Common solution to the Lyapunov equation for 2 × 2 complex matrices

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Abstract

In this work we solve the problem of a common solution to the Lyapunov equation for 2 × 2 complex matrices. We show that necessary and sufficient conditions for the existence of a common solution to the Lyapunov equation for 2 × 2 complex matrices A and B is that matrices (A + iαI)(B + iβI) and (A + iαI)−1(B + iβI) have no negative real eigenvalues for all α, β ∈ R. We show how these results relate to a special class of 4 × 4 real matrices.

Keywords: Lyapunov equation; Lyapunov functions; Stability; Convex cones; Convex invertible cones

1. Introduction

A matrix A ∈ Cn×n is called (Hurwitz) stable if all its eigenvalues lie in the open left half of the complex plane. In this case, the linear-time invariant (LTI) system

\[ \dot{x} = Ax \] (1)

is asymptotically stable.
A classical result of Lyapunov states, that a matrix $A$ is stable if and only if for arbitrary Hermitian positive definite $Q$, the Lyapunov equation

$$AP + PA^* = -Q$$

admits a positive definite solution $P$. The associated form $V(x) = x^TPx$ is called a quadratic Lyapunov function for the system (1).

We shall use the convention that $P > 0$ denotes a Hermitian positive definite matrix and thus, for a given stable matrix $A$, we will denote the set of all solutions to the Lyapunov equation for $A$ by

$$\mathcal{P}(A) = \{ P = P^* > 0 : AP + PA^* < 0 \}.$$

$\mathcal{P}(A)$ is an open convex cone.

Let $A_1, A_2, \ldots, A_k$ be stable matrices in $\mathbb{C}^{n\times n}$ and let $P > 0$ be a common solution to the following Lyapunov equations:

$$A_j P + PA_j^* < 0 \quad \text{for} \quad j = 1, 2, \ldots, k.$$

We say that the matrix $P$ is a common solution to the Lyapunov equation for matrices $A_j, j = 1, 2, \ldots, k$. Accompanying quadratic Lyapunov function $V(x) = x^TPx$ is called a common quadratic Lyapunov function (CQLF) for the LTI systems $\dot{x} = A_jx, j = 1, \ldots, k$.

The problem of deciding whether stable matrices $A_j, j = 1, \ldots, k$, share a common solution to the Lyapunov equation has been extensively studied, but the complete solution is known only in a few special cases. For a source of literature on the problem, we refer the reader to the following works and the citations that appear in them [1–7]. The problem has a wide variety of applications in systems and control theory and elsewhere.

In [8] Loewy considered the following question. Given a stable matrix $A \in \mathbb{C}^{n\times n}$, for what matrices $B$ does $\mathcal{P}(A) = \mathcal{P}(B)$ hold. He proved the following result.

**Theorem 1.** Let $A, B \in \mathbb{C}^{n\times n}$ be stable matrices. Then $\mathcal{P}(A) = \mathcal{P}(B)$ if and only if

$$B = \mu(A + i\alpha I) \quad \text{for some} \quad \alpha, \mu \in \mathbb{R} \quad \text{such that} \quad \mu > 0$$

or

$$B = \mu((A + i\alpha_1 I)^{-1} + i\alpha_2 I) \quad \text{for some} \quad \alpha_1, \alpha_2, \mu \in \mathbb{R} \quad \text{such that} \quad \mu > 0.$$
For given matrices $A$ and $B$, the stability of $\hat{\text{cic}}(A, B)$ is difficult to check. Let us introduce two weaker necessary conditions that are easy to verify. If the convex cone $\text{conv}(A, B)$ is stable, then the matrix $A^{-1}B$ has no negative real eigenvalues. Similarly, stability of the convex cone $\text{conv}(A^{-1}, B)$ implies that the matrix $AB$ has no negative real eigenvalues. In some special cases those weaker conditions are sufficient for the existence of a common solution to the Lyapunov equation.

Let $A$ and $B$ be real stable matrices such that the rank of $A - B$ is one. Shorten and Narendra [12] proved that a necessary and sufficient condition for the existence of a common solution to the Lyapunov equation for matrices $A$ and $B$ is that the matrix product $AB$ does not have a real negative eigenvalue. A different proof of this result was presented by King and Nathanson in [13]. In this paper we will show that when $A$ and $B$ are real and rank of $A - B$ is two, even the stability of $\hat{\text{cic}}(A, B)$ is not sufficient for the existence of a common solution to the Lyapunov equation for $A$ and $B$.

