Common solution to the Lyapunov equation for
2 × 2 complex matrices

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Abstract

In this work we solve the problem of a common solution to the Lyapunov equation for 2 × 2 complex matrices. We show that necessary and sufficient conditions for the existence of a common solution to the Lyapunov equation for 2 × 2 complex matrices A and B is that matrices $(A + iαI)(B + iβI)$ and $(A + iαI)^{-1}(B + iβI)$ have no negative real eigenvalues for all $α, β ∈ \mathbb{R}$. We show how these results relate to a special class of 4 × 4 real matrices.

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1. Introduction

A matrix $A ∈ \mathbb{C}^{n×n}$ is called (Hurwitz) stable if all its eigenvalues lie in the open left half of the complex plane. In this case, the linear-time invariant (LTI) system

$$\dot{x} = Ax$$

is asymptotically stable.
A classical result of Lyapunov states, that a matrix $A$ is stable if and only if for arbitrary Hermitian positive definite $Q$, the Lyapunov equation

$$AP + PA^* = -Q$$

admits a positive definite solution $P$. The associated form $V(x) = x^TPx$ is called a quadratic Lyapunov function for the system (1).

We shall use the convention that $P > 0$ denotes a Hermitian positive definite matrix and thus, for a given stable matrix $A$, we will denote the set of all solutions to the Lyapunov equation for $A$ by

$$\mathcal{P}(A) = \{P = P^* > 0 : AP + PA^* < 0\}.$$ 

$\mathcal{P}(A)$ is an open convex cone.

Let $A_1, A_2, \ldots, A_k$ be stable matrices in $\mathbb{C}^{n \times n}$ and let $P > 0$ be a common solution to the following Lyapunov equations:

$$A_j P + PA_j^* < 0 \quad \text{for} \quad j = 1, 2, \ldots, k.$$ 

We say that the matrix $P$ is a common solution to the Lyapunov equation for matrices $A_j$, $j = 1, 2, \ldots, k$. Accompanying quadratic Lyapunov function $V(x) = x^TPx$ is called a common quadratic Lyapunov function (CQLF) for the LTI systems $\dot{x} = A_jx$, $j = 1, \ldots, k$.

The problem of deciding whether stable matrices $A_j$, $j = 1, \ldots, k$, share a common solution to the Lyapunov equation has been extensively studied, but the complete solution is known only in a few special cases. For a source of literature on the problem, we refer the reader to the following works and the citations that appear in them [1–7]. The problem has a wide variety of applications in systems and control theory and elsewhere.

In [8] Loewy considered the following question. Given a stable matrix $A \in \mathbb{C}^{n \times n}$, for what matrices $B$ does $\mathcal{P}(A) = \mathcal{P}(B)$ hold. He proved the following result.

**Theorem 1.** Let $A, B \in \mathbb{C}^{n \times n}$ be stable matrices. Then $\mathcal{P}(A) = \mathcal{P}(B)$ if and only if

$$B = \mu(A + i\alpha I) \quad \text{for some} \quad \alpha, \mu \in \mathbb{R} \text{ such that} \quad \mu > 0$$

or

$$B = \mu((A + i\alpha_1 I)^{-1} + i\alpha_2 I) \quad \text{for some} \quad \alpha_1, \alpha_2, \mu \in \mathbb{R} \text{ such that} \quad \mu > 0.$$ 

Let $\mathcal{C}$ be a nonempty set in $\mathbb{C}^{n \times n}$. We say that $\mathcal{C}$ is nonsingular (stable) if all matrices $M \in \mathcal{C}$ are nonsingular (stable).

For $A, B \in \mathbb{C}^{n \times n}$ we will denote by $\text{conv}(A, B)$ the convex cone generated by $A$ and $B$.

Convex invertible cone is a convex cone that is closed under matrix inversion. By $\text{cic}(A, B)$ we will denote the convex invertible cone generated by $A$ and $B$. Finally, $\hat{\text{cic}}(A, B)$ will denote the smallest convex invertible cone that contains $A$ and $B$ and has the following property: $M + i\alpha I \in \hat{\text{cic}}(A, B)$ for every $M \in \text{cic}(A, B)$ and $\alpha \in \mathbb{R}$. Clearly:

$$\text{conv}(A, B) \subseteq \text{cic}(A, B) \subseteq \hat{\text{cic}}(A, B).$$


Theorem 1 implies that stability of $\hat{\text{cic}}(A, B)$ is a necessary condition for the existence of a common solution to the Lyapunov equation for matrices $A$ and $B$. This condition is in general not sufficient.
For given matrices $A$ and $B$, the stability of $\widetilde{\text{cic}}(A, B)$ is difficult to check. Let us introduce two weaker necessary conditions that are easy to verify. If the convex cone $\text{conv}(A, B)$ is stable, then the matrix $A^{-1}B$ has no negative real eigenvalues. Similarly, stability of the convex cone $\text{conv}(A^{-1}, B)$ implies that the matrix $AB$ has no negative real eigenvalues. In some special cases those weaker conditions are sufficient for the existence of a common solution to the Lyapunov equation.

Let $A$ and $B$ be real stable matrices such that the rank of $A - B$ is one. Shorten and Narendra [12] proved that a necessary and sufficient condition for the existence of a common solution to the Lyapunov equation for matrices $A$ and $B$ is that the matrix product $AB$ does not have a real negative eigenvalue. A different proof of this result was presented by King and Nathanson in [13]. In this paper we will show that when $A$ and $B$ are real and rank of $A - B$ is two, even the stability of $\text{cic}(A, B)$ is not sufficient for the existence of a common solution to the Lyapunov equation for $A$ and $B$.

Necessary and sufficient conditions for the existence of a common solution to the Lyapunov equation for $2 \times 2$ real matrices $A$ and $B$ is that matrices $AB$ and $A^{-1}B$ do not have a real negative eigenvalue. In this case those conditions are equivalent to the stability of the convex cones $\text{conv}(A, B)$ and $\text{conv}(A^{-1}, B)$. The proof of this result can be found in [10]. The special case of stable matrices was proved earlier in [14] and in [15].

In this work we will investigate the existence of a common solution to the Lyapunov equation for $2 \times 2$ complex matrices. We will show that necessary and sufficient conditions for the existence of a common solution to the Lyapunov equation for $2 \times 2$ complex matrices $A$ and $B$ is that convex cones $\text{conv}((A + i\alpha I_2), B)$ and $\text{conv}((A + i\alpha I_2)^{-1}, B)$ are stable for all $\alpha \in \mathbb{R}$.

