete theory has been established for linear backward stochastic partial differential equations (BSPDEs). See [4,6,5,7] and the references therein. For nonlinear BSPDEs, there are very restricted results, and see for example [12,11,20,24], in addition to the mentioned two works [6,5]. In [24], classical solutions are given for a very general system of BSPDEs, at a cost of strong assumption on the spacial differentiability of the coefficients and their growth. General results existing in the literature on nonlinear BSPDEs often exclude the typical nonlinearities, for example the well-known Stokes nonlinearity, suggested in the classical deterministic nonlinear partial differential equations appearing in mathematical physics, which therefore are worth receiving a separate consideration. In this paper, we concentrate our attentions to study the backward stochastic Navier–Stokes equation (BSNSE).

The standard deterministic Navier–Stokes equation describing the velocity field of an incompressible, viscous fluid motion in a domain of \(\mathbb{R}^d\) \((d = 2 \text{ or } 3)\) takes the following form:

\[
\begin{align*}
\partial_t u - v \Delta u + (u \cdot \nabla)u + \nabla p + f &= 0, & t &\geq 0; \\
\nabla \cdot u &= 0, & u(0) &= u_0,
\end{align*}
\]

(1.1)

where \(u = u(t, x)\) represents the \(d\)-dimensional velocity field of a fluid, \(p = p(t, x)\) is the pressure, \(v \in (0, \infty)\) is the viscosity coefficient, and \(f = f(t, x)\) stands for the external force. Let \((u, p)\) solve Eq. (1.1). By reversing the time and defining \(\tilde{u}(t, x) = -u(T-t, x)\), \(\tilde{p}(t, x) = p(T-t, x)\), for \(t \leq T\), then \((\tilde{u}, \tilde{p})\) satisfies the following backward Navier–Stokes equation:

\[
\begin{align*}
\partial_t \tilde{u} + v \Delta \tilde{u} + (\tilde{u} \cdot \nabla)\tilde{u} + \nabla \tilde{p} + f &= 0, & t &\leq T; \\
\nabla \cdot \tilde{u} &= 0, & \tilde{u}(T) &= \tilde{u}_0.
\end{align*}
\]

(1.2)

Note that the time-reversing makes the original initial value problem of (1.1) become a terminal value problem of (1.2).

We shall study the following two-dimensional backward stochastic Navier–Stokes equations (2D BSNSEs) in \(\mathbb{R}^2\) with a spatially periodic condition and a given terminal condition at time \(T > 0\):
\[
\begin{aligned}
&\frac{du(t, x)}{dt} + \{v \Delta u(t, x) + (u \cdot \nabla)u(t, x) + (\sigma \cdot \nabla)Z(t, x) + \nabla p(t, x)\} dt \\
&= -f(t, x, u, \nabla u, Z) dt + Z(t, x) dW_t, \quad (t, x) \in [0, T] \times \mathbb{R}^2; \\
&\text{div} \ u(t, x) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}^2; \\
u(t, x + ae_i) = u(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^2, \quad i = 1, 2; \\
u(T, x) = \xi(x), \quad (x, \omega) \in \mathbb{R}^2,
\end{aligned}
\]

with \(u, p\) and \(Z\) being the unknown random fields. Here \(\sigma\) is a measure of “correlation” between the Laplace and the Brownian motion, \({e_1, e_2}\) is the canonical basis of \(\mathbb{R}^2\), \(a > 0\) is the period in the \(i\)th direction, \(u = (u_1, u_2)\) is the random two-dimensional velocity field of a fluid in \(\mathbb{R}^2\), \(p\) is the random pressure of a fluid and \(f\) represents the external spatially periodic forces which allow for feedback involving the velocity field \(u\) and the stochastic process \(Z\) and may be inhomogeneous in time. The terminal value of the velocity field is a given spatially periodic random field \(\xi\) on the underlying probability space. For notational convenience, however, the variable \(\omega \in \Omega\) in functions and solutions will often be omitted. If the coefficients \(f, \sigma\) and the terminal value \(\xi\) are all deterministic, then the unknown field \(Z\) vanishes and the above BSNSE (1.3) automatically becomes the deterministic backward Navier–Stokes equation (1.2).

Another motivation for BSNSEs (1.3) is described below in a heuristic way, which is highlighted by the arguments of Tang [24]. For a given suitable predictable spatially periodic field \(p\), consider the following coupled system of forward–backward stochastic differential equation (for instance, see [2]):

\[
\begin{aligned}
dX^{t,x}_s &= Y^{t,x}_s \cdot ds + \nu \cdot d\tilde{W}_s + \sigma_s \cdot dW_s, \quad s \in [t, T]; \\
X^{t,x}_t &= x \in \mathbb{R}^2; \\
-dY^{t,x}_s &= f(s, X^{t,x}_s, Y^{t,x}_s, q^{t,x}_s, \zeta^{t,x}_s - q^{t,x}_s \cdot \sigma_s) ds + \nabla p(s, X^{t,x}_s) ds \\
-\nu q^{t,x}_s \cdot d\tilde{W}_s - \zeta^{t,x}_s \cdot dW_s, \quad s \in [t, T]; \\
Y^{t,x}_T &= \xi(X^{t,x}_T),
\end{aligned}
\]

where \(\{\tilde{W}_t\}_{t \geq 0}\) is a two-dimensional standard Brownian motion and is independent of \(\{W_t\}_{t \geq 0}\). Under suitable regularity and growth assumptions on the data \((f, p, \xi)\) of system (1.4), the forward SDE is expected to define a stochastic flow \(X_s(t, \cdot)\) which admits an inverse \(X^{-1}_s(t, \cdot)\) and the BSDE to define a triplet of random fields \((Y_s^{t,\cdot}, q^{t,\cdot}_s, \zeta^{t,\cdot}_s); s \in [t, T]\). Define

\[
\begin{aligned}
u(t, x) &= Y^{0,0^{-1}(0,0)}_t, \\
Z(t, x) &= \zeta^{0,0^{-1}(0,0)}_t - (\sigma_t \cdot \nabla)u(t, x), \\
\nabla u(t, x) &= q^{0,0^{-1}(0,0)}_t,
\end{aligned}
\]

all of which depend on \(p\). Choose the predictable spatially periodic field \(p\) such that the divergence of the above-defined field \(u\) vanishes. In this way, the resulting predictable field \(p\) and pair of random fields \((u, Z)\) should satisfy BSNSE (1.3). Cruziero and Shamarova [2] studied the Markovian case and associated the strong solution of the deterministic spatially periodic Navier–Stokes equations to the solution of the coupled system of FBSDEs on the group of volume-preserving diffeomorphisms of a flat torus.

It is worth noting that, though sharing the same name, our BSNSE essentially differs from that of Sundar and Yin [23] wherein the BSNSE was viewed as an inverse problem and thus, the sign of the viscous term “\(v \Delta u\)" differs. Furthermore, we allow the external force \(f\) to depend on both unknown fields \(u\) and \(Z\) in a nonlinear way, and the drift term to depend on the gradient of the second unknown field \(Z\).
In [15,16], Cauchy problems are discussed for the (forward) stochastic Navier–Stokes equations in \( \mathbb{R}^d \) driven by a random nonlinear force and a white noise. As a motivation BSNSEs emerge in regard to inverse problems to determine the stochastic noise coefficients from the terminal velocity field as observed. In [13,29], a stochastic representation in terms of Lagrangian paths for the backward incompressible Navier–Stokes equations without forcing is shown and used to prove the local existence of solutions in weighted Sobolev spaces and the global existence results in two dimensions or with a large viscosity. In [22,23], the existence and uniqueness of adapted solutions are given to the backward stochastic Lorenz equations and to the backward stochastic Navier–Stokes equations (1.3) in a bounded domain with \( \sigma \equiv 0, \nu < 0 \) and the external force \( f(t, y, z) \equiv f(t) \).

The rest of the paper is organized as follows. In Section 2, we introduce some notations, assumptions, and preliminary lemmas, and state the main result (see Theorem 2.1). In Section 3, we consider the spectral approximations and give relevant estimates. In Section 4, we prove the existence of an adapted solution to the projected finite dimensional system for our 2D BSNSE. Finally, in Section 5, we give the proof of Theorem 2.1.

2. Preliminaries and the main results

Let \( G = (0, a) \times (0, b) \) be the rectangular of the period. For \( r \in [1, \infty], L^r(G) \text{ and } L^r(\Omega \times [0, T]) \) are the conventional Banach spaces. For any nonnegative integer \( m \), we denote by \( H^m(G) \) the sub-space of \( L^2(G) \), all the derivatives of whose elements up to \( m \) still lie in \( L^2(G) \), and by \( H^m_{pe}(G) \) the space of all functions which belong to \( H^m(\mathbb{R}^2) \) (i.e., whose restriction \( u|_{0} \in H^m(\Omega) \) for every open bounded set \( \Omega \)) and which are periodic with period \( G \). \( H^m_{pe}(G) \) is a Hilbert space for the scalar product and the norm

\[
(u, v)_m = \sum_{[\alpha] \leq m} \int_D D^\alpha u(x) D^\alpha v(x) \, dx, \quad |u|_m = ((u, u)_m)^{1/2},
\]

where \( \alpha = (\alpha_1, \alpha_2), [\alpha] = \alpha_1 + \alpha_2 \), and

\[
D^\alpha = D^\alpha_1 D^\alpha_2 = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}
\]

with \( \alpha_1 \) and \( \alpha_2 \) being two nonnegative integers. The elements of \( H^m_{pe}(G) \) are characterized by their Fourier series expansion:

