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Statistical entropy of two-dimensional dilaton de Sitter space

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Abstract

It has been proposed that a quantum group structure underlies de Sitter/conformal field theory duality. These ideas are used to give a microscopic operator counting interpretation for the entropy of two-dimensional dilaton de Sitter space. This agrees with the Bekenstein–Hawking entropy up to a factor of order unity.

1. Introduction

String perturbation theory is defined using the notion of a finite number of string loops moving in a fixed background spacetime. This modest starting point yields a finite perturbative expansion for scattering amplitudes. It has had remarkable success in elucidating many nonperturbative phenomena, such as strong/weak coupling duality symmetries, the existence of new solitonic objects, D-branes, as well as the microscopic interpretation of black hole entropy in certain cases. Moreover these new ideas have led to the first conjectures for complete (though background dependent) nonperturbative formulations of string theory, such as matrix theory and AdS/CFT.

However it seems clear that we are still missing some key ideas. In the present Letter, we will be mainly concerned with developing the idea that noncommutative geometry and quantum groups should play a much more prominent role than it has to date. This idea has already been extensively studied for both closed (see [1] for a review) and open strings in compact backgrounds (see for example [2]). It has long been known that WZNW models based on the compact group g have an underlying quantum-deformed symmetry $U_q(g)$ where the deformation parameter q is a root of unity, determined by the level number of the CFT. There are a finite number of unitary irreducible representations of the quantum group which correspond to integrable representations of the current algebra. The tensor product structure of the quantum group determines CFT operator product coefficients. If one then attempts to reconstruct the target space geometry by Fourier transforming this finite set of represen-

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tations, one winds up with a noncommutative space on which the quantum group symmetry acts. For the case of D-branes on group manifolds, the noncommutative geometry that emerges makes contact with ideas of Connes [3].

The hope is that quantum group structure may build in from the beginning analogs of spacetime uncertainty principles (see [4] for a review), and thus provide a formulation of string theory applicable to the strong gravity regime where it no longer makes sense to think of single strings moving in smooth spacetime background. In order to make these ideas more precise, it is interesting then to further consider the examples of AdS/CFT and dS/CFT [5] and look for quantum group symmetry underlying these dualities. This was first studied in [6] for the case of AdS/CFT where it was found that the quantum group symmetry of WZNW models gives a natural explanation of the stringy exclusion principle [7] for the case of $AdS_3 \times S_3$. Related ideas appear in [8–14]. From the spacetime viewpoint, this bound is explained by giant gravitons [15], whose maximum size is cutoff by the radius of the sphere.

It was proposed in [16] that similar ideas could be extended to dS/CFT by seeking an underlying quantum group symmetry as a q-deformation of the isometry group of de Sitter/conformal group of the CFT. An important new feature of this construction is the appearance of noncompact groups. It was shown that cyclic unitary representations of the quantum group $U_q(SL(2))$ become unitary principal series representations in a classical limit. These are precisely the representations corresponding to massive particle states in a two-dimensional de Sitter background. For q a nontrivial root of unity the quantum group representations are finite dimensional, so the spectrum of the theory becomes discrete. Thus the quantum group structure introduces both an ultraviolet and an infrared cutoff in an interesting way. These are prerequisites for a microscopic interpretation of the finite horizon entropy of de Sitter. Generalization to three-dimensional de Sitter was discussed in [17] and a formula for the de Sitter entropy was proposed. This was further described from the classical dS/CFT viewpoint in [18]. Related ideas and further developments may be found in [19,20].

In the present Letter these ideas are applied to the case of two-dimensional dilaton de Sitter space [21,22]. This spacetime has a nontrivial Bekenstein–Hawking entropy and temperature. The entropy of this spacetime is accounted for by counting operators in a dual q-CFT.

2. Two-dimensional dilaton de Sitter space

As described in [21] two-dimensional dilaton gravity

$$S = \frac{1}{2}\int \sqrt{-g}\, d^2x\, \Phi\left(R - \frac{2}{\ell^2}\right)$$

admits solutions of de Sitter form

$$ds^2 = -\frac{1}{\frac{t^2}{\ell^2} - a^2}dt^2 + \left(t^2 - a^2\ell^2\right) d\theta^2,$$
$$\Phi = \Phi_0 \frac{t}{\ell}, \tag{1}$$

where a is a dimensionless constant parameterizing a mass deformation, Φ_0 is a dimensionless constant parameterizing the strength of the gravitational constant, ℓ is a length scale that sets the radius of curvature. We take $\theta \in (0, 2\pi)$ to parametrize a spacelike circle and $t \in (-\infty, 0)$. The Penrose diagram for this solution is shown in Fig. 1. The (t, θ) coordinates only cover the lower quadrant of the diagram.

