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Some Cancellation Theorems about Projective Modules over Polynomial Rings

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1. INTRODUCTON

Let R be a commutative noetherian ring of dimension d, let A denote the polynomial ring $R[X_1, ..., X_n]$, and let P be a finitely generated projective A-module. H. Bass [B2, Question $(\text{XIV})_n$] has asked the following question.

Question (Bass). Is every projective A-module P of rank $\geqslant d+1$ cancellative?

R. Swan has shown that when P is stably extended from R then P is cancellative [Sw, Theorem 1.1]. B. Plumstead has given an affirmative answer to the question when $n=1$ [P, Theorem 1]. Moreover he conjectured the affirmative answer to the question for arbitrary n. We gave an affirmative answer to the question (for arbitrary n) when dim $R=1$ or dim $R=2$ and R normal [BR, Corollary 4.9].

In this paper we generalize this result in two directions. First we prove that when dim $R=2$ the question has an affirmative answer (Theorem 3.1) and thus remove the normality assumption when dim $R=2$. In Section 4 we assume that R is normal and give an affirmative answer to the question for arbitrary d (dimension of R) but restrict the number of variables to two (i.e., $n=2$) (Theorem 4.1).

2. PRELIMINARIES

Throughout this paper all rings will be commutative noetherian and all modules will be finitely generated.

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In this section we collect some definitions and results for later use; R will denote a commutative noetherian ring.

(2.1) Given a projective R-module P and an element $p \in P$ we define $O_P(p) = \{\psi(p)/\psi \in \mathrm{Hom}_R(P, R)\}$. We say that p is *unimodular* if $O_P(p) = R$.

(2.2) A projective R-module P is said to be *cancellative* if $P \oplus R \approx Q \oplus R$ implies $P \approx Q$.

(2.3) Given a projective R-module P of constant rank r we denote $\Lambda^r(P)$ by $\det(P)$. Let σ be an endomorphism of P. Then $\det(\sigma)$ will denote $\Lambda^r\sigma$. Note that $\det(\sigma) \in \mathrm{End}_R(\det(P)) = R$. The group of automorphisms σ of P with $\det(\sigma) = 1$ will be denoted by $SL(P)$. Given an ideal K of R, the kernal of the canonical map $SL(P) \to SL(P/KP)$ will be denoted by $SL(P, K)$.

(2.4) Let P be a projective module over $R[X_1, ..., X_n]$. Let $J(R, P)$ be the set of those elements a of R such that P_a is extended from R_a. Then using ideas in the proof of Theorem 1 of [Q] it can be proved that $J(R, P)$ is an ideal of R and $J(R, P) = \sqrt{J(R, P)}$. Moreover by the Quillen–Suslin theorem ht $J(R, P) \geqslant 1$. We refer to $J(R, P)$ as the Quillen ideal of P in R.

(2.5) Given R-modules M and N we write $\mathrm{End}_R(M \oplus N)$ in the matrix form as

$$\mathrm{End}_R(M \oplus N) = \begin{bmatrix} \mathrm{End}_R(M) & \mathrm{Hom}_R(M, N) \\ \mathrm{Hom}_R(N, M) & \mathrm{End}_R(N) \end{bmatrix}.$$

We conclude this section by quoting a result (only for projective modules) of Eisenbud and Evans as stated in [P, Sect. 1].

(2.6) EISENBUD–EVANS THEOREM. *Let P be a projective R-module. Let S be a subset of* $\mathrm{Spec}(R)$ *and d: $S \to \mathbb{N}$ be* generalized dimension function *such that* rank $P \geqslant 1 + d(\mathfrak{p})$ *for all $\mathfrak{p} \in S$. Let $(p, a) \in P \oplus A$ be unimodular. Then there exists $q \in P$ such that $O_P(p + aq)$ is not contained in any member $\mathfrak{p}$ of S.*

3. CANCELLATION OF PROJECTIVE MODULES OVER POLYNOMIAL EXTENSIONS OF TWO-DIMENSIONAL RINGS

In this section we prove the following theorem.

THEOREM 3.1. *Let R be a ring of dimension two. Then every projective $R[X_1, ..., X_n]$-module of rank $\geqslant 3$ is cancellative.*

For the proof of this theorem we shall need some lemmas and the following result which is implicit in the proof of Theorem 4.8 of [BR].

THEOREM 3.2. *Let R be a ring of dimension d. Let P and Q be projective modules of rank $\geqslant d+1$ over $R[X_1, ..., X_n]$ such that $P \oplus R[X_1, ..., X_n] \approx Q \oplus R[X_1, ..., X_n]$. Assume that the Quillen ideal $J(R, P)$ be such that* $\dim R/J(R, P) = 0$. *Then $P \approx Q$. Moreover if* $\operatorname{rank} P \geqslant \max(d+1, 3)$ *then any isomorphism $\sigma' : \bar{P} \xrightarrow{\sim} \bar{Q}$ can be lifted to an isomorphism $\sigma: P \xrightarrow{\sim} Q$, where the bar means* "$\text{mod}(X_1, ..., X_n)$."

