On Finite Groups with a Certain Sylow Normalizer

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The purpose of this paper is to prove the following

**THEOREM A.** Let $G$ be a finite group, $p$ an odd prime and $P$ a Sylow $p$-subgroup of $G$. Assume

(i) $|N_G(P)/P : C_G(P)| = 2$ and $P = [P, t]$ for any $t \in N_G(P) - P \cdot C_G(P)$;

(ii) $P$ is non-cyclic and of exponent $p$, and if $p$ is a Fermat prime, then the class of $P$ is at most $p - 3$ if $p > 5$, and at most 2 if $p = 3$.

Then $G$ is $p$-solvable of $p$-length one.

By the Hall–Wielandt theorem the second part of condition (i) is equivalent (using condition (ii)) to $O_p(G) = G$. The condition imposed when $p$ is a Fermat prime is necessary since we then can consider an irreducible representation of $SL(2, p - 1)$ of degree $p - 2$ over $GF(p)$: the semidirect product gives a non-$p$-solvable group fulfilling all the conditions of the theorem except that $\text{Cl}(P) = p - 2$ (here $\text{Cl}(P)$ denotes the nilpotency class of $P$). The involvement of $SL(2, p - 1)$ is a common feature of all counter-examples of this sort.

We also prove the next theorem which is a key step in the proof of Theorem A.

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**Theorem B.** Let \( G \) be a finite group, \( p \) an odd prime and \( N \) a C-control subgroup of \( G \). Assume \( N/O_p(N) \simeq H \), where \( H \simeq P \times T \) the semidirect product of a non-cyclic group \( P \) of exponent \( p \) and \( T \) of order 2, such that \([P, T] = P\). Then \( G = O_p(G) \cdot N \), namely, \( G \) is \( p \)-solvable of \( p \)-length one.

We recall that \( N \) is a C-control subgroup of \( G \) if it contains a Sylow \( p \)-subgroup of \( G \) and for any non-trivial \( p \)-subgroup \( Q \subseteq N \) and \( g \in G \) such that \( Qg \subseteq N \) we have \( g = c \cdot n \) for some \( c \in C_p(Q) \), \( n \in N \). We define \( W \)-control subgroups similarly for \( W \) any mapping from non-trivial \( p \)-subgroups of \( G \) into subgroups of \( G \).

Denote by \( J(P) \) the Thompson subgroup generated by all Abelian subgroups of \( P \) of maximal order. By Theorem B and Theorem A of [3] we get the following

**Corollary.** Let \( P \) be a Sylow \( p \)-subgroup of a finite group \( G \). Set \( N = N_G(Z(J(P))) \) and assume that \( P \) has exponent \( p \) and \( p \geq 5 \), and \( N/O_p(N) \simeq H \) (as in Theorem B). Then \( G = O_p(G) \cdot N \), i.e., \( G \) is \( p \)-solvable of \( p \)-length one.

In Section 1 we state a more general version of these results that covers the Smith–Tyrer Theorem [9, 10]. We give a complete proof of these results without any reference to it.

**I**

In this section we prove some preliminary lemmas, and give a new more general formulation of the main results of the paper.

**Lemma 1.1.** Let \( P \) be a finite non-cyclic \( p \)-group of exponent \( p \), \( t \) an automorphism of \( P \) of order 2 such that \([P, t] = P\). Suppose \( a \in C_p(t) \); then \([C_p(a), t] \) is non-cyclic.

**Proof.** Let \( P \) be a counterexample of minimal order. Then \( P \) is not Abelian and \( t \) acts on \( Z(P) \). Assume that

\[
x \in Z(P)^* \quad \text{and} \quad x^t = x.
\]

We may apply the lemma to \( P/\langle x \rangle \) and we get \( \alpha, \beta \in P \) such that

\[
[\alpha, a], [\beta, a] \in \langle x \rangle, \quad \alpha^t = \alpha^{-1} \quad \text{and} \quad \beta^t = \beta^{-1},
\]

and \( \langle \alpha, \beta \rangle \langle x \rangle/\langle x \rangle \) is non-cyclic.