This spacetime has an asymptotic symmetry group with a Virasoro algebra that acts [21,22]. As described in [18], states of matter fields in this background fall into representations of this Virasoro algebra.

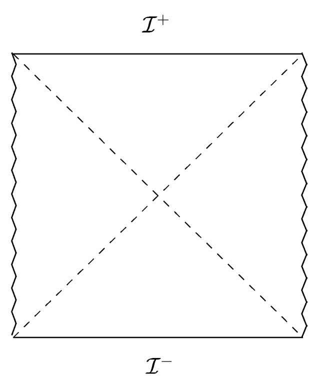

Fig. 1. Penrose diagram for de Sitter-like solution of two-dimensional dilaton gravity. There is a singularity at one point on the spatial circle where the dilaton $\Phi = 0$ and the gravitational coupling diverges. This is analogous to the conical singularity of three-dimensional Schwarzschild de Sitter.

There exists definition of mass M utilizing a space-like Killing vector on $\mathcal{I}^-$ [21]. The mass defined in this way can be positive or negative. On the solution (1)

$$|M| = \frac{a^2}{2\ell}\Phi_0. \tag{2}$$

In the limit $a \to 0$ the singularity becomes null, and the geometry is that of a big crunch or big bang. The $a = 1$ limit describes a pure de Sitter geometry. We expect the region $0 \leqslant a \leqslant 1$ to correspond to sensible semi-classical spacetimes. This can be argued by lifting to three dimensions where these geometries correspond to a positive deficit angle. On the other hand, the $a > 1$ geometries will correspond to negative deficit angle. Since this would arise from a negative energy localized source, we do not expect these geometries to be stable once interactions are included. It is curious the allowed geometries have $|M|$ less than the de Sitter value. However M measures the energy of the complete spacetime, and addition of positive energy sources causes the cosmological horizon to shrink, reducing the overall contribution to $|M|$.

Demanding absence of a singularity on the Euclidean section leads to a Hawking temperature

$$T = \frac{a}{2\pi\ell}. \tag{3}$$

Because the dilaton varies, the relation of Bekenstein–Hawking entropy to area is a bit more subtle than the standard case. The entropy may be derived by computing the Lorentzian action of the solution with appropriate boundary terms included. However the simplest way to obtain the entropy is to take $1/T = \partial S/\partial M$ which leads to

$$S = 2\pi\Phi_0 a, \tag{4}$$

where the constant of integration is fixed so $S = 0$ when $T = 0$. This prescription agrees with results obtained via the semi-classical Lorentzian path integral.

We can try to infer something about the spectrum of microscopic excitations by asking at what temperature do we expect the thermodynamic limit to break down. This happens when $S \sim \mathcal{O}(1)$ so $a \sim 1/\Phi_0$. Therefore an estimate of the mass gap above the $T = 0$ solution is

$$\Delta M \sim \frac{1}{\Phi_0\ell}. \tag{5}$$

3. q-deformed CFT

The proposal for describing two-dimensional dS_2 using a q-deformed version of the dS/CFT correspondence has been studied in [16]. The interpretation of horizon entropy in this framework has been elaborated in [17,18].

Based on those results we postulate the q-deformed CFT is built out of N representations of the q-deformed isometry group $SL(2,\mathbb{R})$ characterized by the parameters $\tau = -1$ and complex number b as described in detail in Appendix A. We set $q = e^{2\pi i/N}$ a root of unity. The q-deformed representations are analogs of the complementary series representations for massless particles in de Sitter space (strictly speaking mass $\to 0^+$, and recall the relation $\tau = -1/2 - \sqrt{1/4 - (m\ell)^2}$ when $m\ell < 1/2$).

We have in mind that the full interacting CFT will be based on a theory with $SU(N)$ gauge symmetry, or something similar, but we will presume there exists a free-field limit where one is left with N fields. On this branch we will assume the gauge symmetry is broken down to the permutation group S_N with the N fields transforming in the defining representation. This is reminiscent of the structure of the CFT's relevant for the D-brane black holes of [23].

The results of [17] carry over for the spectrum of these representations. The generator L_0 is associated with the Killing vector used to define the notion of mass (2) (here we follow the conventions of [16,17], which differ from [18,21]). This is related to the generators defined in Appendix A by the relation $L_0 = (X_+ + X_-)/2$. In particular, for one of these N-dimensional cyclic representations with b real, L_0 has imaginary eigenvalues that are roughly equally spaced ranging from $-i(N-1)/2\ell, \ldots, i(N-1)/2\ell$. In this way we see the q-deformation introduces a ultraviolet and an infrared cutoff versus the continuous unbounded spectrum of the $q = 1$ principal series representation. Note also the spacing of the L_0 eigenvalues does not approach a continuum in the $q \to 1$ limit.