The notation we will use is standard. For example, by $\mathbb{R}$ we will denote the set of real numbers and by $\mathbb{C}$ the set of complex numbers. By $\mathbb{R}^{n \times n}$ we will denote the set of $n \times n$ real matrices and by $\mathbb{C}^{n \times n}$ the set of $n \times n$ complex matrices. We shall write $A^*$ for the conjugate transpose of the matrix $A$ and $A^T$ for the transpose of the matrix $A$. We will denote by $Q_{ij}$ the $(i, j)$th element of the matrix $Q$.

2. Solution to the Lyapunov equation for $2 \times 2$ complex matrices

The main result of this paper gives necessary and sufficient conditions for a pair of complex $2 \times 2$ matrices to have a common solution to the Lyapunov equation. First we will state the result and the remainder of this section will gradually lead us to its proof.

**Theorem 2.** The stable matrices $A \in \mathbb{C}^{2 \times 2}$ and $B \in \mathbb{C}^{2 \times 2}$ have a common solution to the Lyapunov equation if and only if the following conditions are satisfied:

(A1) The convex cone $\text{conv}((A + i\alpha I_2), B)$ is stable for all $\alpha \in \mathbb{R}$.

(A2) The convex cone $\text{conv}((A + i\alpha I_2)^{-1}, B)$ is stable for all $\alpha \in \mathbb{R}$.

**Remark 3.** Let us look at two equivalent ways in which we can express the conditions in Theorem 2. Conditions (A1) and (A2) are equivalent to the conditions:

(B1) Matrix $(A + i\alpha I)(B + i\beta I)$ has no negative real eigenvalues for all $\alpha, \beta \in \mathbb{R}$.

(B2) Matrix $(A + i\alpha I)^{-1}(B + i\beta I)$ has no negative real eigenvalues for all $\alpha, \beta \in \mathbb{R}$.
Indeed to establish this equivalence assume that conditions (A1) and (A2) hold. If (B1) does not hold, then the matrix $(A + i\alpha I)(B + i\beta I)$ has a negative real eigenvalue $-\mu$, $\mu > 0$, for some $\alpha, \beta \in \mathbb{R}$. Hence
\[ \det((A + i\alpha I)(B + i\beta I) + \mu I) = 0 \]
and
\[ \det((B + i\beta I) + \mu (A + i\alpha I)^{-1}) = 0. \]
Therefore the convex cone $\text{conv}((A + i\alpha I)^{-1}, B)$ is not stable. This contradicts the condition (A2). Similarly we can show that the existence of a negative real eigenvalue for the matrix $(A + i\alpha I)^{-1}(B + i\beta I)$ contradicts the condition (A1).

Now assume that conditions (B1) and (B2) are satisfied. If the convex cone $\text{conv}((A + i\alpha I_2, B)$ is not stable, then there exists $\lambda_0 > 0$ such that the matrix $\lambda_0(A + i\alpha I_2) + B$ has a purely imaginary eigenvalue $-i\beta$, $\beta \in \mathbb{R}$. Hence the matrix $(A + i\alpha I)^{-1}(B + i\beta I)$ has a negative eigenvalue $-\lambda_0$, contrary to the condition (B2). In a similar way we get a contradiction to the condition (B1) if the convex cone $\text{conv}((A + i\alpha I_2)^{-1}, B)$ is not stable.

Conditions (B1) and (B2) are clearly equivalent to the conditions:

\begin{enumerate}
\item[(C1)] The convex cone $\text{conv}((A + i\alpha I_2, B)$ is nonsingular for all $\alpha, \beta \in \mathbb{R}$.
\item[(C2)] The convex cone $\text{conv}((A + i\alpha I_2)^{-1}, B + i\beta I)$ is nonsingular for all $\alpha, \beta \in \mathbb{R}$.
\end{enumerate}

First we will consider the existence of a common solution to the Lyapunov equation for matrices of the form $A = D + K$, where $D$ is a diagonal matrix and $K$ is a skew Hermitian matrix. Therefore we will be looking at the matrices of the form:
\[ A = \begin{pmatrix} -a + im & r + is \\ -r + is & -b + in \end{pmatrix} \] (2)
for some real numbers $a, b, m, n, r, s$.

We define the following sets of matrices:
\[ \mathcal{M}_1 = \left\{ A = \begin{pmatrix} -a + im & r + is \\ -r + is & -b + in \end{pmatrix} : (m, n, r, s) \in \mathbb{R}^4, a > 0, b > 0 \right\}, \]
\[ \mathcal{M}_2 = \left\{ A = \begin{pmatrix} im & r + is \\ -r + is & -b + in \end{pmatrix} : (m, n, r, s) \in \mathbb{R}^4, b > 0, r + is \neq 0 \right\}, \]
\[ \mathcal{M}_3 = \left\{ A = \begin{pmatrix} -a + im & r + is \\ -r + is & -i n \end{pmatrix} : (m, n, r, s) \in \mathbb{R}^4, a > 0, r + is \neq 0 \right\} \]
and
\[ \mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3. \]

First we will show that every matrix $A \in \mathcal{M}$ has a solution to the Lyapunov equation of the form:
\[ P = \begin{pmatrix} 1 & h + ik \\ h - ik & 1 \end{pmatrix} \] (3)
for some real numbers $h$ and $k$. 
Proposition 4. Let the matrix $A$ be of the form (2) and the matrix $P$ be of the form (3). Then the following statements hold:

1. If $A \in \mathcal{M}_2$, then $P$ is a solution to the Lyapunov equation for $A$ for all sufficiently small real numbers $h$ and $k$ that satisfy the inequality: $hr + ks < 0$.
2. If $A \in \mathcal{M}_3$, then $P$ is a solution to the Lyapunov equation for $A$ for all sufficiently small real numbers $h$ and $k$ that satisfy the inequality: $hr + ks > 0$.
3. If $A \in \mathcal{M}_1$, then $P$ is a solution to the Lyapunov equation for $A$ for all sufficiently small real numbers $h$ and $k$.