But \([\alpha, a]^t = [\alpha^{-1}, a] = [\alpha, a]^{-1} \) and \([\beta, a]^t = [\beta^{-1}, a] = [\beta, a]^{-1} \). Hence \([\alpha, a] = [\beta, a] = 1 \) and \( a, \beta \in C_p(a) \), a contradiction.
So we have that \( t \) acts fixed point freely on \( Z(P) \). Let \( x \in Z(P)^t \); then \( x^t = x^{-1} \). As before we get \( \alpha, \beta \in P \) such that
\[
[a, a], [\beta, a] \in \langle x \rangle, \quad \alpha^t = \alpha^{-1} \quad \text{and} \quad \beta^t = \beta^{-1},
\]
and \( \langle a, \beta \rangle \langle x \rangle / \langle x \rangle \) is non-cyclic. Set \( B = \langle a, \beta \rangle \). We have a homomorphism
\[
\phi : B^t \langle x \rangle / B^t \langle x \rangle \rightarrow \langle x \rangle,
\]
\[
\phi(y) = [y, a].
\]
Hence some non-trivial class in \( B^t \langle x \rangle / B^t \langle x \rangle \) is contained in \( C_p(a) \). Since it is inverted by \( t \), we have \( \gamma \in C_p(a) - \langle x \rangle \) such that \( \gamma^t = \gamma^{-1} \). Since \( P \) has exponent \( p \), \( \langle \gamma, x \rangle \) is non-cyclic and the Lemma is proved.

We are now in a position to state a slightly more general version of all three main results in the paper as follows:

In Theorem A we may replace conditions (i) and (ii) by:

(Ri) \( P \) is regular and if we set \( H = N_G(P) / O_p(N_G(P)) \), then \( H \) fulfills

(T) \( H = P \rtimes T \) the semi-direct product of a non-cyclic \( p \)-group and a group \( T \) of order 2, \( [P, T] \) is non-cyclic, and for any \( T \)-invariant non-cyclic section \( S \) of \( P \), with \( [S, T] = S \), \( [C_S(u), T] \) is non-cyclic for any \( u \in C_S(T) \); and

(Rii) If \( p \) is a Fermat prime, \( Cl(P) \leq \max(p - 2, 2) \) and for \( p \geq 5 \) no section \( G_i \) of \( G \) satisfies: \( G_i / O_p(G_i) \simeq \text{SL}(2, p - 1) \), \( O_p(G_i) \) is elementary Abelian of order \( p^{n-2} \) and \( G_i / O_p(G_i) \) acts on \( O_p(G_i) \) irreducibly.

In Theorem B and its Corollary we may use simply condition (T) as the condition on \( H \).

We recall that \( P \) is a regular \( p \)-group if for any \( a, b \in P \) we have \( (ab)^p = a^p b^p \cdot S^p \) with \( S \) an appropriate element from the commutator subgroup of the group generated by \( a \) and \( b \). Note that the local condition "\( Cl(P) \leq p - 3 \) when \( p \geq 5 \) is a Fermat prime and \( Cl(P) \leq 2 \) when \( p = 3 \)" implies (Rii).

The next coherence Lemma is due to Lluis Puig. Denote by \( R(H) \) the set of generalized characters of \( H \), by \( R_\chi(H) \) the set of generalized characters that are constant on \( p' \)-elements and by \( X_H \) the set of irreducible characters of \( H \). If \( \overline{H} \) is a homomorphic image of \( H \) we identify \( X_H \) with its corresponding subset of \( X_{\overline{H}} \).

**Lemma 1.2 (Puig).** Let \( L \) be a lattice endowed with a quadratic form holding an orthonormal basis. Assume \( P \) is a finite non-cyclic \( p \)-group, \( T \) has order 2, \( H = P \times T \) and \( P = [P, T] \). Then any isometry \( \tau : R_\chi(H) \rightarrow L \) extends to an isometry \( \sigma : R(H) \rightarrow L \).
Proof. Set \( X_T = \{ 1_T, \varepsilon \} \). We have

\[(*) \quad \text{for any } \lambda \in R(H), \quad \lambda + (1/2)(\lambda(t) - \lambda(1))\varepsilon \in R_c(H).\]

If \( P \) is Abelian a basis for \( R_c(H) \) is made of \( 1_H \) and \( \rho - 1_H - \varepsilon \), where \( \rho \) runs through all irreducible characters of \( X_H - X_T \). Consideration of the scalar product with this basis and the fact that \( |P| \geq 9 \) gives that \( \tau \) can be extended to \( R(H) \) in a unique way.