Necessary and sufficient conditions for the existence of a common solution to the Lyapunov equation for $2 \times 2$ real matrices $A$ and $B$ is that matrices $AB$ and $A^{-1}B$ do not have a real negative eigenvalue. In this case those conditions are equivalent to the stability of the convex cones $\text{conv}(A, B)$ and $\text{conv}(A^{-1}, B)$. The proof of this result can be found in [10]. The special case of stable matrices was proved earlier in [14] and in [15].

In this work we will investigate the existence of a common solution to the Lyapunov equation for $2 \times 2$ complex matrices. We will show that necessary and sufficient conditions for the existence of a common solution to the Lyapunov equation for $2 \times 2$ complex matrices $A$ and $B$ is that convex cones $\text{conv}((A + i\alpha I_2), B)$ and $\text{conv}((A + i\alpha I_2)^{-1}, B)$ are stable for all $\alpha \in \mathbb{R}$.

The notation we will use is standard. For example, by $\mathbb{R}$ we will denote the set of real numbers and by $\mathbb{C}$ the set of complex numbers. By $\mathbb{R}^{n \times n}$ we will denote the set of $n \times n$ real matrices and by $\mathbb{C}^{n \times n}$ the set of $n \times n$ complex matrices. We shall write $A^*$ for the conjugate transpose of the matrix $A$ and $A^T$ for the transpose of the matrix $A$. We will denote by $Q_{ij}$ the $(i, j)$th element of the matrix $Q$.

### 2. Solution to the Lyapunov equation for $2 \times 2$ complex matrices

The main result of this paper gives necessary and sufficient conditions for a pair of complex $2 \times 2$ matrices to have a common solution to the Lyapunov equation. First we will state the result and the remainder of this section will gradually lead us to its proof.

**Theorem 2.** The stable matrices $A \in \mathbb{C}^{2 \times 2}$ and $B \in \mathbb{C}^{2 \times 2}$ have a common solution to the Lyapunov equation if and only if the following conditions are satisfied:

(A1) The convex cone $\text{conv}((A + i\alpha I_2), B)$ is stable for all $\alpha \in \mathbb{R}$.

(A2) The convex cone $\text{conv}((A + i\alpha I_2)^{-1}, B)$ is stable for all $\alpha \in \mathbb{R}$.

**Remark 3.** Let us look at two equivalent ways in which we can express the conditions in Theorem 2. Conditions (A1) and (A2) are equivalent to the conditions:

(B1) Matrix $(A + i\alpha I)(B + i\beta I)$ has no negative real eigenvalues for all $\alpha, \beta \in \mathbb{R}$.

(B2) Matrix $(A + i\alpha I)^{-1}(B + i\beta I)$ has no negative real eigenvalues for all $\alpha, \beta \in \mathbb{R}$.
Indeed to establish this equivalence assume that conditions (A1) and (A2) hold. If (B1) does not hold, then the matrix \((A + i\alpha I)(B + i\beta I)\) has a negative real eigenvalue \(-\mu, \mu > 0\), for some \(\alpha, \beta \in \mathbb{R}\). Hence
\[
\det((A + i\alpha I)(B + i\beta I) + \mu I) = 0
\]
and
\[
\det((B + i\beta I) + \mu(A + i\alpha I)^{-1}) = 0.
\]
Therefore the convex cone \(\text{conv}((A + i\alpha I)^{-1}, B)\) is not stable. This contradicts the condition (A2). Similarly we can show that the existence of a negative real eigenvalue for the matrix \((A + i\alpha I)^{-1}(B + i\beta I)\) contradicts the condition (A1).