\[
H^m_{pe} = \left\{ u : u(x) = \sum_{k \in \mathbb{Z}^2} c_k e^{2\pi i k \cdot x/a}, \bar{c}_k = c_{-k}, \sum_{k \in \mathbb{Z}^2} |k|^{2m} |c_k|^2 < \infty \right\},
\]

and the norm \(|u|_m\) is equivalent to the norm \( \left\{ \sum_{k \in \mathbb{Z}^2} (1 + |k|^{2m}) |c_k|^2 \right\}^{1/2} \). Set a Hilbert subspace of \( H^m_{pe}(G) \):

\[
\dot{H}^m_{pe}(G) = \{ u \in H^m_{pe} : \text{in its Fourier expansion (2.1), } c_0 = 0 \},
\]

with the norm \(|u|_{m,0} = \{ \sum_{k \in \mathbb{Z}^2} |k|^{2m} |c_k|^2 \}^{1/2} \). Actually, through (2.1) and (2.2), we can define \( H^m_{pe}(G) \) and \( \dot{H}^m_{pe}(G) \) for arbitrary \( m \in \mathbb{R} \). Moreover, \( H^m_{pe}(G) \) and \( \dot{H}^m_{pe}(G) \) are in duality for all \( m \in \mathbb{R} \).
As in the framework of treating the deterministic Navier–Stokes equations (cf. [26,27,21]), we set up three phase spaces of functions of the spatial variable $x \in G$ as follows:

$$H = \{ \varphi \in \dot{H}^0_{\text{pe}}(G) \times \dot{H}^0_{\text{pe}}(G) : \text{div} \varphi = 0 \text{ in } \mathbb{R}^2 \},$$

$$V = \{ \varphi \in \dot{H}^1_{\text{pe}}(G) \times \dot{H}^1_{\text{pe}}(G) : \text{div} \varphi = 0 \text{ in } \mathbb{R}^2 \},$$

$$\mathcal{V} = V \bigcap C^\infty(\mathbb{R}^2) \times C^\infty(\mathbb{R}^2),$$

where $C^\infty(\mathbb{R}^2)$ is the set of smooth functions on $\mathbb{R}^2$. Then both $H$ and $V$ are Hilbert spaces equipped with the respective scalar product and norm

$$\langle \phi, \varphi \rangle_H := \sum_{j=1}^2 \langle \phi^i, \varphi^i \rangle_0, \quad \| \phi \|_H := \{ \langle \phi, \phi \rangle_H \}^{1/2}, \quad \phi, \varphi \in H;$$

$$\langle \phi, \varphi \rangle_V := \sum_{i,j=1}^2 \langle D_i \phi^j, D_j \varphi^j \rangle_0, \quad \| \phi \|_V := \{ \langle \phi, \phi \rangle_V \}^{1/2}, \quad \phi, \varphi \in V. \quad (2.3)$$

For simplicity, we denote $\| \cdot \|$ and $\langle \cdot, \cdot \rangle_H$ by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$, respectively. The dual product of $\psi \in \mathcal{V}$ and $\varphi \in \mathcal{V}$ will be denoted by $\langle \psi, \varphi \rangle_{\mathcal{V}}$. We have

$$\langle \phi_1, \phi_2 \rangle_{\mathcal{V}}, \mathcal{V} = \langle \phi_1, \phi_2 \rangle \quad \phi_1 \in \mathcal{V}, \quad \phi_2 \in H.$$

For a little notational abuse, we still denote by $\langle \cdot, \cdot \rangle$ the dual product $\langle \cdot, \cdot \rangle_{\mathcal{V}}, \mathcal{V}$. We shall use $| \cdot |$ to denote the absolute value or the Euclidean norm of $\mathbb{R}^2$. The set of all positive integers will be denoted by $\mathbb{Z}^+$ or $\mathbb{N}$. The Lebesgue measure of the domain $G$ will be denoted by $|G|$. Define $\mathbb{H}^m := \dot{H}^m_{\text{pe}}(G) \times \dot{H}^m_{\text{pe}}(G)$.

For any finite dimensional vector space $F$ and $a, b \in F$, we denote by $a \cdot b$ the scalar product on $F$. Throughout this paper, we assume that the external force term $f(t, u, Z)$ and the terminal value term $\xi$ are $\dot{H}^0$-valued and $\dot{H}^1$-valued, respectively, so the solution pair $(u, Z)$ of (1.3) must be $\dot{H}^0 \times \dot{H}^0$-valued. By applying the projection $P : \mathbb{H}^0 \to H$ (see [27]), since

$$H^\perp = \{ \psi \in \mathbb{H}^0 : \psi = \nabla q \text{ for some } q \in \dot{H}^1_{\text{pe}}(G) \},$$

we can formulate the above terminal value problem of the 2D BSNSE (1.3) into the following problem to solve the backward stochastic Navier–Stokes equation,

$$\begin{cases}
-du(t) = (-vAu(t) + B(u(t)) + JZ(t) + f(t, u(t), Z(t)))dt \\
-Z(t)dW_t, \quad t \in [0, T), \\
u(T) = \xi,
\end{cases} \quad (2.4)$$

where

$$\Pi(u, v) = \mathbb{P}((u \cdot \nabla)v) : V \times V \to V', \quad B(u) = \Pi(u, u) : V \to V',$$

$$JZ = \mathbb{P}((\sigma \cdot \nabla)Z), \quad \sigma(t, x) = (\sigma^1(t, x), \sigma^2(t, x)), \quad ((\sigma \cdot \nabla)Z)^i := \sum_{j=1}^2 \sigma^j Z^i_{x_j},$$

and

$$A\varphi = \mathbb{P}(-\Delta \varphi) = -\Delta \varphi,$$

whose domain is $D(A) = \mathbb{H}^2 \cap H$ and by the Poincaré inequality we can show that $V = D(A^{1/2})$. Accordingly we shall adopt the equivalent norm $\| \varphi \|_V = \| \nabla \varphi \| = \| A^{1/2} \varphi \|$. Then
all $H$, $V$, and $D(A)$ (with the graph norm $\|\cdot\|_{D(A)}$) are separable Hilbert spaces. With a little notational abuse we still use $f$ and $Z$ for the projections $\mathbb{P}(f)$ and $\mathbb{P}(Z)$, respectively.

Note that the operator $A : D(A) \to H$ is positive definite, self-adjoint, and linear, and its resolvent is compact. Therefore, all the eigenvalues of $A$ can be ordered into the increasing sequence $\{\lambda_i\}_{i=1}^\infty$. The corresponding eigenfunctions $\{e_i\}_{i=1}^\infty$ form a complete orthonormal basis for the space $\tilde{H}$, which is also a complete orthogonal (but not orthonormal) basis of the space $V$. With the identification $H = H'$ by the Riesz mapping, one has the triplet structure of compact (consequently continuous) embedding,

$$V \subset H \subset V'.$$

In what follows, $C > 0$ is a constant which may vary from line to line and we denote by $C(a_1, a_2, \ldots)$ or $C_{a_1, a_2, \ldots}$ a constant to depend on the parameters $a_1, a_2, \ldots$.

For a set $\mathbb{L}$ of stochastic processes, we always use $\mathbb{L}_\mathfrak{F}$ to denote the totality of those predictable processes in $\mathbb{L}$. For example, the space $L^p_\mathfrak{F} ([\Omega; L^r ([0, T], H)])$ with $s, r \in [1, \infty]$ consists of all the predictable $H$-valued processes in the conventional space $L^s(\Omega; L^r ([0, T], H))$ of random vector-valued variables.

For Banach space $\mathcal{B}$ and $p > 1$, define

$$L^p_\mathfrak{F} (\Omega; \mathcal{B}) := L^p (\Omega, \mathfrak{F}_T; \mathcal{B});$$

$$\mathcal{L}^p_\mathfrak{F} (0, T; \mathcal{B}) := \{ \phi \in L^p (\Omega \times [0, T]; \mathcal{B}) \mid \{ \phi (\cdot, t) \}_{0 \leq t \leq T} \text{ is a predictable process} \}.$$

Define

$$M[0, T] := L^2_\mathfrak{F} (\Omega; C ([0, T]; H)) \cap \mathcal{L}^2_\mathfrak{F} (0, T; V)$$

and

$$\mathcal{M} := \mathcal{M}[0, T] := M[0, T] \times \mathcal{L}^2_\mathfrak{F} (0, T; H)$$

equipped with the norm

$$\|(u, Z)\|_{\mathcal{M}} = \left\{ E \left[ \sup_{t \in [0, T]} \| u(t) \|^2 \right] + E \left[ \int_0^T \| u(t) \|^2 dV \right] + E \left[ \int_0^T \| Z(t) \|^2 d\mathfrak{F} \right] \right\}^{1/2}.$$

We make the following three assumptions.

**Assumption A1.** The $H$-valued mapping $f$ is defined on $\Omega \times [0, T] \times V \times H$ and for any $(u, z) \in V \times H$, $f(\cdot, u, z)$ is a predictable and $H$-valued process. Moreover, there exist a nonnegative constant $\beta$ and a nonnegative adapted process $g \in L^\infty_\mathfrak{F} ([\Omega; L^1 ([0, T])]$ such that the following conditions hold for all $v, v_1, v_2 \in V, \phi, \phi_1, \phi_2 \in H$ and $(\omega, t) \in \Omega \times [0, T]$:

1. (semi-continuous) the map $s \mapsto \langle f(t, v_1 + s v_2, \phi), v \rangle$ is continuous on $\mathbb{R}$;
2. (locally monotone)

$$\langle f(t, v_1, \phi_1) - f(t, v_2, \phi_2), v_1 - v_2 \rangle \leq \rho (v_2) \| v_1 - v_2 \|^2 + \| v_1 - v_2 \| (\| \phi_1 - \phi_2 \| + \| v_1 - v_2 \| v)$$

where $\rho : V \to (0, +\infty)$ is measurable and locally bounded.
(3) (coercivity)
\[ \langle f(t, v, \phi), v \rangle \leq g(t) + \varepsilon \|\phi\|^2 + C(\|v\|) \|v\|, \]
where \( \varepsilon : (0, 1) \to \mathbb{R}^+ \) is continuous and decreasing;

(4) (growth)
\[ \|f(t, v, \phi)\|^2 \leq \left[ g(t) + \beta(\|v\| + \|\phi\|) \right] \rho_1(v), \]
where \( \rho_1 : H \to (0, +\infty) \) is measurable and locally bounded.