Related issues arise in the fat-black hole limit when one considers black holes with D-brane charge [24]. In that case the mass gap one would infer by having N distinct D-strings wrapping a one-cycle is much larger than the lowest inverse length scale in the system. There the resolution is that the N distinct D-strings coalesce into a single multiply wrapped string. This picture produces the expected mass gap.

An analogous phenomena can happen for the cyclic representations. We assume the dynamics is such that instead of all N representations having the same value for b that they differ by a phase $e^{2\pi ik/N}$ where $k = 1, \ldots, N$ with k labeling the N different cyclic representations. This possibility was explored in [17] for the case of the principal series. These results also carry over to the complementary series described here. The end result is one obtains a reducible representation with a spectrum for L_0 ranging from approximately $-i(N-1)/2\ell, \ldots, i(N-1)/2\ell$ with a typical level spacing of order $1/N\ell$. Therefore we match the expected spacing (5) provided we identify N with the strength of the gravitational coupling

$$N \sim \Phi_0. \tag{6}$$

This gives the expected continuous spectrum for L_0 as $N \to \infty$.

Now we are ready to compute the de Sitter entropy arising from this microscopic description, following [17,18]. A comoving observer in de Sitter sees a thermal density matrix due to a trace over modes outside their horizon. In [17,18] this notion was carried over to the CFT description. In particular, the partition function becomes a sum over a operators in the CFT with Boltzmann weights. The operators that appear in the sum are those corresponding to the one-particle modes with positive imaginary L_0 eigenvalue, together with their tensor products. We begin by assuming Bose statistics for these modes. The expectation value of the mass then collapses to a sum over single particle modes with the Bose–Einstein distribution function appearing

$$M = \sum \frac{L_0}{e^{L_0/T} - 1} \approx N\ell \int_0^{(N-1)/2} dL_0 \frac{L_0}{e^{L_0/T} - 1} \sim \ell N T^2$$

for $N \gg 1$, $T\ell \gg 1/N$, and where we have identified T with the Hawking temperature.

Plugging in to get the entropy we find

$$S \sim \ell N T.$$

Therefore, recalling Eqs. (3), (4) and (6) we find agreement with the Bekenstein–Hawking entropy up to a factor of order unity. In the limit $N \gg 1$, replacing the Bose–Einstein distribution by Fermi–Dirac changes the overall coefficient by a constant factor of order unity.

4. Discussion

In this Letter we have provided a statistical computation of entropy in a de Sitter-like spacetime. We computed the microscopic entropy of a two-dimensional version of de Sitter space that depends on two nontrivial parameters: a gravitational constant and a mass parameter. In the usual semi-classical picture, this entropy diverges, but by q-deforming the spacetime this entropy is rendered finite and can be given a dual interpretation as an operator counting problem in a q-deformed CFT.

Clearly much remains to be done in further developing the correspondence between q-deformed CFT's and quantum de Sitter spacetimes. Thus far it has been established that unitary CFT's of this type exist, and that these appear to account for the entropy of de Sitter space in a straightforward way. It would be very interesting to find versions of string theory that live in these backgrounds and higher-dimensional generalizations, and to give a complete specification of the dual boundary CFT's.

To conclude, let us offer the following speculation. Suppose a CFT of the type outlined in the present work can be shown to be a complete self-consistent theory dual to a gravitational theory in asymptotic de Sitter space in four spacetime dimensions. This presumably will belong to a family of different theories, labeled by the relevant gauge symmetry (let us say $SU(N)$). From the bulk point of view, this will correspond to a family of disconnected string vacua. While its possible we started out in some special state in the past (for example special states with much larger effective cosmological constants leading to inflation), as time evolves we expect to evolve into the most likely type of microstate accessible to us. Therefore if we rule out the unstable spacetimes analogous to the $a > 1$ backgrounds considered here, the CFT description predicts this will be the macrostate with the largest available entropy, which is de Sitter with a small positive cosmological constant determined by N. This matches well with current observations and leads to the prediction that dark energy density will asymptote to a constant determined by the fundamental constants of the theory.