Proof. To prove the first item, we take $A \in \mathcal{M}_2$ and matrix $P$ of the form (2). The matrix $Q = AP + PA^*$ will be negative definite if and only if the following inequalities are satisfied: $Q_{11} < 0$, $Q_{22} < 0$, and $\det(Q) > 0$. We compute:

\[
Q_{11} = 2(hr + ks),
Q_{22} = -2(b + hr + ks),
\det Q = -4b(hr + ks) + \gamma_1 h^2 + \gamma_2 k^2 + \gamma_3 hk,
\]

where $\gamma_1, \gamma_2$ and $\gamma_4$ are expressions in $a, b, r, s, m, n$ and do not depend on $h$ and $k$. Observe that every pair of sufficiently small numbers $h$ and $k$ that satisfies inequality $Q_{11} < 0$ also satisfies inequalities $Q_{22} < 0$ and $\det(Q) > 0$.

Similar arguments give us the second item.

For $A \in \mathcal{M}_1$ we have:

\[
A + A^* = \begin{pmatrix} -2a & 0 \\ 0 & -2b \end{pmatrix} < 0,
\]

therefore the identity is a solution to the Lyapunov equation for $A$. Since $\mathcal{P}(A)$ is an open set, the third item holds. \[\square\]

We have found a solution to the Lyapunov equation for each of the matrices in the set $\mathcal{M}$, hence the following corollary clearly holds.

Corollary 5. All matrices in the set $\mathcal{M}$ are stable.

The next proposition gives conditions when the set of matrices from $\mathcal{M}$ has a common solution to the Lyapunov equation.

Proposition 6. Let matrices $A_j$ be of the form:

\[
A_j = \begin{pmatrix} -a_j + im j & r_j + is j \\ -r_j + is j & -b_j + in j \end{pmatrix}
\]

and let $A_j \in \mathcal{M}_1$ for $j = 1, \ldots, l_1$, $A_j \in \mathcal{M}_2$ for $j = l_1 + 1, \ldots, l_2$ and $A_j \in \mathcal{M}_3$ for $j = l_2 + 1, \ldots, l$.

Matrices $A_j$, $j = 1, \ldots, l$, have a common solution to the Lyapunov equation if and only if there exist real numbers $h$ and $k$ that satisfy the following inequalities:

\[
h r_j + ks_j < 0 \quad \text{for } i = l_1 + 1, \ldots, l_2
\]
and \[ hr_j + ks_j > 0 \quad \text{for } j = l_2 + 1, \ldots, l. \] (5)

**Proof.** From Proposition 4 it follows that matrix \( P \) of the form (3) will be a common solution to the Lyapunov equation for matrices \( A_j, j = 1, \ldots, l \), for all sufficiently small numbers \( h \) and \( k \) that satisfy inequalities (4) and (5).

To prove the other implication we assume that there exists a solution to the Lyapunov equation \( P \) for matrices \( A_j, j = 1, \ldots, l \). Without loss of generality we can take \( P \) to be of the form:
\[
P = \begin{pmatrix} 1 & ik \\ h - ik & h + ik \end{pmatrix}
\]
for some real numbers \( h, k \) and \( z \). Set \( Q_j = A_j P + PA_j^* \). For \( j = l_1 + 1, \ldots, l_2 \) we have \( Q_{11} = 2(hr_j + ks_j) \), hence \( hr_j + ks_j < 0 \). For \( j = l_2 + 1, \ldots, l \) we have \( Q_{22} = -2(hr_j + ks_j) \), hence \( hr_j + ks_j > 0 \). Therefore \( h \) and \( k \) satisfy inequalities (4) and (5) and the proof is complete. \( \square \)

From Proposition 6 we can obtain conditions for the existence of the common solution to the Lyapunov equation for two matrices in \( \mathbb{M} \).

**Corollary 7.** For matrices \( A_1 \) and \( A_2 \) in the form as in Proposition 6 the following statements hold:

1. Matrices \( A_1 \in \mathbb{M}_1 \) and \( A_2 \in \mathbb{M}_2 \) have a common solution to the Lyapunov equation.
2. Matrices \( A_1 \in \mathbb{M}_1 \) and \( A_2 \in \mathbb{M}_2 \) have a common solution to the Lyapunov equation unless \( (r_1, s_1) = -\alpha(r_2, s_2) \) for some \( \alpha > 0 \).
3. Matrices \( A_1 \in \mathbb{M}_3 \) and \( A_2 \in \mathbb{M}_3 \) have a common solution to the Lyapunov equation unless \( (r_1, s_1) = -\alpha(r_2, s_2) \) for some \( \alpha > 0 \).
4. Matrices \( A_1 \in \mathbb{M}_2 \) and \( A_2 \in \mathbb{M}_2 \) have a common solution to the Lyapunov equation unless \( (r_1, s_1) = \alpha(r_2, s_2) \) for some \( \alpha > 0 \).

**Proof.** The first item follows directly from Proposition 6. To prove the second item, we observe that there exist numbers \( h \) and \( k \) that satisfy inequalities \( hr_j + ks_j > 0 \) for \( j = 1, 2 \), if and only if \( (r_1, s_1) \neq -\alpha(r_2, s_2) \). We can apply similar arguments to prove the rest of the corollary. \( \square \)

Now we will consider the existence of a common solution to the Lyapunov equation for matrices \( A \) and \( B \), where \( A \in \mathbb{M} \) and \( B \) is a matrix for which \( B + B^* \) is a real negative semidefinite matrix with zero determinant.

**Proposition 8.** Let
\[
A = \begin{pmatrix} -a + im & r + is \\ -r + is & -a + in \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -c + ip & t + u + iv \\ t - u + iv & -d + iq \end{pmatrix},
\]
where \( r, s, m, n, t, u \) and \( v \) are real numbers, \( a, c \) and \( d \) are positive numbers and \( cd - t^2 = 0 \). Matrices \( A \) and \( B \) do not have a common solution to the Lyapunov equation if and only if \( u = 0, r = 0 \) and \( s = -\alpha(t(p - q) + v(c - d)) \) for some positive number \( \alpha \).