Now assume that conditions (B1) and (B2) are satisfied. If the convex cone \(\text{conv}(A + i\alpha I_2, B)\) is not stable, then there exists \(\lambda_0 > 0\) such that the matrix \(\lambda_0(A + i\alpha I_2) + B\) has a purely imaginary eigenvalue \(-i\beta, \beta \in \mathbb{R}\). Hence the matrix \((A + i\alpha I)^{-1}(B + i\beta I)\) has a negative eigenvalue \(-\lambda_0\), contrary to the condition (B2). In a similar way we get a contradiction to the condition (B1) if the convex cone \(\text{conv}((A + i\alpha I_2)^{-1}, B)\) is not stable.

Conditions (B1) and (B2) are clearly equivalent to the conditions:

1. The convex cone \(\text{conv}(A + i\alpha I_2, B + i\beta I_2)\) is nonsingular for all \(\alpha, \beta \in \mathbb{R}\).
2. The convex cone \(\text{conv}((A + i\alpha I_2)^{-1}, B + i\beta I_2)\) is nonsingular for all \(\alpha, \beta \in \mathbb{R}\).

First we will consider the existence of a common solution to the Lyapunov equation for matrices of the form \(A = D + K\), where \(D\) is a diagonal matrix and \(K\) is a skew Hermitian matrix. Therefore we will be looking at the matrices of the form:

\[
\begin{pmatrix}
-a + im & r + is \\
-r + is & -b + in
\end{pmatrix}
\]

(2)

for some real numbers \(a, b, m, n, r, s\).

We define the following sets of matrices:

\[
\mathcal{M}_1 = \left\{ A = \begin{pmatrix}
-a + im & r + is \\
-r + is & -b + in
\end{pmatrix} : (m, n, r, s) \in \mathbb{R}^4, a > 0, b > 0 \right\},
\]

\[
\mathcal{M}_2 = \left\{ A = \begin{pmatrix}
im & r + is \\
-r + is & -b + in
\end{pmatrix} : (m, n, r, s) \in \mathbb{R}^4, b > 0, r + is \neq 0 \right\},
\]

\[
\mathcal{M}_3 = \left\{ A = \begin{pmatrix}
-a + im & r + is \\
-r + is & -in
\end{pmatrix} : (m, n, r, s) \in \mathbb{R}^4, a > 0, r + is \neq 0 \right\}
\]

and

\[
\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3.
\]

First we will show that every matrix \(A \in \mathcal{M}\) has a solution to the Lyapunov equation of the form:

\[
P = \begin{pmatrix}
1 & h + ik \\
h - ik & 1
\end{pmatrix}
\]

(3)

for some real numbers \(h\) and \(k\).
Proposition 4. Let the matrix $A$ be of the form (2) and the matrix $P$ be of the form (3). Then the following statements hold:

1. If $A \in \mathcal{M}_2$, then $P$ is a solution to the Lyapunov equation for $A$ for all sufficiently small real numbers $h$ and $k$ that satisfy the inequality: $hr + ks < 0$.
2. If $A \in \mathcal{M}_3$, then $P$ is a solution to the Lyapunov equation for $A$ for all sufficiently small real numbers $h$ and $k$ that satisfy the inequality: $hr + ks > 0$.
3. If $A \in \mathcal{M}_1$, then $P$ is a solution to the Lyapunov equation for $A$ for all sufficiently small real numbers $h$ and $k$.

Proof. To prove the first item, we take $A \in \mathcal{M}_2$ and matrix $P$ of the form (2). The matrix $Q = AP + PA^*$ will be negative definite if and only if the following inequalities are satisfied: $Q_{11} < 0$, $Q_{22} < 0$ and $\det(Q) > 0$. We compute:

\[ Q_{11} = 2(hr + ks), \]
\[ Q_{22} = -2(b + hr + ks), \]
\[ \det Q = -4b(hr + ks) + \gamma_1 h^2 + \gamma_2 k^2 + \gamma_3 hk, \]

where $\gamma_1$, $\gamma_2$ and $\gamma_4$ are expressions in $a$, $b$, $r$, $s$, $m$, $n$ and do not depend on $h$ and $k$. Observe that every pair of sufficiently small numbers $h$ and $k$ that satisfies inequality $Q_{11} < 0$ also satisfies inequalities $Q_{22} < 0$ and $\det(Q) > 0$.

Similar arguments give us the second item.