Hence we may assume that \( P \) is non-Abelian. Let \( Z \) be a \( T \)-stable subgroup of \( Z(P) \cap P' \) of order \( p \), and set \( \overline{H} = H/Z \). We have by \((*)\),

\[ R(H) = R(\overline{H}) + R_c(H) \quad \text{and} \quad R_c(\overline{H}) = R(\overline{H}) \cap R_c(H). \]

Let \( \bar{\tau} \) be the restriction of \( \tau \) to \( R_c(\overline{H}) \). By induction \( \bar{\tau} \) extends to \( \bar{\sigma}: R(\overline{H}) \to L \). So there exists a unique \( Z \)-linear map \( \sigma: R(H) \to L \), which extends both \( \bar{\sigma} \) and \( \tau \).

To prove that \( \sigma \) is an isometry, we only need to show

\[ (\sigma(\varepsilon), \sigma(\chi))_L = 0 \quad \text{for any } \chi \in X_H - X_H. \]

Let \( \chi \in X_H - X_H \) and set \( (\sigma(\varepsilon), \sigma(\chi))_L = x \), and assume \( x \neq 0 \). By \((*)\) we have

\[ (\sigma(\chi), \sigma(\chi))_L = 1 + (\chi(1) - \chi(t))x \]

\[ (\sigma(\chi), \sigma(\lambda))_L = (1/2)(\lambda(1) - \lambda(t))x \quad \text{for any } \lambda \in X_H. \]

Therefore we have

\[ 1 + (\chi(1) - \chi(t))x \geq \sum_{\lambda \in X_H} (1/4)(\lambda(1) - \lambda(t))^2 x^2 \]

\[ \geq (1/2) (|P| + |C_H(T)|) x^2 \geq (1/2) (|P| + 1) x^2. \]

Since all terms are integers and \( x \neq 0 \) we get

\[ (***) \quad \chi(1) + 1 - \chi(t) \geq (|P| + 1)/2 \]

and equality is only possible if \( \sigma(\chi) \in \sigma(R(\overline{H})). \)

If \( \chi(t) \neq 0, \chi|_P \) is irreducible and therefore \( \chi(1)^2 \leq |P| \). Hence by \((***)\) \( \chi(t) > 0 \) implies \( \chi(1)^2 > |P| \), a contradiction.

Assume that \( \chi(t) < 0 \); since \( |\chi(t)| \leq \chi(1) - 2 \), \((***)\) gives \( 4 \geq \chi(1) + 3/\chi(1) \), so we have equality and \( \sigma(\chi) \in \sigma(R(\overline{H})). \) \( \varepsilon \chi \) is an irreducible character and \( (\varepsilon \chi)(t) > 0 \). Hence \( \sigma(\varepsilon \chi) \) is orthogonal to \( \sigma(\varepsilon) \) and therefore to \( \sigma(R(\overline{H})). \) In consequence \( (\sigma(\varepsilon \chi), \sigma(\chi))_L = 0 \) and by \((*)\)

\[ (\sigma(\varepsilon \chi), \sigma(\chi))_L = (1/2)(\chi(1) + \chi(t))x, \]

a contradiction.
So we may assume that $\chi(t) = 0$. Then we have $\mu \in X_p - X_p$ such that $\mu' = \chi$ and $\mu' \neq \mu$. From (***) we get 

$$1 + 2\mu(1) \geq (|P| + 1)/2,$$

and hence $\mu(1) = 3$ and $|P| = 9$. But now, $P$ is extraspecial of order 27 and since $P = [P, T]$ we have $Z = P' = C_p(T)$. Since $\mu$ vanishes on $P - Z$ we have $\mu' = \mu$, which a contradiction and proves the Lemma.

In this section we prove Theorem A assuming Theorem B. In this section we assume that $G$ is a counterexample to Theorem A of minimal order. We set $F = GF(p)$.

**Lemma 2.1.** (i) $O_p(G) = 1$ and $p \geq 5$.

(ii) $O_p(G) \neq 1$, is elementary Abelian. We set $\bar{G} = G/O_p(G)$. Then $\bar{G}$ acts faithfully and irreducibly on $O_p(G)$.