**Proof.** In Proposition 4 we have already seen that matrix \( P \) of the form (3) will be a solution to the Lyapunov equation for \( A \) for all sufficiently small numbers \( h \) and \( k \) that satisfy inequality \( hr + ks > 0 \).
Let $Q = BP + PB^*$. The inequalities

$$Q_{11} = 2(-c + h(t + u) + kv) < 0$$

and

$$Q_{22} = 2(-d + h(t - u) - kv) < 0$$

hold for all sufficiently small $h$ and $k$. We compute:

$$\det(Q) = 4hu(c - d) + 4k(t(p - q) + v(c - d)) + h^2\gamma_1 + k^2\gamma_2 + hk\gamma_3,$$

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ do not depend on $h$ and $k$. We see that the matrix $P$ will be a solution to the Lyapunov equation for $B$ for all sufficiently small numbers $h$ and $k$ that satisfy inequality: $hu(c - d) + k(t(p - q) + v(c - d)) > 0$. We conclude that matrices $A$ and $B$ have a common solution to the Lyapunov equation of the form (3) if we can find numbers $h$ and $k$ that satisfy inequalities:

$$hr + ks > 0 \quad \text{and} \quad hu(c - d) + k(t(p - q) + v(c - d)) > 0.$$ 

That is if

$$(r, s) \neq -\alpha(u(c - d), t(p - q) + v(c - d))$$

for some $\alpha > 0$.

Next we consider matrices of the form

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 + z \end{pmatrix}. \quad (6)$$

Let $Q = BP + PB^*$. The inequalities

$$Q_{11} = -2c < 0 \quad \text{and} \quad Q_{22} = -2d(1 + z) < 0$$

hold for all sufficiently small $z$. We compute:

$$\det(Q) = -4tuz - z^2((u + t)^2 + v^2).$$

Since $t \neq 0$ we can choose $z$ such that $\det(Q) > 0$ as long as $u \neq 0$. In this case a matrix of the form (6) will be a solution to the Lyapunov equation for $B$.

The set $\mathcal{P}(B)$ is open, hence a matrix of the form

$$P = \begin{pmatrix} 1 & h + ik \\ h - ik & 1 + z \end{pmatrix} \quad (7)$$

will be a solution to the Lyapunov equation for $B$ for all sufficiently small numbers $h$, $k$. We conclude that a matrix $P$ of the from (7) will be a common solution to the Lyapunov equation for $A$ and $B$ for all sufficiently small numbers $z$, $h$ and $k$ that satisfy inequality $hr + ks > 0$.

We have proved that matrices $A$ and $B$ have a common solution to the Lyapunov equation unless $u = 0$, $r = 0$ and $s = -\alpha(t(p - q) + v(c - d))$ for some positive number $\alpha$. Now we will show that if those relations hold, the matrices $A$ and $B$ do not have a common solution to the Lyapunov equation.

Assume that they have a common solution $P$. Without loss of generality we can assume that $P$ is of the form:

$$P = \begin{pmatrix} 1 & h + ik \\ h - ik & 1 + z \end{pmatrix}.$$
Let \( Q_A = AP + PA^* \) and \( Q_B = BP + PB^* \). We compute:

\[
\det(Q_A) = 2aks(2 + z) - (h(m - n) + zs) - h^2a^2 - k^2(a^2 + (m - n)^2 + 4s^2)
\]

and

\[
\det(Q_B) = 2k(t(p - q) + v(c - d)) - h(c^2 + (p - q)^2 + 4v^2).
\]

Since we want \( \det(Q_A) > 0 \) and \( \det(Q_B) > 0 \) we need to satisfy the inequalities:

\[
2aks(2 + z) > 0 \quad \text{and} \quad 2k(t(p - q) + v(c - d))(2 + z) > 0.
\]

Those inequalities do not hold for any choice of \( k \) and \( z \), since we have assumed that \( s = -\alpha(t(p - q) + v(c - d)) \), \( \alpha > 0 \). We conclude that in this case matrices \( A \) and \( B \) do not have a common solution to the Lyapunov equation. \( \square \)

Before we give the proof of Theorem 2 we need a couple of lemmas. The first lemma is well known and easy to check.

**Lemma 9.** Let \( A \in \mathbb{C}^{n \times n} \) be stable and \( T \in \mathbb{C}^{n \times n} \) invertible. Then \( P \) is a solution to the Lyapunov equation for \( A \) if and only if \( T^{-1}AT \) is a solution to the Lyapunov equation for \( T^{-1}PT \).

In particular, matrices \( A \) and \( B \) have a common solution to the Lyapunov equation if and only if matrices \( T^{-1}AT \) and \( T^{-1}BT \) have.

**Lemma 10.** Let \( A \) and \( B \) be stable matrices such that the matrices \( A - \epsilon I \) and \( B \) have a common solution to the Lyapunov equation for every \( \epsilon > 0 \). Then there exists a positive definite matrix \( P \) such that the matrices \( AP + PA^* \) and \( BP + PB^* \) are negative semidefinite.

**Proof.** For every \( \epsilon > 0 \) there exists positive definite matrix \( P_\epsilon \) with norm 1 such that matrices \( (A - \epsilon I)P_\epsilon + P_\epsilon(A - \epsilon I)^* \) and \( BP_\epsilon + P_\epsilon B^* \) are negative definite. The matrices \( P_\epsilon \) are contained in the compact set, hence there exists a convergent sequence \( \{P_n; n = 1, 2, \ldots\} \) contained in the set \( \{P_\epsilon; \epsilon > 0\} \). The matrix \( P = \lim_{n \to \infty} P_n \) is nonzero positive semidefinite matrix with norm 1 such that the matrices \( AP + PA^* \) and \( BP + PB^* \) are negative semidefinite. Since the matrices \( A \) and \( B \) are invertible, matrices \( AP + PA^* \) and \( BP + PB^* \) are nonzero.

Among all matrices \( P \) that have this property we choose one for which \[
\text{rank}(AP + PA^*) + \text{rank}(BP + PB^*)
\]

is maximal. We will prove that such a matrix \( P \) must be positive definite.

Suppose that \( P \) is singular. Using unitary similarity and Lemma 9 we can assume that \( P \) is of the form:

\[
P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}
\]

for some positive definite matrix \( P_1 \). Let

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}
\]

be the corresponding partitions of the matrices \( A \) and \( B \).
Since the matrix
\[ AP + PA^* = \begin{pmatrix} A_{11}P_1 + P_1A_{11}^* + P_1A_{21}^* \\ A_{21}P_1 \end{pmatrix} \]
is negative semidefinite, we have \( A_{21}P_1 = 0 \) and consequently \( A_{21} = 0 \). Similar argument gives us \( B_{21} = 0 \). Hence the matrices \( A_{22} - \epsilon I \) and \( B_{22} \) have a common solution to the Lyapunov equation for every \( \epsilon > 0 \).

The previous argument applied to the matrices \( A_{22} \) and \( B_{22} \) gives us the existence of a nonzero positive semidefinite matrix \( P_2 \) such that matrices \( A_{22}P_2 + P_2A_{22}^* \) and \( B_{22}P_2 + P_2B_{22}^* \) are nonzero negative semidefinite.