For $A \in \mathcal{M}_1$ we have:

\[ A + A^* = \begin{pmatrix} -2a & 0 \\ 0 & -2b \end{pmatrix} < 0, \]

therefore the identity is a solution to the Lyapunov equation for $A$. Since $\mathcal{P}(A)$ is an open set, the third item holds. \( \square \)

We have found a solution to the Lyapunov equation for each of the matrices in the set $\mathcal{M}$, hence the following corollary clearly holds.

Corollary 5. All matrices in the set $\mathcal{M}$ are stable.

The next proposition gives conditions when the set of matrices from $\mathcal{M}$ has a common solution to the Lyapunov equation.

Proposition 6. Let matrices $A_j$ be of the form:

\[ A_j = \begin{pmatrix} -a_j + im_j & r_j + is_j \\ -r_j + is_j & -b_j + in_j \end{pmatrix} \]

and let $A_j \in \mathcal{M}_1$ for $j = 1, \ldots, l_1$, $A_j \in \mathcal{M}_2$ for $j = l_1 + 1, \ldots, l_2$ and $A_j \in \mathcal{M}_3$ for $j = l_2 + 1, \ldots, l$.

Matrices $A_j$, $j = 1, \ldots, l$, have a common solution to the Lyapunov equation if and only if there exist real numbers $h$ and $k$ that satisfy the following inequalities:

\[ hr_j + ks_j < 0 \quad \text{for } i = l_1 + 1, \ldots, l_2 \]
and
\[ hr_j + ks_j > 0 \quad \text{for } j = l_2 + 1, \ldots, l. \]

**Proof.** From Proposition 4 it follows that matrix \( P \) of the form (3) will be a common solution to the Lyapunov equation for matrices \( A_j, j = 1, \ldots, l \), for all sufficiently small numbers \( h \) and \( k \) that satisfy inequalities (4) and (5).

To prove the other implication we assume that there exists a solution to the Lyapunov equation \( P \) for matrices \( A_j, j = 1, \ldots, l \). Without loss of generality we can take \( P \) to be of the form:
\[
P = \begin{pmatrix}
1 & h + ik
h - ik & z
\end{pmatrix}
\]
for some real numbers \( h, k \) and \( z \). Set \( Q_j = A_j P + P A_j^* \). For \( j = l_1 + 1, \ldots, l_2 \) we have \( Q_{11} = 2(hr_j + ks_j) \), hence \( hr_j + ks_j < 0 \). For \( j = l_2 + 1, \ldots, l \) we have \( Q_{22} = -2(hr_j + ks_j) \), hence \( hr_j + ks_j > 0 \). Therefore \( h \) and \( k \) satisfy inequalities (4) and (5) and the proof is complete. \( \square \)

From Proposition 6 we can obtain conditions for the existence of the common solution to the Lyapunov equation for two matrices in \( \mathcal{H} \).

**Corollary 7.** For matrices \( A_1 \) and \( A_2 \) in the form as in Proposition 6 the following statements hold:

1. Matrices \( A_1 \in \mathcal{H}_1 \) and \( A_2 \in \mathcal{H} \) have a common solution to the Lyapunov equation.
2. Matrices \( A_1 \in \mathcal{H}_2 \) and \( A_2 \in \mathcal{H}_2 \) have a common solution to the Lyapunov equation unless \((r_1, s_1) = -\alpha(r_2, s_2)\) for some \( \alpha > 0 \).
3. Matrices \( A_1 \in \mathcal{H}_3 \) and \( A_2 \in \mathcal{H}_3 \) have a common solution to the Lyapunov equation unless \((r_1, s_1) = -\alpha(r_2, s_2)\) for some \( \alpha > 0 \).
4. Matrices \( A_1 \in \mathcal{H}_2 \) and \( A_2 \in \mathcal{H}_2 \) have a common solution to the Lyapunov equation unless \((r_1, s_1) = \alpha(r_2, s_2)\) for some \( \alpha > 0 \).

**Proof.** The first item follows directly from Proposition 6. To prove the second item, we observe that there exist numbers \( h \) and \( k \) that satisfy inequalities \( hr_j + ks_j > 0 \) for \( j = 1, 2 \), if and only if \((r_1, s_1) \neq -\alpha(r_2, s_2)\). We can apply similar arguments to prove the rest of the corollary. \( \square \)

Now we will consider the existence of a common solution to the Lyapunov equation for matrices \( A \) and \( B \), where \( A \in \mathcal{H} \) and \( B \) is a matrix for which \( B + B^* \) is a real negative semidefinite matrix with zero determinant.