LEMMA 3.3. *Let B be a ring and K be a nilpotent ideal of B. Let P be a projective $B[X_1, ..., X_n]$-module of constant rank. Assume that P contains a unimodular element. Then the canonical map $SL(P, (X_1, ..., X_n)) \to SL(P/KP, (X_1, ..., X_n))$ is surjective.*

Proof. Let τ' be an element of $SL(P/KP, (X_1, ..., X_n))$. We show that τ' can be lifted to an automorphism τ of P such that $\tau \equiv I_P \text{mod}(X_1, ..., X_n)$ and $\det(\tau) = 1$.

Let $L = KB[X_1, ..., X_n] \cap (X_1, ..., X_n)$. Then we have the *Cartesian square* of rings

$$\begin{array}{ccc} B[X_1, ..., X_n]/L & \twoheadrightarrow & B[X_1, ..., X_n]/(X_1, ..., X_n) \\ \downarrow & & \downarrow \\ B/K[X_1, ..., X_n] & \twoheadrightarrow & B/K[X_1, ..., X_n]/(X_1, ..., X_n). \end{array}$$

Since all the maps are surjective and $\tau' \in SL(P/KP, (X_1, ..., X_n))$, τ' and the identity automorphism $I_{P/(X_1, ..., X_n)P}$ can be patched together to get an element τ'' of $SL(P/LP)$.

Since L is a nilpotent ideal of $B[X_1, ..., X_n]$ and P has a unimodular element, the canonical map $SL(P) \to SL(P/LP)$ is surjective. Let $\tau \in SL(P)$ be a preimage of τ'' in $SL(P)$. Then by the construction of τ'' it follows that τ is a lift of τ' and $\tau \in SL(P, (X_1, ..., X_n))$.

LEMMA 3.4. *Let B be a ring and P be a projective $B[X_1, ..., X_n]$-module of rank $\geqslant 3$. Let K be an ideal of B such that P/KP is a free $B/K[X_1, ..., X_n]$-module of rank r $(\geqslant 3)$ and let σ' be an automorphism of P/KP which belongs to $E_r(B/K[X_1, ..., X_n])$, when considered as a matrix with respect to a basis of P/KP. Assume that $\sigma' \equiv I_{p/KP}$* mod $(X_1, ..., X_n)$. *Then σ' can be lifted to an automorphism σ of P such that $\sigma \equiv I_P \text{mod}(X_1, ..., X_n)$.*

Proof. It is easy to see that σ' (as a matrix) is a product of the matrices of the type $\beta' e_{ij}(Yf) \beta'^{-1}$, where $\beta' \in E_r(B/K)$, $f \in B/K[X_1, ..., X_n]$, and

$Y = X_l$ for some X_l. By [BR, Corollary 4.2], β' and $e_{ij}(Yf)$ (as automorphisms of P/KP) can be lifted to automorphisms β and α of P. Moreover, by looking at the proof of Proposition 4.1 of [BR] more carefully, it follows that α can be chosen that $\alpha \equiv I_P \operatorname{mod}(Y)$ and hence $\alpha \equiv I_P \operatorname{mod}(X_1, ..., X_n)$. Obviously $\beta\alpha\beta^{-1}$ is an automorphism of P such that (1) $\beta\alpha\beta^{-1}$ is a lift of $\beta' e_{ij}(Yf)\, \beta'^{-1}$ and (2) $\beta\alpha\beta^{-1} \equiv I_P \operatorname{mod}(X_1, ..., X_n)$.

Hence σ' can be lifted to an automorphism σ of P such that $\alpha \equiv I_P \operatorname{mod}(X_1, ..., X_n)$.

LEMMA 3.5. *Let $R \subsetneq R_1$ be a finite extension of reduced rings. Assume that the canonical map* $\operatorname{Spec}(R_1) \to \operatorname{Spec}(R)$ *is bijective and for every prime ideal* $\mathfrak{p}$ *of* R_1 *the inclusion map* $R/\mathfrak{p} \cap R \subsetneq R_1/\mathfrak{p}$ *is birational. Let C be the conductor ideal of R in R_1. Then there exists a ring S enjoying the properties*

(1) $R \subsetneq S \subsetneq R_1$,

(2) $(R/C)_{\text{red}} = (S/C)_{\text{red}}$,

(3) $\operatorname{ht} C < \operatorname{ht} C_1$, *where C_1 denotes the conductor ideal of S in R_1.*

Proof. Let $K =$ radical of C in R_1. Then by the hypothesis $R/R \cap K$ $(= (R/C)_{\text{red}}) \subsetneq R_1/K$ $(= (R_1/C)_{\text{red}})$ is a finite extension of reduced rings such that $Q(R/R \cap K) = Q(R_1/K)$, where for any reduced noetherian ring B, $Q(B)$ denotes the total quotient ring of B.