(iii) $\bar{G}$ is a non-Abelian simple group with cyclic self-centralizing Sylow $p$-subgroup of order $p$.

**Proof.** $O_p(G) = 1$ is clear. If $p = 3$ since $\text{Cl}(P) \leq 2$, $N_G(P)$ is a $C$-control subgroup of $G$ and by Theorem B we get a contradiction.

By the Corollary to Theorem B, $N_G(Z(J(P)))$ is not $p$-solvable of $p$-length one, so by induction we have $Z(J(P)) \triangleleft G$. Hence $O_p(G) \neq 1$. If for a subgroup $1 \neq M \subset O_p(G)$ $M$ is normal in $G$, then $G/M$ fulfills the hypothesis of the Theorem since $p \mid |\bar{G}|$, and therefore $\bar{G}$ is $p$-solvable of $p$-length one, and by the Hall-Higman Theorem [3], since $P$ is regular, $G$ is $p$-solvable of $p$-length one. So no such $M$ exists and $O_p(G)$ is elementary Abelian and $\bar{G}$ acts irreducibly on $O_p(G)$.

Since $G$ does not fulfill the hypothesis of the Theorem it has a cyclic Sylow $p$-subgroup. Suppose $S \supset O_p(G)$ is a proper normal subgroup of $G$. Since $O^p(G) = G$, set $N = P \cdot S \neq G$. We also have $|N_G(P)/P \cdot C_N(P)| \leq 2$. If $|N_G(P)/P \cdot C_N(P)| = 1$, by the Hall-Wielandt Theorem $N$ and hence $S$ are $p$-nilpotent. Since $O_p(S) \subseteq O_p(G) = 1$, we get that $S$ is a $p$-group and $S \subseteq O_p(G)$. So $|N_G(P)/P \cdot C_N(P)| = 2$; take $t \in N_G(P) - P \cdot C_N(P)$, a 2-element. By construction of $N$, $t \in S$ and we get, since $O^p(G) = G$, $P = [P, t] \subseteq S$. Hence by induction we get $S = O_p(S)$. $N_S(P)$, and since $O_p(S) = 1$, $P$ is characteristic in $S$ and normal in $G$, a contradiction. Hence $\bar{G}$ is simple.

Since $[P, T] = P$ the normal subgroup of $G$, $C_G(O_p(G))$ is proper and hence $C_G(O_p(G)) = O_p(G)$. That completes the proof of (ii).

Now suppose $g \in G$ is a $p'$-element such that $[P, g] \subseteq O_p(G)$. Since
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\[ g \in N_G(P) \text{ and } P \neq O_p(G) \] we get \[ g \in C_G(P) \subseteq C_G(O_p(G)) = O_p(G). \] So \( \bar{G} \) has a self-centralizing Sylow \( p \)-subgroup.

Since \( P \) is regular and \( O_p(G) \) has exponent \( p \), every \( p \)-power in \( P \) commutes with \( O_p(G) \). Hence, since \( \bar{G} \) is faithful on \( O_p(G) \), \(|\bar{P}| = p \). That completes the proof of (iii) and of the Lemma.

**Lemma 2.2.** Set \( M = O_p(G) \). We have \( \dim_p(M) = \text{Cl}(P) \leq p - 1 \).

**Proof.** Set \( K = \bar{G}, Q = \bar{P} \) and \( N = \overline{N_G(P)} \). Since \( M \) is an irreducible \( FK \)-module, by the theory of vertices and sources (see, for example, [1, p. 339]) we know there exists an indecomposable \( FN \)-module \( L \) such that

\[ M \mid L^K \]

\( (M \) is a direct summand of the induced module \( L^K \)). Now restricting to \( Q \) and applying Mackey's decomposition formula [1, p. 497] we get

\[ M \mid_Q \bigoplus \ L^x \mid_N x \cap Q \mid_Q, \]

where the sum is taken over a complete set of representatives of the double cosets \( N \times Q \) in \( K \). Suppose the minimal polynomial of the action on \( M \) of an element \( x \) in \( Q \) was \( (X - 1)^p \). Then there exists \( y \in M \) such that \( [y, x; p - 1] \neq 1 \) (where \( [y, x; 0] = y \), and \( [y, x; n] = [[y, x; n - 1], x] \)) and \( (xy)^p = [y, x; p - 1] \neq 1 \) against the regularity of \( P \). So the minimal polynomial of any element of \( Q \) is a divisor of \( (X - 1)^{p-1} \). We can suppress from the summation in (*) all terms where \( N^x \cap Q = 1 \). We get

\[ M \mid_Q \mid L \mid_Q. \]