**Remark 1.** The technical Assumption A1 is borrowed from the monotone operator theory (for instance see [19, Chapter 4, Page 56]) with a few modifications. An example is given by
\[ \bar{f}^i(v, \phi) = -|v_i| v_i + \psi^i(v, \nabla v_i, \phi), \quad v = (v_1, v_2) \in V, \quad \phi = (\phi_1, \phi_2) \in H, \quad i = 1, 2; \]
\[ f = \mathbb{P} \bar{f} \quad \text{with} \quad \psi^i \in C^\infty_c(\mathbb{R}^6), \]
where \( C^\infty_c(\mathbb{R}^6) \) denotes the set of all the smooth functions with compact support on \( \mathbb{R}^6 \).

**Remark 2.** Conditions (1) and (2) of Assumption A1 imply that \( f(t, x, u, z) \) is locally Lipschitz continuous with respect to \( z \) in the following sense:
\[ \|f(t, u, z) - f(t, u, Z)\|_{W^1} \leq C(\|u\|) \|z - Z\|, \]
for all \((\omega, t) \in \Omega \times [0, T], u \in V \) and \( z, Z \in H \). In fact, for any \( \phi \in V - \{0\} \) and \( \varepsilon \in \mathbb{R}^+ \),
\[ (f(t, u + \varepsilon \phi, z) - f(t, u, Z), \phi) \leq C(\|u\|) \|\phi\| \{\varepsilon \|\phi\| + \|z - Z\|\}. \]
Letting \( \varepsilon \downarrow 0 \), from the arbitrariness of \( \phi \) we conclude that the local Lipschitz continuity holds.

**Assumption A2.** The function \( \sigma : \Omega \times [0, T] \to \mathbb{R}^2 \) is \( \mathcal{F} \)-measurable and essentially bounded such that
\[ \text{ess sup}_{(\omega, t) \in \Omega \times [0, T]} |\sigma(\omega, t)|^2 < 2\nu, \]
which implies that the matrix \((2\nu I_{2 \times 2} - \sigma^* \sigma)\) is uniformly positive and therefore that there exist two constants \( \lambda > 0 \) and \( \hat{\lambda} > 1 \) such that
\[ 2\nu \|\xi\|^2 - \hat{\lambda}^2 (\sigma(\omega, t) : \xi)^2 \geq 2\lambda |\xi|^2 \]
holds almost surely for all \((t, \xi) \in [0, T] \times \mathbb{R}^2 \).

**Remark 3.** Note that, in Assumption A2, our \( \sigma \) is defined independent of the spatial variable \( x \), so we have
\[ IZ = \mathbb{P}((\sigma \cdot \nabla)Z) = (\sigma \cdot \nabla)(\mathbb{P}Z). \]
Throughout the paper, define the random map \( \Phi : \Omega \times [0, T] \times V \times H \to V' \),
\[ \Phi(t, \phi, \varphi) := -vA\phi + B(\phi) + J\varphi + f(t, \phi, \varphi), \quad (t, \phi, \varphi) \in [0, T] \times V \times H. \]

**Definition 1 (Weak Solutions).** For \( \xi \in L^\infty(\Omega; H) \) given, we say that \( (u, Z) \in \mathcal{M} \) is a weak solution to (2.4) if for any \( \varphi \in \mathcal{V} \), there holds almost surely
\[ \langle u(t), \varphi \rangle = \langle \xi, \varphi \rangle + \int_t^T \langle \Phi(s, u(s), Z(s)), \varphi \rangle ds - \int_t^T \langle Z(s), \varphi \rangle dW_s, \quad \forall t \in [0, T]. \]
**Definition 2** (Strong Solutions). For \( \xi \in L^\infty_{\mathcal{F}_T}(\Omega; V) \) given, we say that \((u, Z)\) is a strong solution to (2.4) if \((u, Z)\) is a weak solution and
\[
(u, Z) \in \left( L^2_{\mathcal{F}_T}(\Omega; C([0, T]; V)) \cap \mathcal{L}^2_{\mathcal{F}_T}(0, T; D(A)) \right) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T; V).
\]

**Remark 4.** Assume that \((u, Z) \in (L^2_{\mathcal{F}_T}(\Omega; C([0, T]; V)) \cap \mathcal{L}^2_{\mathcal{F}_T}(0, T; D(A)) \times \mathcal{L}^2_{\mathcal{F}_T}(0, T; V)\)
and that
\[
u(t) = \xi + \int_t^T \Phi(s, u(s), Z(s)) \, ds - \int_t^T Z(s) \, dW_s \quad \text{a.s. in } H.
\]
By the stochastic Fubini theorem (see, [18, Theorem 4.18]), we can check that \((u, Z)\) is a strong solution to (2.4).

The main result of the paper is stated in the following theorem.

**Theorem 2.1.** Let Assumptions A1 and A2 hold and \( \xi \in L^\infty_{\mathcal{F}_T}(\Omega; V) \). Then the 2D BSNSE problem (2.4) admits a unique strong solution such that
\[
\begin{align*}
\text{ess sup}_{(\omega, s) \in \Omega \times [0, T]} \|u(s)\|^2_V + E \left[ \int_0^T \|u(s)\|^2_{D(A)} \, ds + \int_0^T \|Z(s)\|^2_V \, ds \right] \\
\leq C \left\{ \|g\|_{L^\infty_{\mathcal{F}_T}(\Omega; L^1([0, T]))} + \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega; V)} \right\},
\end{align*}
\]
for a constant \( C \) depending on \( \|g\|_{L^\infty_{\mathcal{F}_T}(\Omega; L^1([0, T])), \|\xi\|^2_{L^\infty_{\mathcal{F}_T}(\Omega; V)}, v, \lambda, \bar{\lambda}, \beta, \varrho, \rho_1 \) and \( T \).

For the trilinear mapping
\[
b(u, v, w) := \langle \Pi(u, v), w \rangle = \sum_{i=1}^2 \sum_{j=1}^2 \int_G u_i \frac{\partial v_j}{\partial x_j} w_j \, dx, \quad u, v, w \in H,
\]
we have the following instrumental regularity.

**Lemma 2.2.** The following properties hold, where \( C_G \) is used to denote different constants only depending on \( G \).
\[
\begin{align*}
|b(u, v, w)| &\leq 2^{1/2} \|u\|_{H}^{1/2} \|u\|_{H}^{1/2} \|v\|_{V}^{1/2} \|v\|_{V}^{1/2} \|w\|_{H}^{1/2}, & u, v, w \in V, \\
|b(u, v, w)| &\leq C_G \|u\|_{H}^{1/2} \|u\|_{H}^{1/2} \|v\|_{V} \|v\|_{V} \|w\|_{H}, & u \in D(A), \ v \in V, \ w \in H, \\
|b(u, v, w)| &\leq C_G \|u\|_{H}^{1/2} \|u\|_{H}^{1/2} \|v\|_{V} \|v\|_{V} \|A_v\|_{H}^{1/2} \|w\|_{H}, & u \in V, \ v \in D(A), \ w \in H, \\
|b(u, v, w)| &\leq C_G \|u\|_{H} \|v\|_{V} \|v\|_{V} \|A_v\|_{H} \|w\|_{H}^{1/2}, & u \in H, \ v \in V, \ w \in D(A).
\end{align*}
\]
Moreover,
\[
\langle \Pi(u, v), w \rangle = -\langle \Pi(u, w), v \rangle \quad \text{for } u, v, w \in V, \\
\langle \Pi(u, v), v \rangle = 0 \quad \text{for } u, v \in V.
\]
For \( u \in D(A) \), we have \( B(u) \in H \),
\[
\|B(u)\|_{H} \leq C_G \|u\|_{H}^{1/2} \|u\|_{V} \|A_v\|_{H}^{1/2},
\]
and especially, we have
\[(\Pi(v, v), \Delta v) = 0, \text{ for } v \in D(A) \text{ (cf. [27, Lemma 3.1, Page 19])}.\]

The proof of Lemma 2.2 can be found in [25,26]. The following lemma is used. It shows the
\(L^4\)-integrability of functions in \(H^1_0(G)\) for a 2D domain \(G\), whose proof is available in [25].

**Lemma 2.3.** For any two-dimensional open set \(G\), we have
\[\|v\|_{L^4(G)} \leq 2^{1/4}\|v\|_{L^2(G)}^{1/2}\|\nabla v\|_{L^2(G)}^{1/2}, \quad v \in H^1_0(G).\] (2.12)

We have a stochastic version of the Gronwall–Bellman inequality, whose proof is referred to [8, Corollary B1, Page 386].