**Proposition 8.** Let
\[
A = \begin{pmatrix}
-a + im & r + is
-r + is & in
\end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix}
-c + ip & t + u + iv
-t - u + iv & -d + iq
\end{pmatrix},
\]
where \( r, s, m, n, t, u \) and \( v \) are real numbers, \( a, c \) and \( d \) are positive numbers and \( cd - t^2 = 0 \). Matrices \( A \) and \( B \) do not have a common solution to the Lyapunov equation if and only if \( u = 0, r = 0 \) and \( s = -\alpha(t(p - q) + v(c - d)) \) for some positive number \( \alpha \).

**Proof.** In Proposition 4 we have already seen that matrix \( P \) of the form (3) will be a solution to the Lyapunov equation for \( A \) for all sufficiently small numbers \( h \) and \( k \) that satisfy inequality \( hr + ks > 0 \).
Let \( Q = BP + PB^* \). The inequalities
\[
Q_{11} = 2(-c + h(t + u) + kv) < 0
\]
and
\[
Q_{22} = 2(-d + h(t - u) - kv) < 0
\]
hold for all sufficiently small \( h \) and \( k \). We compute:
\[
det(Q) = 4hu(c - d) + 4k(t(p - q) + v(c - d)) + h^2\gamma_1 + k^2\gamma_2 + hky_3,
\]
where \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) do not depend on \( h \) and \( k \). We see that the matrix \( P \) will be a solution to the Lyapunov equation for \( B \) for all sufficiently small numbers \( h \) and \( k \) that satisfy inequality:
\[
hr + ks > 0 \quad \text{and} \quad hu(c - d) + k(t(p - q) + v(c - d)) > 0.
\]
That is if
\[
(r, s) \neq -\alpha(u(c - d), t(p - q) + v(c - d))
\]
for some \( \alpha > 0 \).

Next we consider matrices of the form
\[
P = \begin{pmatrix}
1 & 0 \\
0 & 1 + z
\end{pmatrix}.
\]

Let \( Q = BP + PB^* \). The inequalities
\[
Q_{11} = -2c < 0 \quad \text{and} \quad Q_{22} = -2d(1 + z) < 0
\]
hold for all sufficiently small \( z \). We compute:
\[
det(Q) = -4tuz - z^2((u + t)^2 + v^2).
\]
Since \( t \neq 0 \) we can choose \( z \) such that \( det(Q) > 0 \) as long as \( u \neq 0 \). In this case a matrix of the form (6) will be a solution to the Lyapunov equation for \( B \).

The set \( \mathcal{P}(B) \) is open, hence a matrix of the form
\[
P = \begin{pmatrix}
1 & h + ik \\
h - ik & 1 + z
\end{pmatrix}
\]
will be a solution to the Lyapunov equation for \( B \) for all sufficiently small numbers \( h, k \). We conclude that a matrix \( P \) of the from (7) will be a common solution to the Lyapunov equation for \( A \) and \( B \) for all sufficiently small numbers \( z, h \) and \( k \) that satisfy inequality \( hr + ks > 0 \).

We have proved that matrices \( A \) and \( B \) have a common solution to the Lyapunov equation unless \( u = 0, r = 0 \) and \( s = -\alpha(t(p - q) + v(c - d)) \) for some positive number \( \alpha \). Now we will show that if those relations hold, the matrices \( A \) and \( B \) do not have a common solution to the Lyapunov equation.