By the structure of the indecomposable \( p \)-modules of the dihedral group \( N \) (see, for example, [11]), \( L \mid_Q \) is indecomposable, and hence \( M \mid_Q = L \mid_Q \), and \( \dim_q(L) \) is the degree of the minimal polynomial of a non-trivial element of \( Q \), which is easily seen to be \( \text{Cl}(P) \). We also have seen this to be at most \( p - 1 \), and we have the result.

**Lemma 2.3.** If Theorem B is true then so is Theorem A.

**Proof.** By Lemmas 2.1 and 2.2, \( K \) fulfills the hypothesis of a Theorem of Feit, Theorem 5.1 in [2], and we see that \( p > 5 \) is a Fermat prime and \( K \approx SL(2, p - 1) \). But now it follows that, since \( M \) is not of defect zero; \( \dim_q(M) = p - 2 \) (and \( M \) comes from the reduction "modulo p" of an ordinary irreducible representation of \( SL(2, p - 1) \)). This contradicts condition (Rii) and completes the proof of Theorem A.
In this section we prove Theorem B without quoting any of the results of Section 2. For this section let $G$ be a counterexample to Theorem B of minimal order. For any non-trivial $p$-subgroup $A$ of $G$ we set $W(A) = O_p(C_G(A))$. We identify $P$ with a Sylow $p$-subgroup of $G$, and we let $t \in T^*$ be also identified with an element of $G$.

**Lemma 3.1.** (i) If $1 \neq A \subseteq B$ are $p$-subgroups of $G$ we have that $C_G(A)$ is $p$-solvable of $p$-length one and $W(B) = W(A) \cap C_G(B)$. $N$ is a $W$-control subgroup of $G$.

(ii) Suppose $K \triangleleft G$ and $P \subseteq K$ and $W(A) \subseteq K$ for all non-trivial subgroup $A \in P$. Then $K = G$.

(iii) There exists $N_1 < N$ such that $F = N/N_1$ is a Frobenius group of order $2p^2$ with Frobenius kernel $Q = N_0/N_1$ of order $p^2$; and a central extension $\tilde{G}$ of $G$ by $Z$ of order $p$ such that $\tilde{N}$ is a $C$-control subgroup of $\tilde{G}$ and $\tilde{N}_1$ splits into $Z \times N_1$ (identifying $N_1$ with a subgroup of $G$) and $\tilde{N}_1/N_1 = \tilde{Q}$ is an extraspecial group of order $p^3$ and exponent $p$. We also set $\tilde{N}/N_1 = \tilde{F}$.

**Proof.** (i) Take any $a \in P^*$. It is clear that $C_G(a) \neq G$, and that $C_{N}(a)$ is a $C$-control subgroup of $C_G(a)$. We also have

$$C_{N}(a)/O_{p^*}(C_{N}(a)) \simeq C_H(a).$$

But now $C_H(a)$ is either a $p$-group or a group fulfilling $(T)$; hence we have either for example by [7, Corollary 2] or by induction $C_G(a)$ $p$-solvable of $p$-length one. We have for any $A \neq 1$ a $p$-subgroup of $G$,

$$C_G(A) = O_{p}(C_G(A)) \cdot C_N(A). \quad (*)$$

Now by induction we reduce the first equation of (i) to the case $|B : A| = p$, $B/A$ acts on $C_G(A)$ and $O_{p^*}(C_G(B)) = O_{p^*}(C_G(A)) \cap C_G(B)$ since $C_G(A)$ is $p$-solvable. (see also [6, Chap. VI, Proposition 5]). This proves the first part of (i).

Now since $N$ is a $C$-control subgroup of $G$, by $(*)$, $N_G(A) = W(A) \cdot N_N(A)$. Suppose $g \in G$ and $A, A^g \subseteq N$. Then $g = c \cdot n$ for some $c \in C_G(A)$ $n \in N$, since $N$ is a $C$-control subgroup of $G$. But now $c = c_1 \cdot n_1$ with $c_1 \in W(A)$ and $n_1 \in N_N(A)$. So $g = c_1 \cdot (n_1 \cdot n)$ with $c_1 \in W(A)$ and $n_1 \cdot n \in N$, i.e., $N$ is a $W$-control subgroup of $G$.