(The Stochastic Gronwall–Bellman Inequality): Let \((\Omega, \mathcal{F}, F, P)\) be a filtered probability
space whose filtration \(\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}\) satisfies the usual conditions. Suppose \(\{Y_s\}\) and
\(\{X_s\}\) are optional integrable processes and \(\alpha\) is a nonnegative constant. If for all \(t\), the map
\(s \mapsto E[|Y_s|]\) is continuous almost surely and
\[Y_t \leq (\geq)E\left[\int_t^T (X_s + \alpha Y_s) \, ds + Y_T \bigg| \mathcal{F}_t\right],\]
then we have almost surely
\[Y_t \leq (\geq)e^{\alpha(T-t)}E[|Y_T|]\mathcal{F}_t + E\left[\int_t^T e^{\alpha(s-t)} X_s \, ds \bigg| \mathcal{F}_t\right], \quad \forall t \in [0, T].\]

In this paper, we prove the existence and uniqueness of an adapted solution to the terminal value problem (1.3) of a two-dimensional backward stochastic Navier–Stokes equation with nonlinear forcing and the random perturbation driven by the Brownian motion. We use the spectral approximation, combined with the truncation and variational techniques, which is also a kind of compactness method. The methodology features an interactive analysis based on the regularity of the deterministic Navier–Stokes dynamics and the stochastic properties of the Itô-type diffusion processes.

3. Spectral approximations and estimates

In this section, we consider the spectral approximation of the problem (2.4) obtained by
orthogonally projecting the equation and the terminal data on the finite dimensional space
\[H_N = \text{Span}\{e_1, e_2, \ldots, e_N\}.\]

Define
\[P_N : V' \rightarrow H_N, \quad P_N f = \sum_{i=1}^N \langle f, e_i \rangle e_i, \quad f \in V'.\]

Then \(\|P_N f\|^2 = \sum_{i=1}^N |\langle f, e_i \rangle|^2\) and \(P_N\) is the orthogonal projection on \(H_N\), which is called
the spectral projection. It is worth noting that \(\|\cdot\|\) and \(\|\cdot\|_V\) are equivalent in \(H_N\) and that
\[ H_N = \mathbb{V}_N := P_N V. \] Define
\[
A^N = P_N A, \quad B^N(u) = P_N B(u), \quad J^N Z = P_N J Z := \sum_{i=1}^{N} (JZ, e_i)e_i; \tag{3.1}
\]
\[
f^N(\cdots) = P_N f(\cdots), \quad Z^N(t) = P_N Z(t), \quad \text{and} \quad \xi^N = P_N \xi.
\]

Then the projected, \(N\)-dimensional problem of approximation to problem (2.4) is defined to be
\[
\begin{aligned}
\begin{cases}
\text{du}^N(t) = \left( v A^N u^N(t) - B^N(u^N(t)) - J^N Z^N(t) - f^N(t, u^N(t), Z^N(t)) \right) \, dt \\
+ Z^N(t) \, dW_t, \quad t \in [0, T);
\end{cases}
\end{aligned}
\tag{3.2}
\]
\[ u^N(T) = \xi^N. \]

Note that the projection does not affect the Brownian motion \([W_t]_{t \geq 0}\), and also that, the finite dimensional approximation equation (3.2) does not satisfy the conditions listed in [1].

We shall conduct \textit{a priori} estimates for the adapted solution to the finite dimensional approximation problem (3.2).

First, by means of Young’s inequality
\[
ab \leq \frac{1}{p} a^p e^p + \frac{1}{q e^q} b^q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad ab \geq 0, \quad \varepsilon > 0,
\]
under Assumptions A1 and A2, we have
\[
\begin{aligned}
2\langle \Phi(t, \phi, \varphi), \phi \rangle - \left\| \varphi \right\|^2 &= 2\langle -v A \Phi + B(\phi) + J \varphi + f(t, \phi, \varphi), \phi \rangle - \left\| \varphi \right\|^2 \\
&= -2v \langle A \phi, \phi \rangle - 2\langle f(t, \phi, \varphi), \phi \rangle - 2\langle \varphi, (\sigma \cdot \nabla) \phi \rangle - \left\| \varphi \right\|^2 \\
&\leq -2v \| \phi \|_V^2 + 2g(t) + \varepsilon \| \varphi \|_V^2 + \phi(\varepsilon) \| \phi \|_V^2 + \beta \| \phi \|_V \| \phi \|_V \\
&+ 2\frac{1}{\lambda} \| \varphi \|_V \| \tilde{\lambda}(\sigma \cdot \nabla) \phi \| - \left\| \varphi \right\|^2 \quad \text{(with \( \varepsilon \) being small enough)}
\end{aligned}
\]
\[
\begin{aligned}
&\leq -2\lambda \| \phi \|_V^2 - \frac{\tilde{\lambda}^2 - 1}{2\lambda^2} \left\| \varphi \right\|^2 + 2g(t) + \frac{\tilde{\lambda}^2 - 1}{4\lambda^2} \left\| \varphi \right\|^2 + \lambda \| \phi \|_V^2 + C \| \phi \|_V^2 \\
&= -\lambda \| \phi \|_V^2 - \frac{\tilde{\lambda}^2 - 1}{4\lambda^2} \left\| \varphi \right\|^2 + 2g(t) + C \| \phi \|_V^2, \quad (\phi, \varphi) \in V \times H,
\end{aligned}
\tag{3.3}
\]
where the constant \( C \) depends only on \( \lambda, \tilde{\lambda}, \varphi \) and \( \beta \).

\[ \textbf{Lemma 3.1.} \text{ Let the conditions of Theorem 2.1 hold. If } (u^N(\cdot), Z^N(\cdot)) \in \mathcal{M} \text{ is an adapted solution of the problem (3.2), then we have almost surely} \]
\[
\begin{aligned}
\sup_{t \in [0, T]} \left\{ \| u^N(t) \|^2 + E_{\mathcal{F}_t} \left[ \int_t^T \| u^N(s) \|_V^2 + \| Z^N(s) \|_V^2 \, ds \right] \right\} \\
\leq C \left( \| g \|_{L^\infty_{\mathcal{F}}(\Omega; L^1([0, T])))} + \| \xi \|_{L^\infty_{\mathcal{F}}(\Omega; H)}^2 \right), \tag{3.4}
\end{aligned}
\]
where \( C \) is a constant depending only on \( T, v, \lambda, \tilde{\lambda}, \beta \) and \( \varphi \).
Applying the backward Itô formula to the scalar-valued, stochastic process (3.3) where

\[ \text{we have} \]

\[ \text{noting that} \]

**Proof.** Applying the backward Itô formula to the scalar-valued, stochastic process \( \|u^N(t)\|^2 \), and noting that \( (B^N(u^N(t)), u^N(t)) = 0 \), we have

\[
\|u^N(t)\|^2 = \|\xi^N\|^2 - 2\nu \int_t^T \langle A^N u^N(s), u^N(s) \rangle \, ds \\
+ 2 \int_t^T (f^N(s, u^N(s), Z^N(s)), u^N(s)) \, ds + 2 \int_t^T \langle J^N Z^N(s), u^N(s) \rangle \, ds \\
- 2 \int_t^T \langle Z^N(s), u^N(s) \rangle \, dW_s - \int_t^T \|Z^N(s)\|^2 \, ds. 
\]

In view of (3.3), we have

\[
\|u^N(t)\|^2 = \|\xi^N\|^2 + \int_t^T \left( 2\langle \Phi(s, u^N(s), Z^N(s)), u^N(s) \rangle - \|Z^N(s)\|^2 \right) \, ds \\
- 2 \int_t^T \langle Z^N(s), u^N(s) \rangle \, dW_s \\
\leq \|\xi^N\|^2 - 2 \int_t^T \langle Z^N(s), u^N(s) \rangle \, dW_s \\
+ \int_t^T \left( -\lambda \|u^N(s)\|^2_V - \frac{\lambda^2}{4\nu^2} \|Z^N(s)\|^2 + 2g(s) + C\|u^N(s)\|^2 \right) \, ds 
\]

where the constant \( C \) is independent of \( N \). Since

\[
E \left[ \sup_{t \in [T]} \left| \int_t^T \langle Z^N(s), u^N(s) \rangle \, dW_s \right| \right] \leq 2E \left[ \sup_{t \in [T]} \left| \int_t^T \langle Z^N(s), u^N(s) \rangle \, dW_s \right| \right] \\
\leq CE \left[ \left( \int_t^T \|Z^N(s)\|^2 \|u^N(s)\|^2 \, ds \right)^{1/2} \right] \text{ (by BDG inequality)}
\]
Taking the conditional expectation on both sides of the second inequality of (3.6), we obtain

\[
\|u^N(t)\|^2 + \lambda E_{\mathcal{F}_t} \left[ \int_t^T \|u^N(s)\|^2 \, ds \right] + \frac{\bar{\lambda}^2 - 1}{4\lambda^2} E_{\mathcal{F}_t} \left[ \int_t^T \|Z^N(s)\|^2 \, ds \right] \\
\leq E_{\mathcal{F}_t} \left[ \|\dot{x}\|^2 \right] + CE_{\mathcal{F}_t} \left[ \int_t^T \left( g(s) + \|u^N(s)\|^2 \right) \, ds \right], \quad \text{a.s.}
\]

From the stochastic Gronwall–Bellman inequality, it follows that

\[
\sup_{t \in [0, T)} \left\{ \|u^N(t)\|^2 + E_{\mathcal{F}_t} \left[ \int_t^T \|u^N(s)\|^2 \, dv \right] + E_{\mathcal{F}_t} \left[ \int_t^T \|Z^N(s)\|^2 \, dv \right] \right\} \\
\leq C \left( \|g\|_{L^\infty_{\mathcal{F}_T}(\Omega; L^1((0, T)))} + \|\dot{x}\|^2_{L^\infty_{\mathcal{F}_T}(\Omega; H)} \right) \\
\leq C \left( \|g\|_{L^\infty_{\mathcal{F}_T}(\Omega; L^1((0, T)))} + \|\dot{x}\|^2_{L^\infty_{\mathcal{F}_T}(\Omega; H)} \right), \quad \text{a.s.} \tag{3.8}
\]

where \( C \) is a constant depending only on \( T, \nu, \lambda, \bar{\lambda}, \beta \) and \( \varrho \).