Put \( P_0 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \).

Then
\[ \text{rank}(AP_0 + P_0A^*) + \text{rank}(BP_0 + P_0B^*) > \text{rank}(AP + PA^*) + \text{rank}(BP + PB^*). \]

This contradicts the choice of the matrix \( P \). Hence we have proved that the matrix \( P \) must be positive definite. □

In the following lemma we will show that if conditions (A1) and (A2) in Theorem 2 are satisfied for the matrices \( A \) and \( B \), then those conditions are satisfied for the matrices \( A - \gamma I \) and \( B - \delta I \) for all \( \gamma \geq 0 \) and \( \delta \geq 0 \).

**Lemma 11.** Let \( A \) and \( B \) be stable matrices in \( \mathbb{C}^{n \times n} \) such that conditions (A1) and (A2) are satisfied. Then for all \( \gamma \geq 0 \) and \( \delta \geq 0 \) the following conditions hold:

(D1) The convex cone \( \text{conv}((A - \gamma I + i\alpha I), B - \delta I) \) is stable for all \( \alpha \in \mathbb{R} \).

(D2) The convex cone \( \text{conv}((A - \gamma I + i\alpha I)^{-1}, B - \delta I) \) is stable for all \( \alpha \in \mathbb{R} \).

**Proof.** Let conditions (A1) and (A2) hold and suppose that the convex cone \( \text{conv}(A - \gamma I + i\alpha I, B - \delta I) \) is not stable for some \( \gamma \geq 0 \), \( \delta \geq 0 \) and \( \alpha \in \mathbb{R} \).

Then there exists \( \eta > 0 \) such that the matrix \( (A - \gamma I + i\alpha I) + \eta(B - \delta I) \) has a purely imaginary eigenvalue \( -i\lambda \), \( \lambda \in \mathbb{R} \):
\[ \det((A - \gamma I + i\alpha I) + \eta(B - \delta I) + i\lambda I) = 0. \]

It follows that the matrix \( A + \eta B \) has an eigenvalue \( \gamma + \delta - i(\alpha + \lambda) \), hence it is not stable. Therefore the convex cone \( \text{conv}(A, B) \) is not stable, which contradicts condition (A1).

Suppose that the convex cone \( \text{conv}((A - \gamma I + i\alpha I)^{-1}, B - \delta I) \) is not stable for some \( \gamma \geq 0 \), \( \delta \geq 0 \) and \( \alpha \in \mathbb{R} \). Then there exists \( \eta > 0 \) such that the matrix \( (A - \gamma I + i\alpha I)^{-1} + \eta(B - \delta I) \) has a purely imaginary eigenvalue \( -i\lambda \), \( \lambda \in \mathbb{R} \). Hence
\[ \det((A - \gamma I + i\alpha I)^{-1} + \eta(B - \delta I) + i\lambda I) = 0 \]
and
\[ \det((\eta(B - \delta I) + i\lambda I)^{-1} + A - \gamma I + i\alpha I) = 0. \]
Since \((\eta(B - \delta I) + i\lambda I)^{-1} + A\) is not stable, there exists \(\eta_1 > 0\) such that the matrix \((\eta(B - \delta I) + i\lambda I)^{-1} + \eta_1 A\) has a purely imaginary eigenvalue \(-i\lambda_1, \lambda_1 \in \mathbb{R}:
\[
\det((\eta(B - \delta I) + i\lambda I)^{-1} + \eta_1 A + i\lambda_1 I) = 0.
\]

It follows that
\[
\det(\eta(B - \delta I) + i\lambda I + (\eta_1 A + i\lambda_1 I)^{-1}) = 0.
\]

This contradicts the assumption that the convex cone \(\text{conv}((A + i\lambda_1/\eta_1 I)^{-1}, B)\) is stable. \(\square\)

We can now prove the main result of this paper.

**Proof of Theorem 2.** If matrices \(A\) and \(B\) have a common solution to the Lyapunov equation then \(\text{cic}(A, B)\) is stable, hence conditions (A1) and (A2) are satisfied.

We will prove the other implication by contradiction. We suppose that matrices \(A\) and \(B\) do not have a common solution to the Lyapunov equation, but they satisfy conditions (A1) and (A2) in Theorem 2.

Let
\[
\alpha_0 = \inf\{\alpha; \text{ } A - \alpha I_2 \text{ and } B \text{ have a common solution to the Lyapunov equation}\}.
\]

Define \(A_0 = A - \alpha_0 I_2\). Then by Lemma 10 there exists a positive definite matrix \(P\) such that matrices \(A_0 P + PA_0^*\) and \(B P + PB^*\) are negative semidefinite. If either \(A_0 P + PA_0^* > 0\) or \(B P + PB^* > 0\), then matrices \(A_0\) and \(B\) have a common solution to the Lyapunov equation. Hence \(\det(A_0 P + PA_0^*) = 0\) and \(\det(B P + PB^*) = 0\). By Lemma 11, the matrices \(A_0\) and \(B\) satisfy conditions (A1) and (A2).

The matrices \(A_1 = P^{-1/2} A_0 P^{1/2}\) and \(B_1 = P^{-1/2} B P^{1/2}\) are stable, satisfy conditions (A1) and (A2) and \(A_1 + A_1^* \leq 0\) and \(B_1 + B_1^* \leq 0\). Furthermore, Lemma 9 tells us that matrices \(A_1 - \epsilon I\) and \(B_1\) have a common solution to the Lyapunov equation for every \(\epsilon > 0\), but matrices \(A_1\) and \(B_1\) do not have a common solution to the Lyapunov equation.

Let \(U_1\) be a unitary matrix such that
\[
U_1^*(A + A^*)U_1 = \begin{pmatrix}
-2a & 0 \\
0 & 0 \\
\end{pmatrix}
\]
and
\[
U_1^*(B + B^*)U_1 = \begin{pmatrix}
-2c & \gamma \\
\bar{\gamma} & -2d \\
\end{pmatrix}.
\]

We choose a real number \(\theta\) such that \(e^{i\theta} \gamma = 2t > 0\) and define:
\[
D = \begin{pmatrix}
e^{-i\theta} & 0 \\
0 & 1 \\
\end{pmatrix}
\]
and \(U = U_1 D\).