(ii) Since $K \triangleleft W(P)$ we have $P \cdot C_G(P) \subseteq K$. So $N \cap K$ is a $C$-control subgroup of $K$ and hence $N \cap K/O_{p^*}(N \cap K)$ is either a $p$-group or is isomorphic to $H$. In both cases if $K \neq G$ $K$ is $p$-solvable and $G = O_{p^*}(K) \cdot (K \cap N) \cdot N$.

$$O_{p^*}(G) \cdot N.$$
(iii) Since $N$ is a $C$-control subgroup we have $O_p^c(G) \cap N = O_p^c(N)$ and $O_p^c(N)$ is a $C$-control subgroup of $O_p^c(G)$; so we have $P = [P, T]$. Then $P$ is non-cyclic and we may take $N_1$ the antiimage in $N$ of a subgroup $P_1$ of $P$ such that $P_1 \leq P$, $|P/P_1| = p^2$, and $P/P_1$ is non-cyclic. $T$ inverts every element of $P/P'$, so acts on $P/P_1$ in a Frobenius way. So we have that $F = N/N_1$ is a Frobenius group of order $2p^2$.

It is well known that $F$ has a non-split central extension $\hat{F}$ by $Z$ of order $p$, and then $\hat{Q}$ is an extraspecial group of order $p^3$ and exponent $p$. This gives a central extension $\hat{N}$ of $N$, with $\hat{N}_1 = Z \times N_1$ (identifying $N_1$ with a subgroup of $\hat{N}$).

As $N$ is a $C$-control subgroup of $G$, it follows from [8, Lemma 1.2] that there exists a central extension $\hat{G}$ of $G$ by $Z$ which induces the extension $\hat{N}$ of $N$ and where $\hat{N}$ is a $C$-control subgroup.

Given a group $K$ denote by $P(K)$ the $Z$-module of generalized projective characters.

**Lemma 3.2.** There exists a $Z$-module isometry $\sigma : R(H) \rightarrow R(G)$ such that

(i) If $\pi \in P(H)$ then $\sigma(\pi)|_N \in P(N)$.

(ii) If $\chi \in R(H)$ and $x \in G$ is not a $p'$-element and $x_p \in N$, we have $\sigma(\chi)(x) = \chi(n)$, where $n \in C_N(x_p)$ and $\chi n^{-1} \in W(\langle x_p \rangle)$.

(iii) If $\chi \in R(H)$ and $\lambda \in R_c(H)$ then $\sigma(\chi \lambda) = \sigma(\chi)\sigma(\lambda)$.

**Proof.** We denote $R(G, W) = \{\chi \in R(G) \mid$ for any $x \in G$ such that $x_p \neq 1 \chi(x) = \chi(x \cdot w)$ for any $w \in W(\langle x_p \rangle)\}$, and $R_0(H)$ the set of elements of $R_c(H)$ which vanish at 1.

In the notation of [7, Théorème 6], since we have $W(A) \cap N = O_p^c(C_N(A)) = O_p^c(N) \cap C_G(A)$ for any $A \neq 1$ a $p$-subgroup of $G$, we have

$$e_p = (1/|O_p^c(N)|) \sum_{x \in O_p^c(N)} x$$

and $e_p \cdot R(N) = R(H)$; hence $R(N, V) = R(H) + P(N)$ [7, Théorème 3] and $R_c(N, V) = R_c(H)$. In particular, $P(N)$ is the orthogonal lattice of $R_0(H)$ in $R(N, V)$.

By [7, Théorème 6] there exists an isometry $\sigma$ from $R_c(H)$ onto $R_c(G, W)$ such that

$$(*) \text{ for any } \lambda \in R_c(H), \sigma(\lambda)|_N = \lambda.$$
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\[(**) \text{ for any } x \in G_p, \text{ and } \chi \in R(H) \]

\[\sigma(\chi)(x) = \frac{\bar{\chi}(1) + \chi(t)}{2} + \frac{\chi(1) - \chi(t)}{2} \sigma(e)(x).\]

In particular, for any \( \lambda \in R_0(H) \), \( \sigma(\lambda) \) vanishes over \( G_p \), and therefore, [7, Théorème 6] and (*) give

\[(\lambda, \sigma(\pi)_N) = (\sigma(\lambda), \sigma(\pi))_G = (\lambda, \pi)_N = 0 \quad \text{for } \pi \in P(H);\]

Hence we have (i).