On the other hand, as \((B(u), \Delta u) = 0\), using Itô formula, we have

\[
\|u^N(t)\|^2_v = \|x^N\|^2_v - 2v \int_t^T \langle A^N u^N(s), A^N u^N(s) \rangle \, ds \\
+ 2 \int_t^T \langle f^N(s, u^N(s), Z^N(s)), A^N u^N(s) \rangle \, ds \\
+ 2 \int_t^T \langle J^N Z^N(s), A^N u^N(s) \rangle \, ds - 2 \int_t^T \langle Z^N(s), A^N u^N(s) \rangle \, dW_s \\
- \int_t^T \|Z^N(s)\|^2_v \, ds \\
= \|x^N\|^2_v - 2v \int_t^T \|Au^N(s)\|^2 \, ds + 2 \int_t^T \langle f(s, u^N(s), Z^N(s)), Au^N(s) \rangle \, ds \\
- \int_t^T \|Z^N(s)\|^2_v \, ds - 2 \int_t^T \sum_{i=1}^2 \langle \nabla(Z^N)^i(s), (\sigma \cdot \nabla)(u^N)^i(s) \rangle \, ds \\
- 2 \int_t^T \langle Z^N(s), Au^N(s) \rangle dW_s, \quad t \in [0, T].
\]
By the integration-by-parts formula and the fact that the integrals on the boundary \( \partial G \) of \( G \) vanish, we obtain
\[
\| A \phi \|^2 = \sum_{i=1}^{2} \| \nabla \phi_i \|^2_{V}, \quad \forall \phi \in D(A).
\]

It follows that
\[
-2\nu \int_{t}^{T} \| Au^{N}(s) \|^2_{V} \, ds - 2 \int_{t}^{T} \sum_{i=1}^{2} \langle \nabla (Z^{N})^{i}(s), (\sigma \cdot \nabla)(u^{N})^{i}(s) \rangle \, ds
\]
\[
\leq -2\nu \sum_{i=1}^{2} \int_{t}^{T} \| \nabla (u^{N}(s))^{i} \|^2_{V} \, ds + \int_{t}^{T} \sum_{i=1}^{2} \| \bar{\lambda}(\sigma \cdot \nabla)(u^{N})^{i}(s) \|^2 \, ds
\]
\[
+ \bar{\lambda}^{-2} \int_{t}^{T} \| Z^{N}(s) \|^2_{V} \, ds
\]
\[
\leq -2\lambda \sum_{i=1}^{2} \int_{t}^{T} \| \nabla (u^{N}(s))^{i} \|^2_{V} \, ds + \bar{\lambda}^{-2} \int_{t}^{T} \| Z^{N}(s) \|^2_{V} \, ds, \quad \text{a.s., } \forall t \in [0, T).
\]

Therefore, we have
\[
\| u^{N}(t) \|^2_{V} \leq \| \xi \|^2_{V} - 2 \int_{t}^{T} \langle Z^{N}(s), Au^{N}(s) \rangle \, dW_{s} - \bar{\lambda}^{-2} - \frac{1}{2\bar{\lambda}^2} \int_{t}^{T} \| Z^{N}(s) \|^2_{V} \, ds
\]
\[
- 2\lambda \int_{t}^{T} \| Au^{N}(s) \|^2_{V} \, ds + \frac{1}{\lambda} \int_{t}^{T} \| f(s, u^{N}(s), Z^{N}(s)) \|^2 \, ds
\]
\[
\leq \| \xi \|^2_{V} - 2 \int_{t}^{T} \langle Z^{N}(s), Au^{N}(s) \rangle \, dW_{s} - \bar{\lambda}^{-2} - \frac{1}{2\bar{\lambda}^2} \int_{t}^{T} \| Z^{N}(s) \|^2_{V} \, ds
\]
\[
- \lambda \int_{t}^{T} \| Au^{N}(s) \|^2_{V} \, ds + \frac{1}{\lambda} \int_{t}^{T} \left[ g(s) + \beta(\| u^{N}(s) \|^2_{V} + \| Z^{N}(s) \|^2) \right] \rho_{1}(u^{N}) \, ds
\]
\[
\leq \| \xi \|^2_{V} - 2 \int_{t}^{T} \langle Z^{N}(s), Au^{N}(s) \rangle \, dW_{s} - \bar{\lambda}^{-2} - \frac{1}{2\bar{\lambda}^2} \int_{t}^{T} \| Z^{N}(s) \|^2_{V} \, ds
\]
\[
- \lambda \int_{t}^{T} \| Au^{N}(s) \|^2_{V} \, ds + C \int_{t}^{T} \left[ g(s) + \beta(\| u^{N}(s) \|^2_{V} + \| Z^{N}(s) \|^2) \right] \, ds,
\]
\[\text{(3.9)}\]

with the constant depending on \( \| g \|_{L_{\mathcal{F}_{\infty}}^{\infty}(\Omega; L^{1}([0, T]))}, \| \xi \|_{L_{\mathcal{F}_{T}}^{\infty}(\Omega; H)}, T, \nu, \lambda, \bar{\lambda}, \beta, \rho_{1} \) and \( \varphi \). In a similar way to (3.7), we have
\[
E \left[ \sup_{\tau \in [t, T]} \left| \int_{\tau}^{T} \langle Z^{N}(s), Au^{N}(s) \rangle \, dW_{s} \right| \right]
\]
\[
\leq C(N) \left\{ E \left[ \sup_{s \in [t, T]} \| u^{N}(s) \|^2_{V} \right] + E \left[ \int_{t}^{T} \| Z^{N}(s) \|^2_{V} \, ds \right] \right\}.
\]
\[\text{(3.10)}\]
Taking conditional expectation on both sides of (3.9), we obtain
\[
\|u^N(t)\|_V^2 + \lambda E_{\mathcal{F}_t}\left[\int_t^T \|Au^N(s)\|^2 \, ds\right] + \frac{\bar{\lambda}^2 - 1}{2\lambda^2} E_{\mathcal{F}_t}\left[\int_t^T \|Z^N(s)\|_V^2 \, ds\right] \\
\leq E_{\mathcal{F}_t}\left[\|\xi\|_V^2\right] + CE_{\mathcal{F}_t}\left[\int_t^T (g(s) + \|u^N(s)\|_V^2 + \|Z^N(s)\|^2) \, ds\right].
\]
(3.11)

By the stochastic Gronwall–Bellman inequality, we conclude that, with probability 1,
\[
\sup_{t \in [0, T]} \left\{\|u^N(t)\|_V^2 + E_{\mathcal{F}_t}\left[\int_t^T \|Au^N(s)\|^2 \, ds\right] + E_{\mathcal{F}_t}\left[\int_t^T \|Z^N(s)\|^2 \, ds\right]\right\} \\
\leq C \left(\|g\|_{L^\infty(\Omega; L^1([0, T]))} + \|\xi\|_{L^\infty_T(\Omega; V)}\right),
\]
(3.12)
where \(C\) is a constant depending only on \(\|g\|_{L^\infty(\Omega; L^1([0, T])))}, \|\xi\|_{L^\infty_T(\Omega; V)}, v, \bar{\lambda}, \lambda, \beta, \varphi, \varphi_1\) and \(T\).

**Lemma 3.2.** For any \(u, v \in V\) and \(\phi, \varphi \in H\),
\[
|\langle B(u) - B(v), u - v \rangle| \leq \frac{\lambda}{4} \|u - v\|_V^2 + \frac{2}{\lambda} \|v\|_V \|u - v\|^2.
\]
(3.13)

Moreover, under Assumptions A1 and A2, there exists a positive constant \(K\) depending on \(\bar{\lambda}\) such that
\[
-2\langle \Phi(t, u, \phi) - \Phi(t, v, \varphi), w \rangle + \|w\|^2 \left(K + \frac{4}{\lambda} \|v\|_V^2 + K\rho^2(v)\right) \\
+ \frac{\bar{\lambda}^2 + 1}{2\lambda^2} \|\bar{w}\|^2 \geq \lambda \|w\|^2_2,
\]
(3.14)
holds almost surely for any \(t \in [0, T]\), \(u, v \in V\) and \(\phi, \varphi \in H\) with \(w := u - v, \bar{w} := \phi - \varphi,\) and \(\Phi\) being defined by (2.6). Define
\[
r_1(t) = \int_0^t \left(K + \frac{4}{\lambda} \|u(s)\|_V^2 + K\rho^2(u(s))\right) \, ds
\]
and
\[
r_2(t) = \int_0^t \left(K + \frac{4}{\lambda} \|v(s)\|_V^2 + K\rho^2(v(s))\right) \, ds,
\]
for arbitrary \(u, v \in L^2(\Omega; L^2(0, T; V))\), and let \(w(\cdot) = u(\cdot) - v(\cdot).\) Then for any \(\phi, \varphi \in L^2(\Omega; L^2(0, T; H))\) and \(\bar{w}(\cdot) := \Phi(\cdot) - \varphi(\cdot),\) we have for \(i = 1, 2\)
\[
-\left(2\Phi(t, u, \phi) - 2\Phi(t, v, \varphi) + \frac{dr_i(t)}{dt} w, w\right) + \frac{1 + \bar{\lambda}^2}{2\lambda^2} \|\bar{w}\|^2 \geq 0, \quad a.s.
\]
(3.15)