Then
\[
U^*(A + A^*)U = \begin{pmatrix}
-2a & 0 \\
0 & 0 \\
\end{pmatrix}
\]
and
\[
U^*(B + B^*)U = \begin{pmatrix}
-2c & 2t \\
2t & -2d \\
\end{pmatrix}.
\]

Set \(A_2 = U^* A_1 U\) and \(B_2 = U^* B_1 U\).

Then
\[
A_2 = \begin{pmatrix}
-a + im & r + is \\
-r + is & in \\
\end{pmatrix}
\]
and
\[
B_2 = \begin{pmatrix}
-c + ip & t - u + iv \\
t + u + iv & -d + iq \\
\end{pmatrix},
\]

where \(r, s, m, n, t, u\) and \(v\) are real numbers, \(a, b\) and \(c\) are nonnegative numbers and \(cd - t^2 = 0\).

Now we look at the conditions for the common solution to the Lyapunov equation. We consider several cases.
If \( t = 0, c = 0 \) and \( d \neq 0 \), then \( A_2 \in \mathcal{M}_3 \) and \( B_2 \in \mathcal{M}_2 \). Matrices \( A_2 \) and \( B_2 \) do not have a common solution to the Lyapunov equation if and only if \( (u, v) = \alpha(r, s) \) for some positive \( \alpha \), by Corollary 7. Hence:

\[
A_2 = \begin{pmatrix}
-a + im & r + is \\
-r + is & i n
\end{pmatrix}
\quad \text{and} \quad
B_2 = \begin{pmatrix}
\alpha p & \alpha(r + is) \\
\alpha(-r + is) & -d + iq
\end{pmatrix}.
\]

A short computation gives us:

\[
\alpha(r^2 + s^2)(A - i n I)^{-1} + B_2 = \begin{pmatrix}
-ip & 0 \\
0 & \alpha(-a + i(m - n)) - d + iq
\end{pmatrix}.
\]

We see that the matrix \( \alpha(r^2 + s^2)(A - i n I)^{-1} + B_2 \) is not stable, contrary to condition (A2).

If \( t = 0, c \neq 0 \) and \( d = 0 \), then \( A_2 \in \mathcal{M}_3 \) and \( B_2 \in \mathcal{M}_3 \). By Corollary 7 they do not have a common solution to the Lyapunov equation if and only if \( (u, v) = -\alpha(r, s) \) for some positive \( \alpha \). Since in this case

\[
\alpha A_2 + B_2 = \begin{pmatrix}
\alpha(-a + im) - c + ip & 0 \\
0 & i(\alpha n + q)
\end{pmatrix},
\]

we have a contradiction to the stability of the convex cone \( \text{conv}(A_2, B_2) \).

Finally, let \( t \neq 0 \). Then \( c \neq 0 \) and \( d \neq 0 \). In this case Proposition 8 tells us that matrices \( A_2 \) and \( B_2 \) do not have a common solution to the Lyapunov equation if and only if \( u = 0, r = 0, \) and \( s = -\alpha(t(p - q) + v(c - d)) \) for some positive \( \alpha \). Therefore:

\[
A_2 = \begin{pmatrix}
-a + im & -i\alpha(t(p - q) + v(c - d)) \\
-i\alpha(t(p - q) + v(c - d)) & i n
\end{pmatrix}
\]

and

\[
B_2 = \begin{pmatrix}
-c + ip & t + iv \\
t + iv & -d + iq
\end{pmatrix},
\]

where \( cd = t^2 \). Now take

\[
\beta = \frac{c^2(ct(p - q) + (c - t)(c + t)v)^2(t^2 + v^2)}{t^4},
\]

\[
\gamma = -p - \frac{c}{t} v \quad \text{and} \quad \delta = \frac{c^3(t^2 + v^2)}{t^2 \alpha} > 0.
\]

A simple computation gives us

\[
\beta(B_2 + iyI_2)^{-1} + \delta A_2 = \begin{pmatrix}
\zeta & 0 \\
0 & \eta
\end{pmatrix},
\]

where \( \zeta \in \mathbb{C} \) and \( \eta \in \mathbb{R} \). We conclude that \( \text{conv}((A_2 - \frac{y}{\delta}I_2)^{-1}, B_2) \) is not stable. \[\square\]

3. Examples

We present an example to show that stability of the convex invertible cone \( \text{cic}(A, B) \) is not sufficient for the existence of a common solution to the Lyapunov equation for \( 2 \times 2 \) complex matrices \( A \) and \( B \).
Example 12. Take matrices $A \in \mathcal{M}_3$ and $B \in \mathcal{M}_2$, that do not have a common solution to the Lyapunov equation:

$$A = \begin{pmatrix} -\alpha + im & r + is \\ -r + is & in \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} ip & \alpha(r + is) \\ \alpha(-r + is) & -d + iq \end{pmatrix}.$$ 

Assume that $n \neq 0$, $p \neq 0$ and $\alpha \neq mp/(r^2 + s^2)$. We will show that in this case the convex invertible cone $\text{cic}(A, B)$ is stable.

It is easy to see that the matrix $M + M^*$ is negative semidefinite for every matrix $M \in \text{cic}(A, B)$.

We define the following sets:

$$\mathcal{N}_1 = \{\alpha A + \beta B + \gamma A^{-1} + \delta B^{-1}; \alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta \geq 0\}$$

and

$$\mathcal{N}_j = \{M_1 + M_2 + M_3^{-1}; M_1, M_2, M_3 \in \mathcal{N}_{j-1}\}.$$ 

Then $\mathcal{N}_j \subseteq \mathcal{N}_{j+1}$ and $\bigcup_{j=1}^{\infty} \mathcal{N}_j = \text{cic}(A_1, A_2)$, since $\bigcup_{j=1}^{\infty} \mathcal{N}_j$ is closed under addition, multiplication by a positive scalar and inversion.

Using induction we will show that the matrices of the form $\alpha A, \alpha B, \alpha A^{-1}$ or $\alpha B^{-1}$ for some $\alpha \geq 0$ are the only matrices in $\mathcal{N}_j$ for which the identity matrix is not a solution to the Lyapunov equation.

It is easy to see that the identity matrix is a solution to the Lyapunov equation for matrices $A + \alpha B, A^{-1} + \alpha B, A + \alpha A^{-1}, A^{-1} + \alpha B^{-1}, A + \alpha A^{-1}$ and $B + \alpha B^{-1}$ for every $\alpha > 0$. Hence the statement holds for $\mathcal{N}_1$.