Assume that they have a common solution \( P \). Without loss of generality we can assume that \( P \) is of the form:
\[
P = \begin{pmatrix}
1 & h + ik \\
h - ik & 1 + z
\end{pmatrix}.
\]
Let $Q_A = AP + PA^*$ and $Q_B = BP + PB^*$. We compute:

$$
\det(Q_A) = 2aks(2 + z) - (h(m - n) + zs)^2 - h^2a^2 - k^2(a^2 + (m - n)^2 + 4s^2)
$$

and

$$
\det(Q_B) = 2k(t(p - q) + v(c - d))(2 + z) - (h(c - d) - tz)^2 - (h(p - q) + vz)^2
- k^2((c + d)^2 + (p - q)^2 + 4v^2).
$$

Since we want $\det(Q_A) > 0$ and $\det(Q_B) > 0$ we need to satisfy the inequalities:

$$2aks(2 + z) > 0 \text{ and } 2k(t(p - q) + v(c - d))k(2 + z) > 0.$$ 

Those inequalities do not hold for any choice of $k$ and $z$, since we have assumed that $s = -\alpha(t(p - q) + v(c - d)), \alpha > 0$. We conclude that in this case matrices $A$ and $B$ do not have a common solution to the Lyapunov equation. □

Before we give the proof of Theorem 2 we need a couple of lemmas. The first lemma is well known and easy to check.

**Lemma 9.** Let $A \in \mathbb{C}^{n \times n}$ be stable and $T \in \mathbb{C}^{n \times n}$ invertible. Then $P$ is a solution to the Lyapunov equation for $A$ if and only if $T^* PT$ is a solution to the Lyapunov equation for $T^{-1} AT$.

In particular, matrices $A$ and $B$ have a common solution to the Lyapunov equation if and only if matrices $T^{-1} AT$ and $T^{-1} BT$ have.

**Lemma 10.** Let $A$ and $B$ be stable matrices such that the matrices $A - \epsilon I$ and $B$ have a common solution to the Lyapunov equation for every $\epsilon > 0$. Then there exists a positive definite matrix $P$ such that the matrices $AP + PA^*$ and $BP + PB^*$ are negative semidefinite.

**Proof.** For every $\epsilon > 0$ there exists positive definite matrix $P_\epsilon$ with norm 1 such that matrices $(A - \epsilon I)P_\epsilon + P_\epsilon (A - \epsilon I)^*$ and $BP_\epsilon + P_\epsilon B^*$ are negative definite. The matrices $P_\epsilon$ are contained in the compact set, hence there exists a convergent sequence $\{P_n; n = 1, 2, \ldots\}$ contained in the set $\{P_\epsilon; \epsilon > 0\}$. The matrix $P = \lim_{n \to \infty} P_n$ is nonzero positive semidefinite matrix with norm 1 such that the matrices $AP + PA^*$ and $BP + PB^*$ are negative semidefinite. Since the matrices $A$ and $B$ are invertible, matrices $AP + PA^*$ and $BP + PB^*$ are nonzero.

Among all matrices $P$ that have this property we choose one for which

$$\text{rank}(AP + PA^*) + \text{rank}(BP + PB^*)$$

is maximal. We will prove that such a matrix $P$ must be positive definite.

Suppose that $P$ is singular. Using unitary similarity and Lemma 9 we can assume that $P$ is of the form:

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix}$$

for some positive definite matrix $P_1$. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

be the corresponding partitions of the matrices $A$ and $B$. 
Since the matrix 
\[ AP + PA^* = \begin{pmatrix} A_{11}P_1 + P_1A_{11}^* & P_1A_{21}^* \\ A_{21}P_1 & 0 \end{pmatrix} \]
is negative semidefinite, we have \( A_{21}P_1 = 0 \) and consequently \( A_{21} = 0 \). Similar argument gives us \( B_{21} = 0 \). Hence the matrices \( A_{22} - \epsilon I \) and \( B_{22} \) have a common solution to the Lyapunov equation for every \( \epsilon > 0 \).

The previous argument applied to the matrices \( A_{22} \) and \( B_{22} \) gives us the existence of a nonzero positive semidefinite matrix \( P_2 \) such that matrices \( A_{22}P_2 + P_2A_{22}^* \) and \( B_{22}P_2 + P_2B_{22}^* \) are nonzero negative semidefinite.

Put 
\[ P_0 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}. \]
Then
\[ \text{rank}(AP + P_0A^*) + \text{rank}(BP_0 + P_0B^*) > \text{rank}(AP + PA^*) + \text{rank}(BP + PB^*). \]
This contradicts the choice of the matrix \( P \). Hence we have proved that the matrix \( P \) must be positive definite. □

In the following lemma we will show that if conditions (A1) and (A2) in Theorem 2 are satisfied for the matrices \( A \) and \( B \), then those conditions are satisfied for the matrices \( A - \gamma I \) and \( B - \delta I \) for all \( \gamma \geq 0 \) and \( \delta \geq 0 \).