**Proof.** Let \(w = u - v.\) Then
\[
\langle B(u) - B(v), u - v \rangle = \langle II(u, w), u \rangle + \langle II(v, w), v \rangle \\
= \langle II(u, w), u \rangle + \langle II(v, w), v \rangle = -\langle B(w), v \rangle.
\]
By the first inequality in Lemma 2.2, we can get
\[ |\langle B(u) - B(v), u - v \rangle_{\mathcal{V}, \mathcal{V}}| = |\langle B(w), v \rangle| = |\langle II(w, v), w \rangle_{\mathcal{V}, \mathcal{V}}| \]
\[ \leq 2^{1/2}||u - v||_\mathcal{V}||u - v||_\mathcal{V} ||v||_\mathcal{V} \leq \frac{\lambda}{4}||u - v||^2_\mathcal{V} + \frac{2}{\lambda}||u - v||^2_\mathcal{V}. \]

It follows from Assumptions A1 and A2 that
\[ -2v\langle Aw, w \rangle + 2\langle J\tilde{w}, w \rangle + 2\langle f(t, u, \phi) - f(t, v, \phi), w \rangle \]
\[ \leq -2v||w||^2_\mathcal{V} + 2\langle \tilde{w}, (\sigma \cdot \nabla)w \rangle + 2\rho(v)||w||^2 + 2\rho(v)||w||\langle ||w||_\mathcal{V} + ||\tilde{w}|| \rangle \]
\[ \leq -\frac{3\lambda}{2}||w||^2_\mathcal{V} + \frac{1}{\lambda^2}||\tilde{w}||^2 + \left(1 - \frac{1 + \tilde{\lambda}^2}{2\lambda^2}\right)||\tilde{w}||^2 + (C(\tilde{\lambda})\rho^2(v) + 2\rho(v))||w||^2 \]
\[ \leq -\frac{3\lambda}{2}||w||^2_\mathcal{V} + \frac{1 + \tilde{\lambda}^2}{2\lambda^2}||\tilde{w}||^2 + (K + K\rho^2(v))||w||^2, \quad (3.16) \]
where the constant $K$ only depends on $\tilde{\lambda}$. Hence, in view of (3.13), we obtain (3.14).

Then (3.15) for $i = 2$ follows from (3.14) by direct calculation. The case of $i = 1$ in (3.15) is shown in a similar way. □

4. Solutions of the finite dimensional systems

In this section, we consider the existence of an adapted solution to the projected, $N$-dimensional problem (3.2) of the 2D backward stochastic Navier–Stokes equations which we also call the finite dimensional system. To solve the finite dimensional system (3.2), we shall make use of the result of Briand et al. [1].

Consider the following BSDE:
\[ Y(t) = \zeta + \int_t^T g(s, Y(s), q(s)) \, ds - \int_t^T q(s) \, dW_s, \quad (4.1) \]
where $\zeta$ is an $\mathbb{R}^N$-valued $\mathcal{F}_T$-measurable random vector and the random function
\[ g : \Omega \times [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \]
is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^N) \times \mathcal{B}(\mathbb{R}^N)$-measurable.

The following lemma comes from [1, Theorem 4.2].

Lemma 4.1. Assume that $g$ and $\zeta$ satisfy the following four conditions.

(C1) For some $p > 1$, we have
\[ E \left[ |\zeta|^p + \left( \int_0^T |g(t, 0, 0)| \, ds \right)^p \right] < \infty. \]

(C2) There exist constants $\alpha \geq 0$ and $\mu \in \mathbb{R}$ such that almost surely we have for each $(t, y, y', z, z') \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$,
\[ |g(t, y, z) - g(t, y, z')| \leq \alpha |z - z'|, \quad (4.2) \]
\[ (y - y', g(t, y, z) - g(t, y', z)) \leq \mu |y - y'|^2 \quad (\text{monotonicity condition}). \quad (4.3) \]

(C3) The function $y \mapsto g(t, y, z)$ is continuous for any $(t, z) \in [0, T] \times \mathbb{R}^N$.  

(C4) For any $r > 0$, the random process\[\psi_r(t) := \sup_{|y| \leq r} |g(t, y, 0) - g(t, 0, 0)|, \quad t \in [0, T]\]
lies in the space $L^1_{\mathcal{F}}(\Omega \times [0, T])$. Then BSDE (4.1) admits a unique solution $(Y, q) \in L^p_{\mathcal{F}}(\Omega; C([0, T]; \mathbb{R}^N)) \times L^p_{\mathcal{F}}(\Omega; L^2([0, T]; \mathbb{R}^N))$.

**Remark 5.** Note that our finite dimensional system does not satisfy the monotonicity condition (C2). In fact, by Lemma 3.2 our finite dimensional system only satisfies a local monotonicity condition in some sense and by Remark 2 there only holds the local Lipschitz continuity with respect to $Z$, which both prevent us from directly using this lemma to our finite dimensional system.

**Lemma 4.2.** For any $M, N \in \mathbb{Z}^+$, define the function of truncation $R_M(\cdot)$ to be a $C^2$ function on $H_N$ such that for $X = \sum_{i=1}^N x_i e_i,$

$$R_M(X) := \varphi(\|X\|) = \begin{cases} 1, & \text{if } \|X\| \leq M; \\ \in (0, 1), & \text{if } M < \|X\| < M + 1; \\ 0, & \text{if } \|X\| \geq M + 1, \end{cases}$$

with $\varphi \in C^2(\mathbb{R})$. Thus $R_M(\cdot)$ is uniformly Lipschitz continuous. For each $n \in \mathbb{Z}^+$, denote $\varphi_n(z) = zn/\|z\| \vee 1, z \in H_N$ and set

$$\phi_{N,M,n}(t, y, z) = R_M(y) \frac{n}{h_M(t) \vee n} P_N \Phi(t, y, \varphi_n(z)),$$

where

$$h_M(t) = 4 \left \{ \frac{(g(t) + \beta C_N(M + 1)^2) \text{ess sup}_{\|w\| \leq M + 1} |\rho_1(w)| + C_{N,n}(M + 2)^4}{\operatorname{ess sup}_{\|w\| \leq M + 1} \|\Phi(t, w, 0)\|} \right \}^{1/2}$$

and $h_M \in L^1(\Omega \times [0, T])$. Then under Assumptions A1 and A2, $\phi_{N,M,n}$ satisfies the conditions (C2)–(C4) of Lemma 4.1.

**Proof.** Under Assumptions A1 and A2 and Remark 2, we only need to verify (4.3), i.e., there is a uniform constant $C_{N,M,n} > 0$ such that

$$\langle \phi_{N,M,n}(t, X, Z) - \phi_{N,M,n}(t, Y, Z), X - Y \rangle \leq C_{N,M,n} \|X - Y\|^2, \quad \text{a.s.,}$$

for any $X, Y, Z \in H_N$ and all $t \in [0, T]$. For any $X, Y \in H_N$, inequality (4.5) holds trivially if $\|X\| > M + 1$ and $\|Y\| > M + 1$. Thus, it is sufficient to consider the case of $\|Y\| \leq M + 1$. We have

$$\langle \phi_{N,M,n}(t, X, Z) - \phi_{N,M,n}(t, Y, Z), X - Y \rangle$$

$$= R_M(X) \frac{n}{h_M(t) \vee n} \langle \Phi(t, X, \varphi_n(Z)) - \Phi(t, Y, \varphi_n(Z)), X - Y \rangle$$

$$+ \frac{n}{h_M(t) \vee n} (R_M(X) - R_M(Y)) \langle \phi_{N,M,n}(t, Y, \varphi_n(Z)), X - Y \rangle$$

(by (3.14) of Lemma 3.2)

$$\leq \left( K + \frac{4}{\lambda} \|Y\|^2 + K \rho^2(Y)\right) \|X - Y\|^2.$$
\[ + C_M \|X - Y\|^2 \frac{n}{h_M(t) \vee n} \|\phi^{N,M,n}(t, Y, \varphi_n(Z))\| \]
\[ \leq \left( K + \frac{4}{\lambda} \|Y\|^2 + K\rho^2(Y) \right) \|X - Y\|^2 \]
\[ + C_M \|X - Y\|^2 \frac{n}{h_M(t) \vee n} (h_M(t) + C_{N,M} \cdot n) \]
\[ \leq C_{M,N,n} \|X - Y\|^2, \quad (4.6) \]
which completes the proof. \( \square \)

**Theorem 4.3.** Let Assumptions \textbf{A1} and \textbf{A2} hold. For any \( \xi \in L^\infty_\mathcal{F}_T(\Omega; V) \), the projected problem \((3.2)\) admits a unique adapted solution \((u^N(\cdot), Z^N(\cdot)) \in \mathcal{M} \) for each given positive integer \( N \), which satisfies
\[ \|(u^N, Z^N)\|_\mathcal{M}^2 \leq C \left\{ 1 + E \left[ \|\xi\|^2 \right] \right\}, \quad (4.7) \]
where \( C \) is a constant independent of \( N \).