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Appendix A. Complementary series

In [16] a q-deformed version of the principal series representations of $SL(2,\mathbb{R})$ was found. Here we generalize those results to the case of the complementary series. Let us begin by reviewing the complementary series for the case $q=1$. We can realize this representation on the basis $|k\rangle = e^{-ik\theta}$ with k an integer. The action of the generators takes the form

$$\begin{aligned} He^{-ik\theta} &= 2ke^{-ik\theta}, \\ X^+e^{-ik\theta} &= (k-\tau)e^{-i(k+1)\theta}, \\ X^-e^{-ik\theta} &= -(k+\tau)e^{-i(k-1)\theta}, \end{aligned}$$

where for the complementary series $-1<\tau<0$. As described in [25] there is an equivalence of representations under $\tau \to -1-\bar{\tau}$. We will also be interested in the discrete series representation corresponding to $\tau=-1$. For our purposes, this may be considered a continuation of the complementary series.

The complementary series is unitary with respect to the norm

$$\langle\chi|\psi\rangle = \sum_{n=-\infty}^{\infty} \frac{\Gamma(\tau-n+1)}{\Gamma(-\tau-n)} a_n \bar{b}_n, \tag{A.1}$$

where

$$\psi = \sum_{n=-\infty}^{\infty} a_n e^{-in\theta}, \qquad \chi = \sum_{n=-\infty}^{\infty} b_n e^{-in\theta}.$$

The coefficients appearing in the norm (A.1) arise when the Klein–Gordon norm is computed for a field in de Sitter with mass $0 \leqslant m < (d-1)/2$, as shown in Appendix B.

The q-deformed version of the algebra takes the form

$$\begin{aligned} KK^{-1} &= K^{-1}K = 1, \\ KX^{\pm}K^{-1} &= q^{\pm 2}X^{\pm}, \\ [X^+, X^-] &= \frac{K-K^{-1}}{q-q^{-1}}, \end{aligned} \tag{A.2}$$

where the classical limit is obtained by setting $K=q^H$ and taking the limit $q\to 1$. We will be interested in the case where $q=e^{2\pi i/N}$ with N odd. The basic structure of the representations of interest can be carried over from the results of [16],

$$\begin{aligned} K|m\rangle &= q^{-2m}\lambda|m\rangle, \\ X^+|m\rangle &= \left(bc + \frac{q^m-q^{-m}}{q-q^{-1}} \frac{\lambda q^{1-m}-\lambda^{-1}q^{m-1}}{q-q^{-1}}\right) \\ &\quad \times |m-1\rangle, \\ X^-|m\rangle &= |m+1\rangle, \end{aligned}$$

with $m=0,\ldots,N-1$, $\lambda=q^{2l}$, $l=(N-1)/2$ and b, c complex numbers that satisfy

$$bc = \tau^2+\tau-l^2-l. \tag{A.3}$$

These transformations are supplemented by the cyclic operations

$$X_+|0\rangle = b|N-1\rangle, \qquad X_-|N-1\rangle = c|0\rangle.$$

To check unitarity, we need to check positivity of the norm (A.1). The eigenvalues of H are real, since l is an integer. It remains to examine

$$\begin{aligned} &\langle X^+m|X^+m\rangle \\ &\quad = -\langle m|X^-X^+|m\rangle \\ &\quad = -\left(bc + \frac{q^m-q^{-m}}{q-q^{-1}} \frac{\lambda q^{1-m}-\lambda^{-1}q^{m-1}}{q-q^{-1}}\right)\langle m|m\rangle \end{aligned} \tag{A.4}$$

if we use the notion of conjugation defined by the *-structure $X^*_\pm = -X_\mp$, $K^* = K^{-1}$. Substituting in for bc, we need to check whether

$$\begin{aligned} v &= l^2+l-\tau(\tau+1) \\ &\quad - \left(\frac{q^m-q^{-m}}{q-q^{-1}} \frac{\lambda q^{1-m}-\lambda^{-1}q^{m-1}}{q-q^{-1}}\right) \\ &> 0. \end{aligned}$$

This can be expressed as

$$v = l^2 + l - \tau(\tau+1) - \frac{\sin(\frac{2\pi(l-k)}{2l+1})\sin(\frac{2\pi(l+1+k)}{2l+1})}{\sin^2(\frac{2\pi}{2l+1})}$$
$$= l^2 + l - \tau(\tau+1) + \frac{\sin^2(\frac{2\pi(l-k)}{2l+1})}{\sin^2(\frac{2\pi}{2l+1})}.$$

Since $-1 \leqslant \tau \leqslant 0$ this expression is always positive, so the q-deformed representation is unitary.