**Lemma 11.** Let \( A \) and \( B \) be stable matrices in \( \mathbb{C}^{n \times n} \) such that conditions (A1) and (A2) are satisfied. Then for all \( \gamma \geq 0 \) and \( \delta \geq 0 \) the following conditions hold:

1. \((D1)\) The convex cone \( \text{conv}((A - \gamma I + i\alpha I), B - \delta I) \) is stable for all \( \alpha \in \mathbb{R} \).
2. \((D2)\) The convex cone \( \text{conv}((A - \gamma I + i\alpha I)^{-1}, B - \delta I) \) is stable for all \( \alpha \in \mathbb{R} \).

**Proof.** Let conditions (A1) and (A2) hold and suppose that the convex cone \( \text{conv}(A - \gamma I + i\alpha I, B - \delta I) \) is not stable for some \( \gamma \geq 0 \), \( \delta \geq 0 \) and \( \alpha \in \mathbb{R} \).

Then there exists \( \eta > 0 \) such that the matrix \( (A - \gamma I + i\alpha I) + \eta(B - \delta I) \) has a purely imaginary eigenvalue \(-i\lambda, \lambda \in \mathbb{R}\):

\[ \det((A - \gamma I + i\alpha I) + \eta(B - \delta I) + i\lambda I) = 0. \]

It follows that the matrix \( A + \eta B \) has an eigenvalue \( \gamma + \delta - i(\alpha + \lambda) \), hence it is not stable. Therefore the convex cone \( \text{conv}(A, B) \) is not stable, which contradicts condition (A1).

Suppose that the convex cone \( \text{conv}((A - \gamma I + i\alpha I)^{-1}, B - \delta I) \) is not stable for some \( \gamma \geq 0 \), \( \delta \geq 0 \) and \( \alpha \in \mathbb{R} \). Then there exists \( \eta > 0 \) such that the matrix \( (A - \gamma I + i\alpha I)^{-1} + \eta(B - \delta I) \) has a purely imaginary eigenvalue \(-i\lambda, \lambda \in \mathbb{R}\). Hence

\[ \det((A - \gamma I + i\alpha I)^{-1} + \eta(B - \delta I) + i\lambda I) = 0 \]

and

\[ \det((\eta(B - \delta I) + i\lambda I)^{-1} + A - \gamma I + i\alpha I) = 0. \]
Since \( (\eta(B - \delta I) + i\lambda I)^{-1} + A \) is not stable, there exists \( \eta_1 > 0 \) such that the matrix \( (\eta(B - \delta I) + i\lambda I)^{-1} + \eta_1 A \) has a purely imaginary eigenvalue \(-i\lambda_1, \lambda_1 \in \mathbb{R}\):

\[
\det((\eta(B - \delta I) + i\lambda I)^{-1} + \eta_1 A + i\lambda_1 I) = 0.
\]

It follows that

\[
\det(\eta(B - \delta I) + i\lambda I + (\eta_1 A + i\lambda_1 I)^{-1}) = 0.
\]

This contradicts the assumption that the convex cone \( \text{conv}((A + i\lambda_1/\eta_1 I)^{-1}, B) \) is stable. \( \Box \)

We can now prove the main result of this paper.

**Proof of Theorem 2.** If matrices \( A \) and \( B \) have a common solution to the Lyapunov equation then \( \text{cic}(A, B) \) is stable, hence conditions (A1) and (A2) are satisfied.

We will prove the other implication by contradiction. We suppose that matrices \( A \) and \( B \) do not have a common solution to the Lyapunov equation, but they satisfy conditions (A1) and (A2) in Theorem 2.