If $R/R \cap K = R_1/K$ then taking $S = R_1$ we are through. So assume that $R/R \cap K$ is a proper subring of R_1/K. Since R_1/K is a finite extension of $R/R \cap K$ having the same total quotient ring, we have $\operatorname{ht} C' \geqslant 1$, where C' denotes the conductor ideal of $R/R \cap K$ in R_1/K.

Let $S = R + K$. Then obviously $R \subsetneq S \subsetneq R_1$. Now $K =$ the radical ideal of C in S. Therefore $(S/C)_{\text{red}} = S/K = R/R \cap K = (R/C)_{\text{red}}$. Let C_1 be the conductor ideal of S in R_1. Then obviously $K \subset C_1$ and $C' = C_1/K$. Since $\operatorname{ht} C' \geqslant 1$ we have $\operatorname{ht} C < \operatorname{ht} C_1$.

LEMMA 3.6. *Let $R \subsetneq S$ be a finite extension of reduced rings of dimension 2 such that $Q(R) = Q(S)$ and $(R/C)_{\text{red}} = (S/C)_{\text{red}}$, where C is the conductor ideal of R in S. Let A denote $R[X_1, ..., X_n]$. Let P and Q be A-projective modules of (constant) rank $\geqslant 3$ such that $P \oplus A \approx Q \oplus A$. Assume that* $\operatorname{ht} J(S, S \otimes_R P) \geqslant 2$. *Let $\sigma' : \bar{P} \xrightarrow{\sim} \bar{Q}$ be an isomorphism (where the bar means "$\operatorname{mod}(X_1, ..., X_n)$"). Then there exist isomorphisms $\sigma_1 : S \otimes_R P \xrightarrow{\sim} S \otimes_R Q$, $\sigma_2 : R/C \otimes_R P \xrightarrow{\sim} R/C \otimes_R Q$ such that*

(1) $\bar{\sigma}_1 = 1_S \otimes_R \sigma'$,

(2) $\bar{\sigma}_2 = 1_{R/C} \otimes_R \sigma'$,

(3) $1_{(S/C)_{\text{red}}} \otimes_{R/C} \sigma_2 = 1_{(S/C)_{\text{red}}} \otimes_S \sigma_1$.

Proof. Since $\operatorname{ht} J(S, S\otimes_R P)\geqslant 2$ and $\dim S=2$, by Theorem 3.2 there exists an isomorphism $\sigma_1: S\otimes_R P \cong S\otimes_R Q$ of projective $S[X_1, ..., X_n]$-modules such that $\bar{\sigma}_1 = 1_S\otimes_R \sigma'$.

Since $Q(R)=Q(S)$ we have $\operatorname{ht} C\geqslant 1$ and therefore $\dim R/C\leqslant 1$. Therefore by Theorem 3.2 and (2.4) there exists an isomorphism $\tilde{\sigma}_2: R/C\otimes_R P\cong R/C\otimes_R Q$ of projective $R/C[X_1, ..., X_n]$-modules such that $\bar{\tilde{\sigma}}_2 = 1_{R/C}\otimes_R \sigma'$.

Let $\theta=(1_{(S/C)_{\text{red}}}\otimes_{R/C}\tilde{\sigma}_2^{-1})(1_{(S/C)_{\text{red}}}\otimes_S \sigma_1)$. Then $\theta\equiv I_{(S/C)_{\text{red}}}\otimes_R P \bmod (X_1, ..., X_n)$. Therefore $\theta\in SL((S/C)_{\text{red}}\otimes_R P, (X_1, ..., X_n))$.

Since $(R/C)_{\text{red}}=(S/C)_{\text{red}}$ and by [BR, Theorem] P has a unimodular element, by Lemma 3.3, θ can be lifted to an element $\tilde{\theta}$ of $SL(R/C\otimes_R P, (X_1, ..., X_n))$. Now we are through if we put $\sigma_2=\tilde{\sigma}_2\tilde{\theta}$.

LEMMA 3.7. *Let B be a reduced ring and let P be a projective $B[X_1, ..., X_n]$-module of* (constant) *rank* $r\geqslant 3$ *such that* $\det(P)$ *is extended from B. Let L be an ideal of B such that* $\dim B/L=1$. *Let s be an element of B such that P_s is free and* $\operatorname{ht}(L+sB)>\operatorname{ht}(L)$. *Let $\theta'\in SL(P/LP, (X_1, ..., X_n))$ be such that $\theta'\equiv I_{P/LP}$* $\bmod(\sqrt{L}/L)$. *Then θ' can be lifted to an element θ of $SL(P, (X_1, ..., X_n))$.*

Proof. We first note that by [BR, Theorem 3.1], $P/LP=\det(P/LP)\oplus F$, where F is a free $B/L[X_1, ..., X_n]$-module. Moreover, since $\det(P)$ is extended from B, $\det(P/LP)$ is extended from B/L.