Assuming that it is true for $\mathcal{N}_j$ we will prove it for $\mathcal{N}_{j+1}$. Take

$$M = M_1 + M_2 + M_3^{-1} \in \mathcal{N}_{j+1},$$

where matrices $M_1, M_2$ and $M_3$ lie in $\mathcal{N}_j$. Matrices $M_1 + M_1^*, M_2 + M_2^*$ and $M_3 + M_3^*$ are negative semidefinite. If the identity matrix is a solution to the Lyapunov equation for either $M_1$, $M_2$ or $M_3$, then it is also a solution for $M$. If not, then $M_1, M_2$ and $M_3$ are of the form $\alpha A, \alpha B, \alpha A^{-1}$ or $\alpha B^{-1}$ for some $\alpha \geq 0$ by induction hypothesis. This implies that $M$ lies in $\mathcal{N}_1$.

We have proved that the only matrices in $\text{cic}(A, B)$ for which the identity matrix is not a solution to the Lyapunov equation are the matrices of the form $\alpha A, \alpha B, \alpha A^{-1}$ or $\alpha B^{-1}$ for some $\alpha \geq 0$. Since matrices $A, B$ are stable, this implies that $\text{cic}(A, B)$ is stable.

In the following example we look at a special case of the previous example. We show that stability of $\text{cic}(A, B)$ does not imply the existence of a common solution to the Lyapunov equation for complex $2 \times 2$ matrices $A$ and $B$ even in the case when rank of the matrix $A - B$ is one.

Example 13. Let $m, n, r$ be real numbers such that $r$ is not equal to 0 or 1, $m \neq 0$, $n \neq 0$ and $r(r - 1) \neq m^2$. Let

$$R = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$ 

Consider the following matrices:

$$A = \begin{pmatrix} -1 + im & r \\ -r & in \end{pmatrix} \quad \text{and} \quad B = A + R.$$ 

Corollary 7 tells us matrices $A$ and $B$ have a common solution to the Lyapunov equation if and only if $r$ lies in the interval $(0,1)$ and Example 12 tells us that $\text{cic}(A, B)$ is stable.
4. Real case

In this section we will explain how can results for \(2 \times 2\) complex matrices be related to a class of \(4 \times 4\) real matrices. We will use the standard embedding of \(\mathbb{C}^{n \times n}\) into \(\mathbb{R}^{2n \times 2n}\).

We will write matrix \(A\) in \(\mathbb{C}^{n \times n}\) in the following way:

\[
A = A_{\text{Re}} + iA_{\text{Im}}
\]

where \(A_{\text{Re}}\) and \(A_{\text{Im}}\) are matrices in \(\mathbb{R}^{n \times n}\).

Denote \(\tilde{A} = A_{\text{Re}} - iA_{\text{Im}}\) and

\[
\hat{A} = \begin{pmatrix}
A_{\text{Re}} & A_{\text{Im}} \\
-A_{\text{Im}} & A_{\text{Re}}
\end{pmatrix}.
\]

Since the spectrum of the matrix \(\hat{A}\) is the union of the spectra of matrices \(A\) and \(\tilde{A}\), the matrix \(\hat{A}\) is stable if and only if the matrix \(A\) is stable.

**Proposition 14.** There exists a common solution to the Lyapunov equation \(P \in \mathbb{C}^{n \times n}\) for matrices \(A\) and \(B\) if and only if there exists a common solution \(\hat{P} \in \mathbb{R}^{2n \times 2n}\) for matrices \(\hat{A}\) and \(\hat{B}\).

**Proof.** Let

\[
T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix},
\]

where \(I_n\) is the identity matrix in \(\mathbb{R}^{n \times n}\).

Observe that:

\[
\tilde{A} = T^* \hat{A} T = \begin{pmatrix} A & 0 \\ 0 & \bar{A} \end{pmatrix} \quad \text{and} \quad \tilde{B} = T^* \hat{B} T = \begin{pmatrix} B & 0 \\ 0 & \bar{B} \end{pmatrix}.
\]

Assume that \(A\) and \(B\) have a common solution to the Lyapunov equation \(P \in \mathbb{C}^{n \times n}\). Then

\[
\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}
\]

is a common solution to the Lyapunov equation for \(\tilde{A}\) and \(\tilde{B}\). Lemma 9 tells us that

\[
\hat{P} = T \tilde{P} T^* = \begin{pmatrix} P_{\text{Re}} & P_{\text{Im}} \\ -P_{\text{Im}} & P_{\text{Re}} \end{pmatrix} \in \mathbb{R}^{2n \times 2n}
\]

is a common solution to the Lyapunov equation for matrices \(\hat{A}\) and \(\hat{B}\).

To prove the other implication, we suppose that \(P_1 \in \mathbb{R}^{2n \times 2n}\) is a common solution to the Lyapunov equation for matrices \(A\) and \(B\). Then

\[
\tilde{P}_1 = T^* P_1 T = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{pmatrix}
\]

is a common solution to the Lyapunov equation for matrices \(\tilde{A}\) and \(\tilde{B}\). Now it is easy to see that \(P_{11}\) is a common solution to the Lyapunov equation for matrices \(A\) and \(B\). \(\Box\)

We are ready to state the conditions for the existence of a common solution to the Lyapunov equation for real \(4 \times 4\) matrices that correspond to \(2 \times 2\) complex matrices. Note that matrix

\[
J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}
\]

corresponds to the matrix \(iI_2\) in the standard embedding.

**Theorem 15.** Let \(A_1, A_2, B_1\) and \(B_2\) be real \(2 \times 2\) matrices such that the matrices:

\[
A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix}
\]

have a common solution to the Lyapunov equation.
are stable. Then $A$ and $B$ have a common solution to the Lyapunov equation if and only if the following conditions are satisfied:

\((D1)\) The convex cone $\text{conv}((A + \alpha J), B)$ is stable for all $\alpha \in \mathbb{R}$.

\((D2)\) The convex cone $\text{conv}((A + \alpha J)^{-1}, B)$ is stable for all $\alpha \in \mathbb{R}$.

**Proof.** By Proposition 14 matrices $A$ and $B$ have a common solution to the Lyapunov equation if and only if matrices $A_1 + iA_2$ and $B_1 + iB_2$ have. Therefore we have to show that conditions \((A1)\) and \((A2)\) hold for matrices $A_1 + iA_2$ and $B_1 + iB_2$ if and only if the conditions \((D1)\) and \((D2)\) hold for matrices $A$ and $B$.