Appendix B. Klein–Gordon norm

In this appendix we demonstrate the coefficients appearing in the norm (A.1) arise from the Klein–Gordon norm when $0 \leqslant m\ell \leqslant 1/2$. The Klein–Gordon norm is

$$(\psi, \phi) = -i \int d\Sigma^\mu \, (\psi \overleftrightarrow{\partial_\mu} \phi^*). \tag{B.1}$$

Rather than working with the coordinate patch (1), we will instead work in global coordinates

$$ds^2 = -dT^2 + \ell^2 \cosh^2 T \, d\theta^2$$

to make direct contact with previous work [16,17] and take θ to have 2π periodicity. The dilaton vanishes at $\theta = 0$ in these coordinates. In the mass range of interest the mode expansion for a free minimally coupled scalar field takes the form

$$\phi = \sum_{n=-\infty}^{\infty} f_n(T) e^{-in\theta},$$

where ϕ satisfies the equation

$$\Box\phi = m^2\phi.$$

The solutions take the form

$$f_n = A_n P_{-\frac{1}{2}-n}^{-\tau-\frac{1}{2}}\left(\tanh\frac{T}{\ell}\right) + B_n Q_{-\frac{1}{2}-n}^{-\tau-\frac{1}{2}}\left(\tanh\frac{T}{\ell}\right),$$

where $\tau = -1/2 - \sqrt{1/4 - (m\ell)^2}$ and the coefficients are chosen so the modes are orthonormal with respect to the Klein–Gordon norm (B.1). This imposes the condition

$$A_n^* B_n - A_n B_n^* = \frac{i}{\pi^{5/2}} \frac{\Gamma(\tau - n + 1)}{\Gamma(-\tau - n)}.$$

The n dependent factor in this expression reproduces the nontrivial n dependent factor that appear in the norms of the complementary series (A.1).

References

[1] J. Frohlich, O. Grandjean, A. Recknagel, in: A. Connes, K. Gawedzki, J. Zinn-Justin (Eds.), Les Houches Proceedings, North-Holland, Amsterdam, 1995, pp. 221–385, hep-th/9706132.
[2] A.Y. Alekseev, A. Recknagel, V. Schomerus, Mod. Phys. Lett. A 16 (2001) 325, hep-th/0104054.
[3] A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.
[4] M. Li, T. Yoneya, hep-th/9806240.
[5] A. Strominger, JHEP 0110 (2001) 034, hep-th/0106113.
[6] A. Jevicki, S. Ramgoolam, JHEP 9904 (1999) 032, hep-th/9902059.
[7] J.M. Maldacena, A. Strominger, JHEP 9812 (1998) 005, hep-th/9804085.
[8] V. Dobrev, in: H. Doebner (Ed.), Proceedings of Workshop on Generalized Symmetries in Physics, Clausthal, July 1993, World Scientific, Singapore, ISBN 981-02-1771-4, 1994.
[9] V.K. Dobrev, R. Floreanini, J. Phys. A 27 (1994) 4831.
[10] V.K. Dobrev, P.J. Moylan, Phys. Lett. B 315 (1993) 292.
[11] H. Steinacker, Commun. Math. Phys. 192 (1998) 687, q-alg/9611009.
[12] H. Steinacker, Adv. Theor. Math. Phys. 4 (2000) 155, hep-th/9910037.
[13] P.-M. Ho, S. Ramgoolam, R. Tatar, Nucl. Phys. B 573 (2000) 364, hep-th/9907145.
[14] A. Jevicki, M. Mihailescu, S. Ramgoolam, hep-th/0008186.
[15] J. McGreevy, L. Susskind, N. Toumbas, JHEP 0006 (2000) 008, hep-th/0003075.
[16] A. Güijosa, D.A. Lowe, Phys. Rev. D 69 (2004) 106008, hep-th/0312282.
[17] D.A. Lowe, Phys. Rev. D 70 (2004) 104002, hep-th/0407188.
[18] A. Güijosa, D.A. Lowe, J. Murugan, Phys. Rev. D 72 (2005) 046001, hep-th/0505145.
[19] P. Pouliot, Class. Quantum Grav. 21 (2004) 145, hep-th/0306261.
[20] T. Banks, hep-th/0503066.
[21] M. Cadoni, P. Carta, M. Cavaglia, S. Mignemi, Phys. Rev. D 66 (2002) 065008, hep-th/0205211.
[22] A.J.M. Medved, Phys. Rev. D 67 (2003) 084016, hep-th/0209029.
[23] A. Strominger, C. Vafa, Phys. Lett. B 379 (1996) 99, hep-th/9601029.
[24] J.M. Maldacena, L. Susskind, Nucl. Phys. B 475 (1996) 679, hep-th/9604042.
[25] N. Vilenkin, A. Klimyk, Representation of Lie Groups and Special Functions, Kluwer Academic, Dordrecht, 1991.