Let $T=1+sB$. If $T\cap L\neq\varnothing$ then P_s is free implies P/LP is free. Therefore, since $\theta'\equiv I_{P/LP}\bmod(\sqrt{L}/L)$ and $\det\theta'=1$, $\theta'\in E_r(B/L[X_1, ..., X_n])$ when considered as a matrix with respect to some basis of P/LP. Therefore, since B is reduced, by Lemma 3.4, θ' can be lifted to an element θ of $SL(P, (X_1, ..., X_n))$.

Now we assume that $T\cap L=\varnothing$. Then $(B/L)_T$ is a semilocal ring and hence $(P/LP)_T$ is a free module over $(B/L)_T[X_1, ..., X_n]$. Therefore as before, by Lemma 3.4, θ'_T can be lifted to an element θ_1 of $SL(P_T, (X_1, ..., X_n))$.

Since P_s is free, by Lemma 3.4, θ'_s can be lifted to an element θ_2 of $SL(P_s, (X_1, ..., X_n))$.

Let $\tau=(\theta_2^{-1})_T(\theta_1)_s$. Then τ is an automorphism of free $B_{sT}[X_1, ..., X_n]$-module P_{sT} such that (1) $\tau\equiv I_{P_{st}}\bmod(X_1, ..., X_n)$ and (2) $\tau\equiv I_{P_{sT}}\bmod(L_{sT})$. But then by [P, Lemma 2 of Sect. 2], $\tau=(\delta_2)_T(\delta_1^{-1})_s$, where δ_1 (resp. δ_2) is an element of $SL(P_T, (X_1\cdots X_n)\cap LB_T[X_1, ..., X_n])$ (resp. $SL(P_s, (X_1, ..., X_n)\cap LB_s[X_1, ..., X_n])$). Therefore $\theta_1\delta_1$ and $\theta_2\delta_2$ patch up together to give an element θ of $SL(P, (X_1, ..., X_n))$ which is a lift of θ'.

Proof of Theorem 3.1. Let A denote $R[X_1, ..., X_n]$. In what follows the bar will denote "modulo$(X_1, ..., X_n)$."

Let P and Q be projective A-modules of rank $\geqslant 3$ such that $P \oplus A \approx Q \oplus A$. Then $\bar{P} \oplus \bar{A} \approx \bar{Q} \oplus \bar{A}$. Since $\bar{A} = R$ and $\dim R = 2$, by the Bass cancellation theorem [B1, Corollary 3.5, p. 184], $\bar{P} \approx \bar{Q}$. Let $\sigma' : \bar{P} \xrightarrow{\sim} \bar{Q}$ be an isomorphism. We shall show that σ' can be lifted to an isomorphism $\sigma : P \xrightarrow{\sim} Q$.

Without loss of generality we can assume that R is reduced and P is of constant rank r ($\geqslant 3$). If $\operatorname{ht} J(R, P) \geqslant 2$ then we can appeal to Theorem 3.2. So we assume that $\operatorname{ht} J(R, P) \leqslant 1$. Therefore by (2.4), $\operatorname{ht} J(R, P) = 1$.

Since $P \oplus A \approx Q \oplus A$ we have $\det(P) \approx \det(Q)$. Therefore the isomorphism $\Lambda^r \sigma' : \det(\bar{P}) \xrightarrow{\sim} \det(\bar{Q})$ can be lifted to an isomorphism $\psi : \det(P) \xrightarrow{\sim} \det(Q)$.

Let $Q(R)$ denote the total quotient ring of R. Then there exists a ring R' such that (1) $R \subsetneq R' \subsetneq Q(R)$, (2) R' is a finite R-module, and (3) the projective $R'[X_1, ..., X_n]$-module $R' \otimes_R \det(P)$ is extended from R'.

Let R_1 be the seminormalization of R in R'. Then since R_1 is seminormal in R' and $R' \otimes_{R_1} (R_1 \otimes_R \det(P))$ is extended from R', by [I, Theorem 9] the projective $R_1[X_1, ..., X_n]$-module $R_1 \otimes_R \det(P)$ is extended from R_1. Therefore by [BR, Theorem 3.1], $\operatorname{ht} J(R_1, R_1 \otimes_R P) \geqslant 2$.