Let $T$ be the matrix (8) defined in the proof of Proposition 14. Since

\[
T^*(A + \alpha J)T = \begin{pmatrix}
A_1 + i(A_2 + \alpha I_2) & 0 \\
0 & A_1 - i(A_2 + \alpha I_2)
\end{pmatrix}
\]

and

\[
T^*(A + \alpha J)^{-1}T = \begin{pmatrix}
(A_1 + i(A_2 + \alpha I_2))^{-1} & 0 \\
0 & (A_1 - i(A_2 + \alpha I_2))^{-1}
\end{pmatrix}
\]

the conditions are clearly equivalent. \(\square\)

**Corollary 16.** Let $A_1, A_2, B_1$ and $B_2$ be real $2 \times 2$ matrices such that the matrices:

\[
A = \begin{pmatrix}
A_1 & A_2 \\
-A_2 & A_1
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
B_1 & B_2 \\
-B_2 & B_1
\end{pmatrix}
\]

are stable. Then $A$ and $B$ have a common solution to the Lyapunov equation if and only if the following conditions are satisfied:

\((E1)\) The convex cone $\text{conv}((A + i\alpha I_4), B)$ is stable for all $\alpha \in \mathbb{R}$.

\((E1)\) The convex cone $\text{conv}((A + i\alpha I_4)^{-1}, B)$ is stable for all $\alpha \in \mathbb{R}$.

**Proof.** We observe that

\[
T^*(iI_{2n})T = iI_{2n}
\]

for the matrix $T$ defined in (8). The rest of the proof is similar to the proof of Theorem 15. \(\square\)

**Remark 17.** In the previous section we have seen that for real $2 \times 2$ matrices $A$ and $B$, stability of the convex cones $\text{conv}(A, B)$ and $\text{conv}(A^{-1}, B)$ implies the stability of convex cones $\text{conv}(A + i\alpha I_2, B + i\beta I_2)$ and $\text{conv}(A + i\alpha I_2)^{-1}, B + i\beta I_2)$ for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. We see that this is not true for real $4 \times 4$ matrices. Therefore stability of these cones is a necessary condition for the existence of a common solution to the Lyapunov equation for matrices $A$ and $B$ that is stronger than the conditions that the convex cones $\text{conv}(A, B)$ and $\text{conv}(A^{-1}, B)$ are stable.

We can use Example 13 to show that the stability of $\text{cic}(A, B)$ is not sufficient for the existence of a common solution to the Lyapunov equation for real $4 \times 4$ matrices. In particular, this is not true even in the case when rank of the matrix $A - B$ is two.

**Example 18.** Let $m, n, r, s$ be real numbers such that $r$ is not equal to 0 or 1, $m \neq 0$, $n \neq 0$ and $r(r - 1) \neq m^2$. Let
Consider the following matrices:

\[ \hat{R} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}. \]

The matrices \( \hat{A} \) and \( \hat{B} \) do not have a common solution to the Lyapunov equation if and only if \( r \) does not lie in the interval \((0, 1)\). However, \( \text{cic}(\hat{A}, \hat{B}) \) is stable for every \( r \in \mathbb{R} \).

Theorem 19. Let \( A \) and \( B \) be real \( n \times n \) matrices, such that the matrices \( A^k \) and \( B^l \) commute with both \( A \) and \( B \) for some \( k, l \in \{1, 2, 3, 4\} \). Then matrices \( A \) and \( B \) have a common solution to the Lyapunov equation if and only if the following conditions are satisfied:

(A1) The convex cone \( \text{conv}(\lambda I + A^1) \) is stable for all \( \lambda \in \mathbb{R} \).
(A2) The convex cone \( \text{conv}(\lambda I + A^{k-1}) \) is stable for all \( \lambda \in \mathbb{R} \).

Proof. If \( k = 1 \) or \( l = 1 \) then matrices \( A \) and \( B \) commute. We observe that for a stable matrix \( A \) the set of matrices that commute with \( A^2 \) is the same as the set of matrices that commute with \( A \). Thus matrices \( A \) and \( B \) commute if \( k = 2 \) or \( l = 2 \). Commuting matrices have a common solution to the Lyapunov equation.

Now we consider the case when \( k, l \in \{3, 4\} \). Assume that conditions (A1) and (A2) hold. Let \( \mathcal{S} \) be a simple component of the algebra over \( \mathbb{C} \) generated by matrices \( A \) and \( B \) and let \( \mathcal{A}_\mathcal{S} \) and \( \mathcal{B}_\mathcal{S} \) be the images of matrices \( A \) and \( B \) in \( \mathcal{S} \). To prove our statement it suffices to show that matrices \( \mathcal{A}_\mathcal{S} \) and \( \mathcal{B}_\mathcal{S} \) have a common solution to the Lyapunov equation.

Matrices \( \mathcal{A}_\mathcal{S} \) and \( \mathcal{B}_\mathcal{S} \) are central in \( \mathcal{S} \). Hence \( \mathcal{A}_\mathcal{S} = \alpha I \) and \( \mathcal{B}_\mathcal{S} = \beta I \) for some \( \alpha \in \mathbb{C} \) and \( \beta \in \mathbb{C} \). It follows that minimal polynomial of \( A \) divides polynomial \( q(x) = x^k - \alpha \). For \( k = 3 \) or \( k = 4 \) at most two \( k \)th roots of \( \alpha \) have negative real part, hence the minimal polynomial of the matrix \( A \) is linear or quadratic. The same argument tells us that the matrix \( \mathcal{B}_\mathcal{S} \) has a linear or a quadratic minimal polynomial. Laffey [17] proved that this implies that \( \mathcal{S} \) is isomorphic to \( \mathbb{C} \) or \( \mathbb{C}^2 \times \mathbb{C} \).

If \( \mathcal{S} \) is isomorphic to \( \mathbb{C} \), then the matrices \( \mathcal{A}_\mathcal{S} \) and \( \mathcal{B}_\mathcal{S} \) commute. Thus they have a common solution to the Lyapunov equation. Observe that matrices \( \mathcal{A}_\mathcal{S} \) and \( \mathcal{B}_\mathcal{S} \) satisfy conditions (A1) and (A2). Therefore we can use Corollary 16 to prove the statement when \( \mathcal{S} \) is isomorphic to \( \mathbb{C}^2 \times \mathbb{C} \).

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