**Proof.** \textbf{Step 1.} Let us verify the uniqueness part. Suppose \((u^N, Z^N)\) and \((v^N, Y^N)\) are two solutions of the projected problem \((3.2)\). Note that the a priori estimates in Lemma 3.1 hold for both \((u^N, Z^N)\) and \((v^N, Y^N)\). Denote by \((U^N, X^N)\) the pair of processes \((u^N - v^N, Z^N - Y^N)\). Define
\[ r(t) := \int_0^t \left[ K + \frac{4}{\lambda} \|v^N(s)\|_V^2 + K\rho^2(v^N(s)) \right] ds. \]
An application of Itô formula and Lemma 3.2 yield
\[ e^{r(t)}\|U^N(t)\|^2 = \int_t^T e^{r(s)} \left[ 2 \left( \Phi(s, u^N(s), Z^N(s)) - \Phi(s, v^N(s), Y^N(s)), U^N(s) \right) \right. \]
\[ - \|X^N\|^2 - \|U^N(s)\|^2 \left( K + \frac{4}{\lambda} \|v^N(s)\|_V^2 + K\rho^2(v^N(s)) \right) \] \]
\[ - 2 \int_t^T e^{r(s)} \langle U^N(s), X^N(s) \rangle dW_s \]
\[ \leq - \int_t^T e^{r(s)} \left( \frac{\lambda^2 - 1}{2\lambda^2} \|X^N(s)\|^2 + \lambda \|U^N(s)\|_V^2 \right) ds \]
\[ - 2 \int_t^T e^{r(s)} \langle U^N(s), X^N(s) \rangle dW_s. \]
Taking conditional expectations on both sides, we have for any \( t \in [0, T] \),
\[ e^{r(t)}\|U^N(t)\|^2 + E_{\mathcal{F}_t} \left[ \int_t^T e^{r(s)} \left( \frac{\lambda^2 - 1}{2\lambda^2} \|X^N(s)\|^2 + \lambda \|U^N(s)\|_V^2 \right) ds \right] \leq 0, \quad \text{a.s.,} \]
which implies the uniqueness.

\textbf{Step 2.} For any \( N, M, n \in \mathbb{Z}^+ \), following Lemma 4.2, we can verify that the pair \((\xi^N, \phi^{N,M,n})\) satisfies the conditions (C1)–(C4) in Lemma 4.1. Hence, by Lemma 4.1 there
exists a unique solution \((u^{N,M,n}, Z^{N,M,n}) \in \mathcal{M}\) to the following BSDE:

\[
\begin{align*}
\label{eq:5.8}
& u^{N,M,n}(t) = \xi^N + \int_t^T \phi^{N,M,n}(s, u^{N,M,n}(s), Z^{N,M,n}(s)) \, ds - \int_t^T Z^{N,M,n}(s) \, dW_s. \\
& \text{In a similar way to Lemma 3.1, we deduce that there exists a positive constant } K_1 \text{ which is independent of } N, M \text{ and } n \text{ such that}
\end{align*}
\]

\[
\sup_{t \in [0,T]} \|u^{N,M,n}(t)\|^2 + E \left[ \int_0^T \|Z^{N,M,n}(s)\|^2 \, ds \right] \leq K_1, \quad \text{a.s.} \tag{4.9}
\]

Then, letting \(M = K_1 + 1\) be fixed, we have \(R_M(u^{N,M,n}(s)) \equiv 1\). Now, we write \((u^{N,n}, Z^{N,n})\) instead of \((u^{N,M,n}, Z^{N,M,n})\) below. On the other hand, there exists a positive constant \(K_2\) independent of \(n\) such that

\[
\begin{align*}
& K + \frac{4}{\lambda^2} \|u^{N,n}(t)\|^2 + K\rho^2(u^{N,n}(t)) \leq K_2, \\
& \|\phi(t, u^{N,n}(t), \phi_1) - \phi(t, u^{N,n}(t), \phi_2)\| \leq K_2\|\phi_1 - \phi_2\|, \quad \text{dP} \otimes dt\text{-almost,}
\end{align*}
\]

holds for all \(\phi_1, \phi_2 \in H\) and \(N, n \in \mathbb{Z}^+\).

For \(j \in \mathbb{Z}^+\), set \((U^N, X^N) = (u^{N,n+j} - u^{N,n}, Z^{N,n+j} - Z^{N,n})\). Applying the Itô formula like in Step 1, we get

\[
\begin{align*}
& e^{K_2 s} \|U^N(s)\|^2 + \frac{s^2 - 1}{2\lambda^2} \int_t^T e^{K_2 s} \|X^N(s)\|^2 \, ds \\
& \leq 2 \int_t^T e^{K_2 s} \left( \phi^{N,n+j}(s, u^{N,n}(s), Z^{N,n}(s)) - \phi^{N,n}(s, u^{N,n}(s), Z^{N,n}(s)), U^N(s) \right) \, ds \\
& \quad - 2 \int_t^T e^{K_2 s} \langle U^N(s), X^N(s) \rangle \, dW_s \text{(by (4.9))} \\
& \leq 4K_1 \int_t^T e^{K_2 s} \|\phi^{N,n+j}(s, u^{N,n}(s), Z^{N,n}(s)) - \phi^{N,n}(s, u^{N,n}(s), Z^{N,n}(s))\| \, ds \\
& \quad - 2 \int_t^T e^{K_2 s} \langle U^N(s), X^N(s) \rangle \, dW_s. \tag{4.11}
\end{align*}
\]

On the other hand, by the BDG inequality, we have

\[
\begin{align*}
E \left[ \sup_{\tau \in [t,T]} \left| \int_t^\tau e^{K_2 s} \langle U^N(s), X^N(s) \rangle \, dW_s \right| \right] \\
& \leq \epsilon E \left[ \sup_{s \in [t,T]} \left( e^{K_2 s} \|U^N(s)\|^2 \right) \right] + C\epsilon E \left[ \int_t^T \|X^N(s)\|^2 e^{K_2 s} \, ds \right], \tag{4.12}
\end{align*}
\]

with the positive constant \(\epsilon\) to be determined later. Then choosing \(\epsilon\) to be small enough, we deduce from (4.11) that

\[
\begin{align*}
& \|\langle U^N, X^N \rangle\|_{\mathcal{M}}^2 \\
& \leq CE \left[ \int_0^T \|\phi^{N,n+j}(s, u^{N,n}(s), Z^{N,n}(s)) - \phi^{N,n}(s, u^{N,n}(s), Z^{N,n}(s))\| \, ds \right].
\end{align*}
\]
As
\[ \| \Phi^{N,n+j}(s,u^{N,n}(s),Z^{N,n}(s)) - \Phi^{N,n}(s,u^{N,n}(s),Z^{N,n}(s)) \| \]
\[ \leq 2K_2 \| Z^{N,n}(s) \|_{\| Z^{N,n}(s) \|_{\geq n}} + 2K_2 \| Z^{N,n}(s) \|_{\| h_{K_1}(s) \|_{\geq n}} + 2h_{K_1}(s) \|_{h_{K_1}(s) > n}, \]
(4.13)
in view of (4.9) and \( h_{K_1} \in L^1(\Omega \times [0, T]) \), we conclude that \( (u^{N,n}, Z^{N,n}) \) is a Cauchy sequence in \( \mathcal{M} \) for each \( N \). Denote the limit by \( (u^N, Z^N) \in \mathcal{M} \). It is easily checked that \( (u^N, Z^N) \) is a solution of the projected problem (3.2).

Step 3. Estimate (4.7) follows from Lemma 3.1, which completes the proof. \[ \square \]

5. Proof of Theorem 2.1

Proof of Theorem 2.1. Our proof consists of the following four steps.

Step 1. In Theorem 4.3, we have solved the projected problem (3.2) in \( \mathcal{M} \). By Lemma 3.1, we have
\[ \text{ess sup}_{(\omega,s) \in \Omega \times [0, T]} \| u^N(s) \|_V^2 + E \left[ \int_0^T \| A u^N(s) \|_V^2 + \| Z^N(s) \|_V^2 \, ds \right] \]
\[ \leq C \left( \| g \|_{L^\infty(\Omega; H^1([0, T])))} + \| \xi \|_{L^\infty(\Omega; V)}^2 \right), \]
(5.1)
where \( C \) is a constant depending only on \( \| g \|_{L^\infty(\Omega; H^1([0, T])))} \), \( \| \xi \|_{L^\infty(\Omega; H)} \), \( v, \lambda, \bar{\lambda}, \beta, \varrho, \rho_1 \) and \( T \). Since we get \( \| B(v) \|^2 \leq C_G \| v \| \| v \|_V^2 \| A v \| \) from Lemma 2.2, under Assumptions A1 and A2, we conclude
\[ E \left[ \int_0^T \left( \| B(u^N(s)) \|^2 + \| (\sigma \cdot \nabla) Z^N(s) \|^2 + \| f(s, u^N(s), Z^N(s)) \|^2 \right) \, ds \right] \leq C, \]
and thus,
\[ \| P_N \Phi(\cdot, u^N, Z^N) \|_{L^2(0,T; H)} \leq \| \Phi(\cdot, u^N, Z^N) \|_{L^2(0,T; H)} \leq C. \]
(5.2)
All the constants \( C \)s above are independent of \( N \).

Step 2. Now we consider the weak convergence. Clearly,
\[ \| \xi^N \|_V \leq \| \xi \|_V \quad \text{and} \quad \xi^N \to \xi \quad \text{strongly in} \ V, \quad \text{as} \ N \to \infty, \ a.s. \]
which implies that \( \xi^N \to \xi \) in \( L^p(\Omega; V) \) for any \( p \in (1, +\infty) \). Then the following weak and weak star convergence results in respective spaces hold: there exists a subsequence \( \{N_k\}_{k=1}^\infty \) of \( \{N\} \), such that, as \( k \to \infty \),
\[ u^{N_k}(\cdot) \overset{u}{\longrightarrow} u(\cdot) \quad \text{in} \ L^2(0, T; D(A)), \]
\[ u^{N_k}(\cdot) \overset{w}{\longrightarrow} u(\cdot) \quad \text{in} \ L^\infty(0, T; V), \]
\[ Z^{N_k}(\cdot) \overset{w}{\longrightarrow} Z(\cdot) \quad \text{in} \ L^2(0, T; V), \]
\[ \Phi(\cdot, u^{N_k}, Z^{N_k})(\cdot) \overset{w}{\longrightarrow} \Gamma(\cdot) \quad \text{in} \ L^2(0, T; H), \]
\[ P_N \Phi(\cdot, u^{N_k}, Z^{N_k})(\cdot) \overset{w}{\longrightarrow} \Psi(\cdot) \quad \text{in} \ L^2(0, T; H), \]
(5.3)
where \( u, Z, \Gamma \) and \( \Psi \) are some functions in the respective spaces.
By the Burkholder–Davis–Gundy (BDG) inequality, we have
\[
E \left[ \int_0^T \left\| \int_t^T Z^{N_k}(s) \, dW_s \right\|_V^2 \, dt \right] \leq 4L_1TE \left[ \int_0^T \|Z^{N_k}(s)\|_V^2 \, ds \right],
\]  
where \( L_1 > 0 \) is a uniform constant from the BDG inequality. Hence, as a bounded linear operator on the space \( L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]); V) \), the mapping
\[
T : Z^{N_k}(\cdot) \longmapsto \int_0^T Z^{N_k}(s) \, dW_s
\]
maps the weakly convergent sequence \( \{Z^{N_k}(\cdot)\} \) to a weakly convergent sequence \( \left\{ \int_0^T Z^{N_k}(s) \, dW_s \right\} \) in \( L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]); V) \) such that
\[
\int_0^T Z^{N_k}(s) \, dW_s \xrightarrow{w} \int_0^T Z(s) \, dW_s
\]
in \( L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]); V) \), as \( k \to \infty \).