From the construction of R_1 it follows that the canonical map $\operatorname{Spec}(R_1) \to \operatorname{Spec}(R)$ is bijective and for every prime ideal $\mathfrak{p}$ of R_1 the inclusion $R/R \cap \mathfrak{p} \subsetneq R_1/\mathfrak{p}$ is birational. Let C denote the conductor ideal of R in R_1. Since $C \cap J(R_1, R_1 \otimes_R P) \subset J(R, P)$, $\operatorname{ht} J(R, P) = 1$, and $\operatorname{ht} J(R_1, R_1 \otimes_R P) \geqslant 2$ we have $\operatorname{ht} C = 1$. Then by Lemma 3.5 there exists a ring S such that (1) $R \subsetneq S \subsetneq R_1$, (2) $(R/C)_{\mathrm{red}} = (S/C)_{\mathrm{red}}$, and (3) $\operatorname{ht} C_1 > \operatorname{ht} C = 1$, where C_1 is the conductor ideal of S in R_1.

Since $C_1 \cap J(R_1, R_1 \otimes_R P) \subset J(S, S \otimes_R P)$ we have $\operatorname{ht} J(S, S \otimes_R P) \geqslant 2$. Therefore by Lemma 3.6 there exist isomorphisms

$$\tilde{\sigma}_1 : S \otimes_R P \xrightarrow{\sim} S \otimes_R Q \qquad \text{and} \qquad \sigma_2 : R/C \otimes_R P \xrightarrow{\sim} R/C \otimes_R Q$$

such that (1) $\bar{\tilde{\sigma}}_1 = 1_S \otimes_R \sigma'$, (2) $\bar{\sigma}_2 = 1_{R/C} \otimes_R \sigma'$ and (3) $1_{(S/C)_{\mathrm{red}}} \otimes_S \tilde{\sigma}_1 = 1_{(S/C)_{\mathrm{red}}} \otimes_{R/C} \sigma_2$. Moreover, since by [BR, Theorem 3.1] P has a unimodular element, $\tilde{\sigma}_1$ and σ_2 can be chosen that

$$\Lambda^r(\tilde{\sigma}_1) = 1_S \otimes_R \psi \qquad \text{and} \qquad \Lambda^r \sigma_2 = 1_{R/C} \otimes_R \psi.$$

Now $1_{R_1} \otimes_S \tilde{\sigma}_1 : R_1 \otimes_R P \xrightarrow{\sim} R_1 \otimes_R Q$ is an isomorphism such that $\overline{1_{R_1} \otimes_S \tilde{\sigma}_1} = 1_{R_1} \otimes_R \sigma'$ and $\Lambda^r (1_{R_1} \otimes_S \tilde{\sigma}_1) = 1_{R_1} \otimes_R \psi$.

Let $\theta' = (1_{R_1/C} \otimes_{R/C} \sigma_2^{-1})(1_{R_1/C} \otimes_{R_1} (1_{R_1} \otimes_S \tilde{\sigma}_1))$.

Then it is easy to see that $\theta' \in SL(R_1/C \otimes_R P, (X_1, ..., X_n))$ such that $\theta' \equiv I_{R_1/C \otimes_R P} \bmod (\sqrt{C}/C)$. Therefore by Lemma 3.7, θ' can be lifted to an element θ of $SL(R_1 \otimes_R P, (X_1, ..., X_n))$.

Let $\sigma_1 = (1_{R_1} \otimes_S \tilde{\sigma}_1)\,\theta^{-1}$. Then $1_{R_{1/C}} \otimes_{R_1} \sigma_1 = 1_{R_1/C} \otimes_{R/C} \sigma_2$. Since the following square of rings

$$\begin{array}{ccc} (R[X_1, ..., X_n] =)\, A & \hookrightarrow & R_1 \otimes_R A\ (= R_1[X_1, ..., X_n]) \\ \downarrow & & \downarrow \\ (R/C[X_1, ..., X_n] =)\, R/C \otimes_R A & \hookrightarrow & R_1/C \otimes_R A\ (= R_1/C[X_1, ..., X_n]) \end{array}$$

is *Cartesian* with vertical maps surjective, σ_1 and σ_2 will patch up together to give an isomorphism $\sigma\colon P \xrightarrow{\sim} Q$ such that $\bar{\sigma} = \sigma'$ and $\Lambda^r \sigma = \psi$.

4. Cancellation of Projective Modules over $R[X, Y]$

The aim of this section is to prove the following theorem.

THEOREM 4.1. *Let R be a reduced normal ring of dimension d. Let P be a projective $R[X, Y]$-module of rank $\geqslant d+1$. Let (p, a) be a unimodular element of $P \oplus A$, where $A = R[X, Y]$. Then there exists an automorphism σ of $P \oplus A$ such that $\sigma(p, a) = (0, 1)$.*

For the proof of this theorem we need some lemmas and propositions.