Let \( x \) be an element of \( G - G_p \), such that \( x_p \in N \). We have \( x = w \cdot n \), where \( w \in W(\langle x_p \rangle) \) and \( n \in N - N_p \); so, for any \( \chi \in R(H) \) \( \sigma(\chi)(x) = \sigma(\chi)(n) \), since \( e_w \cdot R(G) \subseteq R(G, W) \). On the other hand, since \( \text{rang}_z(R(H)) = \text{rang}_z(R_0(H) + P(H)) \), (*) gives \( \sigma(\chi)(n) = \chi(n) \). So we have (ii). Now (ii) and (**) give (iii), which completes the proof of the Lemma.

**Lemma 3.3.** Let \( \varphi \) be a non-trivial character of \( Z \). There exists an irreducible character \( \zeta \) of \( \hat{F} \) such that \( \zeta|_Z = \varphi \), and \( \zeta(t) = 1 \). Then \( \sigma \) may be extended to a \( Z \)-linear isometry of \( Z\zeta + R(H) \) into \( R(G) \). From this we deduce that \( \sigma(\varepsilon)(1) = 1 \) and get a contradiction. (Recall that \( \varepsilon \) is the non-trivial linear character of \( H \)).

**Proof.** The first statement is clear.

Let \( R(K, p) = \{ \chi \in R(K) \mid \chi(x) = \chi(x_p) \text{ for all } x \in G \} \). By [7, Théorème 1], there exists \( \alpha \in R(\hat{F}, p) \) such that \( \alpha|_{\hat{F}} = \zeta|_{\hat{F}} \). Set

\[\omega = \frac{p + 1}{2} \varepsilon + \frac{p - 1}{2} \varepsilon \]

and

\[\theta = \frac{p + 1}{2} \varepsilon - \frac{p - 1}{2} \varepsilon - p \varepsilon + \sum_{\lambda \in \hat{F} - \hat{F}} \lambda \]

It is easy to verify that \( p \varepsilon = \alpha \cdot \omega \) and that \( \theta \) vanishes on each element \( \hat{x} \) of \( \hat{F} \) such that \( \hat{x}_p \) and \( \hat{x}_p^{-1} \) are conjugate.

We identify \( R(\hat{F}) \) with its image in \( R(\hat{\mathcal{N}}) \). Since \( R(\hat{F}, p) \subseteq R(\hat{\mathcal{N}}, p) \), there exists \( \alpha' \in R(\hat{G}, p) \) such that \( \alpha'|_{\hat{F}} = \alpha' \) [7, Corollaire 2]. We set \( \zeta' = (1/p) \alpha' \sigma(\omega) \). We first prove that \( \zeta' \) has norm 1. We denote as usual \( \bar{\alpha} \) for the complex conjugate of \( \alpha \). We have

\[(\alpha' \bar{\alpha})|_{\hat{F}} = \alpha \cdot \bar{\alpha}, \quad (\alpha \cdot \bar{\alpha})|_{\hat{F}} = (\zeta \bar{\zeta})|_{\hat{F}} \quad \text{and} \quad \zeta \bar{\zeta} \in R(F).\]
Therefore $\alpha \cdot \bar{a} \in R(F, p) \subseteq R_c(H)$ and by (*) of Lemma 3.2 we get

$$\sigma(\alpha \cdot \bar{a}) = \alpha' \cdot \bar{a}' \text{.}$$

Now by (iii) of Lemma 3.2

$$(\alpha' \cdot \sigma(\omega), \alpha' \cdot \sigma(\omega))_\delta = (\sigma(\omega), \sigma(\alpha \bar{a} \omega))_\delta = (\omega, \alpha \bar{a} \omega)_F$$