Similarly it can be shown that, as \( k \to \infty \),
\[
\int_0^T P_{N_k} \Phi(s, u^{N_k}(s), Z^{N_k}(s)) \, ds \xrightarrow{w} \int_0^T \Psi(s) \, ds
\]
in \( L^2(\Omega \times [0, T], \mathcal{F}_T \otimes \mathcal{B}([0, T]); H) \).

Define
\[
\bar{u}(t) = \xi + \int_t^T \Psi(s) \, ds - \int_t^T Z(s) \, dW_s.
\]  
(5.5)

It is easily checked that \( \bar{u} = u \), \( P \otimes \text{dr}-\text{almost} \). In view of [19, Theorem 4.2.5], we conclude that \( u \in L^\infty(\Omega; C([0, T], V)) \) and by (5.1), we obtain
\[
\text{ess sup}_{(\omega, s) \in \Omega \times [0, T]} \|u(s)\|_V^2 + E \left[ \int_0^T \|u(s)\|_{D(A)}^2 \, ds + \int_0^T \|Z(s)\|_V^2 \, ds \right]
\]
\[
\leq C \left\{ \|g\|_{L^\infty(\Omega; L^1([0, T]))} + \|\xi\|_{L^\infty(\Omega; V)} \right\},
\]  
(5.6)

where \( C \) is a constant depending on \( \|g\|_{L^\infty(\Omega; L^1([0, T]))}, \|\xi\|_{L^\infty(\Omega; V)} \), \( v, \lambda, \bar{\lambda}, \beta, \varrho, \rho_1 \) and \( T \).

Step 3. For a notational convenience, we now use the index \( N \) instead of \( N_k \) for all the relevant subsequences.

As \( \cup_{N=1}^\infty L^2_\mathcal{F}(0, T; P_N^* H) \) is dense in \( L^2_\mathcal{F}(0, T; H) \) and it can be checked that \( \Psi = \Gamma \) on \( \cup_{N=1}^\infty L^2_\mathcal{F}(0, T; P_N^* H) \), by a density argument we have \( \Psi = \Gamma \). Thus, to show that \( (u, Z) \) is a strong solution of the 2D BSNSE problem (2.4), we only need to prove
\[
\Psi(\cdot) = \Phi(\cdot, u, Z), \quad \text{a.s.}
\]  
(5.7)

For any \( v \in L^\infty(\Omega; C([0, T]; V)) \), define
\[
r(t) = r(\omega, t) := \int_0^t \left( K + \frac{4}{\lambda} \|v(\omega, s)\|_V^2 + K\rho^2(v(\omega, s)) \right) \, ds, \quad (\omega, t) \in \Omega \times [0, T],
\]
where the constant $K$ comes from (3.14) in Lemma 3.2. Applying the Itô formula to compute $e^{r(t)}\|u^N(t)\|^2$, we have

$$E \left[ e^{r(t)}\|u^N(t)\|^2 - e^{r(T)}\|u^N(T)\|^2 \right]$$

$$= E \int_t^T e^{r(s)} \left( 2P_N \Phi(s, u^N(s), Z^N(s)), u^N(s) \right) - \|Z^N(s)\|^2$$

$$- \left( K + \frac{4}{\lambda} \|v(s)\|^2 + K\rho^2(v(s)) \right)\|u^N(s) - v(s)\|^2 \right) ds \right]$$

$$= E \int_t^T e^{r(s)} \left( 2\Phi(s, u^N(s), Z^N(s)) - \Phi(s, v(s), Z(s)), u^N(s) - v(s) \right)$$

$$- \|Z^N(s) - Z(s)\|^2 - \left( K + \frac{4}{\lambda} \|v(s)\|^2 + K\rho^2(v(s)) \right)\|u^N(s) - v(s)\|^2 \right) ds \right]$$

$$+ E \int_t^T e^{r(s)} \left( 2\Phi(s, u^N(s), Z^N(s)) - \Phi(s, v(s), Z(s)), v(s) \right)$$

$$+ 2\Phi(s, v(s), Z(s)), u^N(s) - 2\Phi(s, v(s), Z(s)) + \|Z(s)\|^2$$

$$- \left( K + \frac{4}{\lambda} \|v(s)\|^2 + K\rho^2(v(s)) \right) \left( 2\Phi(s, v(s)) - \|v(s)\|^2 \right) ds \right]$$

$$\leq E \int_t^T e^{r(s)} \left( 2\Phi(s, u^N(s), Z^N(s)) - \Phi(s, v(s), Z(s)), v(s) \right)$$

$$+ 2\Phi(s, v(s), Z(s)), u^N(s) - 2\Phi(s, v(s), Z(s)) + \|Z(s)\|^2$$

$$- \left( K + \frac{4}{\lambda} \|v(s)\|^2 + K\rho^2(v(s)) \right) \left( 2\Phi(s, v(s)) - \|v(s)\|^2 \right) ds \right].$$

Letting $N \to \infty$, by Lemma 3.2 and the lower semicontinuity, we have for any nonnegative $\varphi \in L^\infty(0, T)$,

$$E \left[ \int_0^T \varphi(t) \left( e^{r(t)}\|u(t)\|^2 - e^{r(T)}\|u(T)\|^2 \right) dt \right]$$

$$\leq \liminf_{N \to \infty} E \left[ \int_0^T \varphi(t) \left( e^{r(t)}\|u^N(t)\|^2 - e^{r(T)}\|u^N(T)\|^2 \right) dt \right]$$

$$\leq E \left[ \int_0^T \varphi(t) \left( \int_t^T e^{r(s)} \left( 2\Phi(s) - \Phi(s, v(s), Z(s)), v(s) \right) \right.$$

$$+ 2\Phi(s, v(s), Z(s)), u(s) - 2\Phi(s, v(s), Z(s)) + \|Z(s)\|^2$$

$$- \left( K + \frac{4}{\lambda} \|v(s)\|^2 + K\rho^2(v(s)) \right) \left( 2\Phi(s, v(s)) - \|v(s)\|^2 \right) ds \right] dt.$$  (5.8)
while the Itô formula yields

\begin{equation}
E \left[ e^{\psi(t)} \|u(t)\|^2 - e^{\psi(T)} \|u(T)\|^2 \right] = E \left[ \int_t^T e^{\psi(s)} \left( 2 \langle \psi(s), u(s) \rangle - \|Z(s)\|^2 - \left( K + \frac{4}{\lambda} \|v(s)\|^2 + K \rho^2(v(s)) \right) \|u(s) - v(s)\|^2 \right) ds \right].
\end{equation}

(5.9)

By substituting (5.9) into (5.8), we get

\begin{equation}
E \left[ \int_0^T \varphi(t) \left( \int_t^T e^{\psi(s)} \left( 2 \langle \psi(s), u(s) - v(s) \rangle - \left( K + \frac{4}{\lambda} \|v(s)\|^2 + K \rho^2(v(s)) \right) \|u(s) - v(s)\|^2 \right) ds \right) dr \right] \leq 0.
\end{equation}

(5.10)

Take \( v = u - \gamma \phi w \) for \( \gamma > 0 \), \( w \in V \) and \( \phi \in L^\infty_{\mathcal{F}}(\Omega \times [0, T], \mathbb{P}, \mathbb{R}) \). Then we divide by \( \gamma \) and let \( \gamma \to 0 \) to derive that

\begin{equation}
E \left[ \int_0^T \varphi(t) \left( \int_t^T e^{\psi(s)} \phi(s) \left( 2 \langle \psi - \phi(s), u(s), Z(s), w \rangle \right) ds \right) dr \right] \leq 0.
\end{equation}

(5.11)

By the arbitrariness of \( \varphi, \phi, \) and \( w \), we have

\[ \Gamma = \psi = \phi(\cdot, u, Z), \quad \text{a.e. on } \Omega \times [0, T]. \]

In view of (5.5) and keeping in mind the fact \( \tilde{u} = u \, dt \times \mathbb{P}\text{-a.e.} \), we have

\begin{equation}
\tilde{u}(t) = \xi + \int_t^T \phi(s, u(s), Z(s)) \, ds - \int_t^T Z(s) \, dW_s.
\end{equation}

(5.12)

Hence, by Remark 4 we conclude that \((u, Z)\) is a strong solution to the 2D BSNSE problem (2.4).

Step 4. In a similar way to Step 1 of the proof of Theorem 4.3, we prove the uniqueness. This completes our proof. \( \square \)

References