LEMMA 4.2. *Let R be any ring and let A denote $R[X_1, ..., X_n]$. Let P be a projective A-module and let (p, a) be a unimodular element of $P \oplus A$ such that $(\bar{p}, \bar{a}) = (0, 1)$, where the bar means* "mod$(X_1, ..., X_n)$." *Let s_1 and s_2 be elements of R such that $s_1 R + s_2 R = R$. Let σ_i be an automorphism of $P_{s_i} \oplus A_{s_i}$ (for $i = 1, 2$) such that $\sigma_i(p, a) = (0, 1)$ and $\bar{\sigma}_i = I_{\bar{P}_{s_i} \otimes \bar{A}_{s_i}}$. Assume further that $P_{s_1 s_2}$ is extended from $R_{s_1 s_2}$. Then there exists an automorphism σ of $P \oplus A$ such that $\sigma(p, a) = (0, 1)$ and $\bar{\sigma} = I_{\bar{P} \oplus \bar{A}}$.*

Proof. Let $\theta = (\sigma_2)_{s_1}(\sigma_1^{-1})_{s_2}$. Then θ is an automorphism of $P_{s_1 s_2} \oplus A_{s_1 s_2}$ such that $\theta(0, 1) = (0, 1)$ and $\bar{\theta} = I_{\bar{P}_{s_1 s_2} \oplus \bar{A}_{s_1 s_2}}$. Therefore

$$\theta = \begin{bmatrix} \alpha & \psi \\ 0 & 1_{A_{s_1 s_2}} \end{bmatrix},$$

where α is an automorphism of $P_{s_1 s_2}$ and ψ is an element of $\mathrm{Hom}_{A_{s_1 s_2}}(P_{s_1 s_2}, A_{s_1 s_2})$ $(= P^*_{s_1 s_2})$ such that $\bar{\alpha} = I_{\bar{P}_{s_1 s_2}}$ and $\bar{\psi} = 0$.

Since $\bar{\alpha} = I_{\bar{P}_{s_1 s_2}}$ by [P, Lemma 2 of Sect. 2], $\alpha = (\alpha_2^{-1})_{s_1}(\alpha_1)_{s_2}$, where for $i = 1, 2$, α_i is an automorphism of P_{s_i} such that $\bar{\alpha}_i = I_{\bar{P}_{s_i}}$. Now $\psi(\alpha_1^{-1})_{s_2}$ is an element of $P^*_{s_1 s_2}$ such that $\overline{\psi(\alpha_1^{-1})_{s_2}} = 0$. Therefore since $s_1 R + s_2 R = R$,

$\psi(\alpha_1^{-1})_{s_2} = (\psi_1)_{s_2} - (\psi_2)_{s_1}$, where ψ_i is an element of $P_{s_i}^*$ (for $i = 1, 2$) such that $\overline{\psi_i} = 0$. Let, for $i = 1, 2$,

$$\tau_i = \begin{bmatrix} \alpha_i & \psi_i \alpha_i \\ 0 & 1_{A_{s_i}} \end{bmatrix}.$$

It is easy to see that τ_i is an automorphism of $P_{s_i} \oplus A_{s_i}$ such that $\bar{\tau}_i = I_{\bar{P}_{s_i} \oplus \bar{A}_{s_i}}$ and $\tau_i(0, 1) = (0, 1)$. Moreover $(\tau_1 \sigma_1)_{s_2} = (\tau_2 \sigma_2)_{s_1}$. Therefore $\tau_1 \sigma_1$ and $\tau_2 \sigma_2$ patch up together to give an automorphism σ of $P \oplus A$ such that $\sigma(p, a) = (0, 1)$ and $\bar{\sigma} = I_{\bar{P} \oplus \bar{A}}$.

Proposition 4.3. *Let R be a ring of dimension $d > 1$ and let $A = R[X_1, ..., X_n]$. Let P be a projective A-module of rank $\geqslant d + 1$. Let s be an element of R such that P_s is free. Let (p, a) be a unimodular element of $P \oplus A$ such that $(p, a) \equiv (0, 1) \bmod (sX_n)$. Then there exists an automorphism σ of $P \oplus A$ such that $\sigma(p, a) = (0, 1)$ and $\sigma \equiv I_{P \oplus A} \bmod (X_n)$.*

Proof. In what follows the bar will denote "modulo X_n" and the tilde will denote "modulo (sX_n)."

Since P_s is free, by [Su, Theorem 2.6] there exists an (elementary) automorphism σ_1 of $P_s \oplus A_s$ such that $\sigma_1(p, a) = (0, 1)$. Moreover since $(\bar{p}, \bar{a}) = (0, 1)$, σ_1 can be so chosen that $\bar{\sigma}_1 = I_{\bar{P} \oplus \bar{A}}$. We are going to apply Lemma 4.2 with $s_1 = s$ and σ_1 thus chosen. The rest of the proof will be devoted to defining s_2 and σ_2.

For z in P we denote the map $A \xrightarrow{1 \to z} P$ by λ_z. By [BR, Theorem 3.1] there exists p_1 in P and ψ in P^* $(= \mathrm{Hom}_A(P, A))$ such that $\psi(p_1) = 1$.