$$= p^2(\zeta, \zeta)_F = p^2 \text{.}$$

We next prove that $\zeta'|_E$ is a generalized character for any nilpotent subgroup $E$ of $G$ which will show that $\zeta'$ is a simple character of $G$. We may assume that $Z \subseteq E$. Let $E = E/Z$, $S = O_p(E)$ and $D = O_p(S)$. We may assume $\rho \supseteq S$. If $S \subseteq Z \times N_1$, $S = Z \times R$ with $R \subseteq N_1$, and $\alpha'|_{Z \times D \times R} = p \rho \otimes 1_{D \times R}$

so that $\zeta'|_E$ is a generalized character in this case. So we have $S = S \cdot N_1/N_1$

non-trivial and $C_H(S) \subseteq H$. Hence by the structure of $C_H(S)$ we get $D \subseteq W(S/Z)$. Now Lemmas 3.1(i) and 3.2(ii) give $\sigma(\omega)(x) = \omega(x_p) = p$, for any $x \in E - D$. Therefore there exists $\gamma \in R(D)$ such that

$$\sigma(\omega)|_E = p \cdot 1_E + \gamma.$$

But now

$$\zeta'|_E = \alpha'|_E + (\gamma \otimes \varphi)|_E,$$

where $\gamma \otimes \varphi$ is a character of $D \times Z$. This completes the proof that the isometry $\sigma$ can be extended.

Now we set $\theta' = \sigma(\theta)$. We claim that $\theta'$ vanishes on each element $\hat{x}$ of $G$

such that $\hat{x}_p$ and $\hat{x}_p^{-1}$ are conjugate. Indeed, let $x$ be the image of $\hat{x}$ in $G$. If $x_p = 1$, $\hat{x}$ is a p-element and so $\zeta'(\hat{x}) = \sigma(\omega)(x)$, thus using (**) of the previous Lemma we get $\theta'(\hat{x}) = 0$. Therefore we may assume that $1 \neq x_p \in N_1$; since $O_p(C_G(\hat{x}_p))$ maps onto $O_p(C_G(x_p))$ we get using the structure of $C_G(x_p)$, $\hat{x} = \hat{w} \cdot \hat{n}$ with $\hat{w} \in O_p(C_G(\hat{x}_p))$ and $\hat{n} \in N$. Therefore $\hat{x}_p = \hat{n}_p$ and hence $\alpha'(\hat{x}) = \alpha'(\hat{n})$. Also we have $\sigma(\lambda)(\hat{x}) = \lambda(\hat{n})$ for any $\lambda \in R(F)$. Hence we have the claim.

Now let $u$ be an involution of $G$, $\theta'$ vanishes over the set $\{\hat{u}\hat{z}\hat{u}^{-1} | \hat{z}, \hat{u} \in G \}$ and we get

$$\frac{p + 1}{2} - \frac{p - 1}{2} \frac{\sigma(\epsilon)(\hat{u})^2}{\sigma(\epsilon)(1)} - p \frac{\zeta'(\hat{u})^2}{\zeta'(1)} + \sum_{\lambda \epsilon x_{p} - x_{p}} \frac{\sigma(\lambda)(\hat{u})^2}{\sigma(\lambda)(1)} = 0.$$
By (** of Lemma 3.2 we have \( P_a(b) = 0 \). It is not hard to show that in fact \( a \) is a double root of \( P_a(X) \). So

\[
P_a(X) = \left( \frac{X}{a} - 1 \right)^2 P_a(0).
\]

Therefore either \( P_a(0) = 0 \) or \( a = b \). On one hand, a simple computation shows that \( P_a(0) = 0 \) implies \( a = 1 \). On the other hand \( a = b \) implies \( \hat{a} \in K = \text{Ker}(\sigma(\varepsilon)) \), namely, \( K \neq 1 \). Since obviously \( O_p(G) = 1 \), \( K \cap P \) contains a non-trivial element \( x \), and by Lemma 3.2(ii) \( a = \sigma(\varepsilon)(x) = \varepsilon(x) = 1 \). Hence \( \sigma(\varepsilon)(1) = 1 \) in all cases.

But now Lemma 3.2(ii) gives that \( P \leq K = \text{Ker}(\sigma(\varepsilon)) \) and further that \( W(A) \leq K \) for any \( A \leq P \) non-trivial. But by Lemma 3.1(ii) we get a contradiction. This completes the proof of the Lemma and of the Theorem.

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**REFERENCES**