Now we define automorphisms of $P \oplus A$ as

$$\alpha_1 = \begin{bmatrix} I_P & 0 \\ \lambda_{p_1} & 1_A \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} I_P & -\psi \\ 0 & 1 \end{bmatrix}.$$

Then $\alpha_2 \alpha_1(p, a) = (p_2, a_2)$, where $p_2 = p + ap_1$ and $a_2 = -\psi(p)$. Since $\bar{p}_2$ is a unimodular element of $\bar{P}$, the element $(p_2, a_2 X_n)$ of $P \oplus A$ is unimodular. Therefore by (2.6) there exists an element q in P such that $\mathrm{ht}\, O_P(p_2 + a_2 X_n q) \geqslant \mathrm{rank}\, P \geqslant d + 1$. Therefore by [BR, Lemma 2.5] we can find a change of variables $X_i \to X_i' = X_i + X_n^{l_i}$ (for $1 \leqslant i \leqslant n - 1$) and $X_n \to X_n$ such that $O_P(p_2 + a_2 X_n q)$ contains a monic polynomial $f(X_n)$ with coefficients in B, where B denotes $R[X_1', ..., X_{n-1}']$.

Let $T = 1 + sB$. Then, since $\widetilde{p_2 + a_2 X_n q} = \tilde{p}_1$ (note that $\tilde{p} = 0$ and hence $\tilde{a}_2 = \tilde{\psi}(\tilde{p}) = 0$) by [BR, Lemma 2.3], $p_2 + a_2 X_n q$ is a unimodular element of P_T. Therefore there exists an element φ of P_T^* such that $\varphi(p_2 + a_2 X_n q) = 1$. Let $\psi(p_2 + a_2 X_n q) = c$. Then it is obvious that $\bar{c} = 1$.

Let $\theta=(1-c)\varphi+\psi_T$. Then θ is an element of P_T^* such that $\theta(p_2+a_2X_nq)=1$ and $\bar{\theta}=\bar{\psi}_T$.

Consider the automorphisms of $P_T\oplus A_T$

$$\alpha_3=\begin{bmatrix} I_{P_T} & 0 \\ \lambda_{X_nq} & 1_{A_T}\end{bmatrix},\qquad \alpha_4=\begin{bmatrix} I_{P_T} & (1-a_2)\theta \\ 0 & 1_{A_T}\end{bmatrix}$$

$$\alpha_5=\begin{bmatrix} I_{P_T} & 0 \\ \lambda_{-(p_2+a_2X_nq)} & 1_{A_T}\end{bmatrix}.$$

Then $\alpha_5\alpha_4\alpha_3(p_2,a_2)=(0,1)$.

Let $\sigma'=\alpha_5\alpha_4\alpha_3(\alpha_2)_T(\alpha_1)_T$. Then $\sigma'(p,a)=(0,1)$. We claim that $\bar{\sigma}'=I_{\bar{P}_T\oplus\bar{A}_T}$. To see this we first observe that $\bar{\alpha}_3=I_{\bar{P}_T\oplus\bar{A}_T}$. Moreover since $\bar{p}=0$, $\bar{a}_2=0$, and $\bar{\theta}=\bar{\psi}_T$, we have $(\bar{\alpha}_2)_T=\bar{\alpha}_4^{-1}$ and $(\bar{\alpha}_1)_T=\bar{\alpha}_5^{-1}$. Thus our claim is proved.

Now it is easy to see that there exists an element s_2 in T and an automorphism σ_2 of $P_{s_2}\oplus A_{s_2}$ such that $\sigma_2(p,a)=(0,1)$ and $\bar{\sigma}_2=I_{\bar{P}_{s_2}\oplus\bar{A}_{s_2}}$.

Since $s_1B+s_2B=B$ and $A=B[X_n]$, by Lemma 4.2 there exists an automorphism σ of $P\oplus A$ such that $\sigma(p,a)=(0,1)$ and $\bar{\sigma}=I_{\bar{P}\oplus\bar{A}}$.

COROLLARY 4.4. *Let R be an affine algebra of dimension d over a finite field. Let $A=R[X,Y]$ and (p,a) be a unimodular element of $P\oplus A$, where P is a projective A-module of rank $\geqslant d+1$. Then there exists an automorphism σ of $P\oplus A$ such that $\sigma(p,a)=(0,1)$.*

Proof. Without loss of generality we can assume that R is reduced and P is of constant rank. Moreover in view of [BR, Corollary 4.9] we can assume that $d\geqslant 2$.

By the Quillen–Suslin theorem there exists a non-zero-divisor s in R such that P_s is free.

In what follows the tilde means "mod(sY)."

Since $\tilde{A}$ is an affine algebra of dim $d+1$ over a *finite field* and rank$(\tilde{P})\geqslant d+1\geqslant 3$, by a result of [MMR] there exists an automorphism τ of $P\oplus A$ such that $\tilde{\tau}(\tilde{p},\tilde{a})=(0,1)$. Now we are through due to Proposition 4.3.

PROPOSITION 4.5. *Let R be a ring of dimension d such that* ht $J(R)$ *$(=$Jacobson radical of $R)\geqslant 2$. Let $A=R[X,Y]$. Let P be a projective A-module of rank $\geqslant d+1$ and let (p,a) be a unimodular element of $P\oplus A$ such that $(p,a)\equiv(0,1)$* mod(X,Y). *Then there exists an automorphism σ of $P\oplus A$ such that $\sigma(p,a)=(0,1)$ and $\sigma\equiv I_{P\oplus A}$* mod$(X,Y)$.

Proof. Without loss of generality we assume that R is reduced and P is of constant rank.

By the Quillen–Suslin theorem there exists a non-zero-divisor s such that P_s is free. We can assume that s is in $J(R)$.

In what follows the tilde denotes "mod(sY)."

Note that $\tilde{A}$ $(=R[Y]/(sY)[X])$ has generalized dimension (see [P, Sect 1] for definition) $\leqslant d$. Therefore by (2.6) there exists q in P such that $(\tilde{p}+\tilde{a}\tilde{q})$ is a unimodular element of $\tilde{P}$. Let ψ be an element of P^* such that $\tilde{\psi}(\tilde{p}+\tilde{a}\tilde{q})=1$. Now consider the automorphisms of $P\oplus A$

$$\alpha_1=\begin{bmatrix} I_P & 0 \\ \lambda_q & I_A \end{bmatrix}, \qquad \alpha_2=\begin{bmatrix} I_P & (1-a)\psi \\ 0 & I_A \end{bmatrix}, \qquad \alpha_3=\begin{bmatrix} I_P & 0 \\ \lambda_{-(p+aq)} & I_A \end{bmatrix},$$

where for z in P, λ_z denote the map $A \to^{1 \to z} P$.

Let $\tau=\alpha_3\alpha_2\alpha_1$ and let $\tau(p, a)=(p_1, a_1)$. Then it is obvious that $(\tilde{p}_1, \tilde{a}_1)=(0, 1)$. Moreover since $(p, a) \equiv (0, 1) \operatorname{mod}(X, Y)$, $\tau \equiv I_{P\oplus A} \operatorname{mod}(X, Y)$.

Applying Proposition 4.3 to the unimodular (p_1, a_1) we can find an automorphism θ of $P\oplus A$ such that $\theta(p_1, a_1)=(0, 1)$ and $\theta \equiv I_{P\oplus A} \operatorname{mod}(X, Y)$. Now we are through if we put $\sigma=\theta\tau$.

Proof of Theorem 4.1. Let the bar denote "modulo (X, Y)." Then since $\dim R=d$ and $\operatorname{rank} P \geqslant d+1$, by the Bass cancellation theorem [B1, Theorem 3.4, p. 183] there exists an automorphism τ of $P\oplus A$ such that $(\bar{p}_1, \bar{a}_1)=(0, 1)$, where $\tau(p, a)=(p_1, a_1)$.

Since R is normal, $\det(P)$ is extended from R (without loss of generality we can assume that P is of constant rank). Therefore by [BR, Theorem 3.1], ht $J(R, P) \geqslant 2$, where $J(R, P)$ denotes the Quillen ideal of P.

Applying Proposition 4.5 to the ring $R_{1+J(R,P)}$ and the unimodular element (p_1, a_1) we can find an automorphism θ_2' of $(P\oplus A)_{1+J(R,P)}$ such that $\theta_2'(p_1, a_1)=(0, 1)$ and $\bar{\theta}_2'=I_{(\bar{P}\oplus\bar{A})_{1+J(R,P)}}$. Therefore there exists an element s_1 of $J(R, P)$ and an automorphism θ_2 of $P_{s_2}\oplus A_{s_2}$ such that $\theta_2(p_1, a_1)=(0, 1)$ and $\bar{\theta}_2=I_{\bar{P}_{s_2}\oplus\bar{A}_{s_2}}$, where $s_2=1+s_1$.

Since s_1 is an element of $J(R, P)$, P_{s_1} is extended from R_{s_1}. Therefore, since $(\bar{p}_1, \bar{a}_1)=(0, 1)$ we can find an automorphism θ_1 of $P_{s_1}\oplus A_{s_1}$ such that $\theta_1(p_1, a_1)=(0, 1)$ and $\bar{\theta}_1=I_{\bar{P}_{s_1}\oplus\bar{A}_{s_1}}$.

Therefore by Lemma 4.2 there exists an automorphism θ of $P\oplus A$ such that $\theta(p_1, a_1)=(0, 1)$. Put $\sigma=\theta\tau$. Obviously σ is an automorphism of $P\oplus A$ such that $\sigma(p, a)=(0, 1)$.

Note added in proof. The question of Bass stated in the Introduction of our paper has been answered in the affirmative by R. A. Rao and later by H. Lindel (by different techniques).

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