Let

\[
\alpha_0 = \inf\{\alpha; \ A - \alpha I_2 \text{ and } B \text{ have a common solution to the Lyapunov equation}\}.
\]

Define \( A_0 = A - \alpha_0 I_2 \). Then by Lemma 10 there exists a positive definite matrix \( P \) such that matrices \( A_0 P + PA_0^* + BP + PB^* \) are negative semidefinite. If either \( A_0 P + PA_0^* > 0 \) or \( BP + PB^* > 0 \), then matrices \( A_0 \) and \( B \) have a common solution to the Lyapunov equation. Hence \( \det(A_0 P + PA_0^*) = 0 \) and \( \det(BP + PB^*) = 0 \). By Lemma 11, the matrices \( A_0 \) and \( B \) satisfy conditions (A1) and (A2).

The matrices \( A_1 = P^{-1/2}A_0P^{1/2} \) and \( B_1 = P^{-1/2}BP^{1/2} \) are stable, satisfy conditions (A1) and (A2) and \( A_1 + A_1^* \leq 0 \) and \( B_1 + B_1^* \leq 0 \). Furthermore, Lemma 9 tells us that matrices \( A_1 - \epsilon I \) and \( B_1 \) have a common solution to the Lyapunov equation for every \( \epsilon > 0 \), but matrices \( A_1 \) and \( B_1 \) do not have a common solution to the Lyapunov equation.

Let \( U_1 \) be a unitary matrix such that

\[
U_1^*(A + A^*)U_1 = \begin{pmatrix} -2a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U_1^*(B + B^*)U_1 = \begin{pmatrix} -2c & \gamma \\ \bar{\gamma} & -2d \end{pmatrix}.
\]

We choose a real number \( \theta \) such that \( e^{i\theta} \gamma = 2t > 0 \) and define:

\[
D = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad U = U_1 D.
\]

Then

\[
U^*(A + A^*)U = \begin{pmatrix} -2a & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad U^*(B + B^*)U = \begin{pmatrix} -2c & 2t \\ 2t & -2d \end{pmatrix}.
\]

Set \( A_2 = U^*A_1U \) and \( B_2 = U^*B_1U \).

Then

\[
A_2 = \begin{pmatrix} -a + im & r + is \\ -r + is & in \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} -c + ip & t - u + iv \\ t + u + iv & -d + iq \end{pmatrix},
\]

where \( r, s, m, n, t, u \) and \( v \) are real numbers, \( a, b \) and \( c \) are nonnegative numbers and \( cd - r^2 = 0 \).

Now we look at the conditions for the common solution to the Lyapunov equation. We consider several cases.
If \( t = 0, c = 0 \) and \( d \neq 0 \), then \( A_2 \in \mathcal{M}_3 \) and \( B_2 \in \mathcal{M}_2 \). Matrices \( A_2 \) and \( B_2 \) do not have a common solution to the Lyapunov equation if and only if \( (u, v) = \alpha(r, s) \) for some positive \( \alpha \), by Corollary 7. Hence:

\[
A_2 = \begin{pmatrix} -a + im & r + is \\ -r + is & in \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} ip & \alpha(r + is) \\ \alpha(-r + is) & -d + iq \end{pmatrix}.
\]

A short computation gives us:

\[
\alpha(r^2 + s^2)(A - inI)^{-1} + B_2 = \begin{pmatrix} ip & 0 \\ 0 & \alpha(-a + i(m - n)) - d + iq \end{pmatrix}.
\]

We see that the matrix \( \alpha(r^2 + s^2)(A - inI)^{-1} + B_2 \) is not stable, contrary to condition (A2).

If \( t = 0, c \neq 0 \) and \( d = 0 \), then \( A_2 \in \mathcal{M}_3 \) and \( B_2 \in \mathcal{M}_3 \). By Corollary 7 they do not have a common solution to the Lyapunov equation if and only if \( (u, v) = -\alpha(r, s) \) for some positive \( \alpha \). Since in this case

\[
\alpha A_2 + B_2 = \begin{pmatrix} \alpha(-a + im) - c + ip & 0 \\ 0 & i(\alpha n + q) \end{pmatrix},
\]

we have a contradiction to the stability of the convex cone \( \text{conv}(A_2, B_2) \).

Finally, let \( t \neq 0 \). Then \( c \neq 0 \) and \( d \neq 0 \). In this case Proposition 8 tells us that matrices \( A_2 \) and \( B_2 \) do not have a common solution to the Lyapunov equation if and only if \( u = 0, r = 0 \), and \( s = -\alpha(t(p - q) + v(c - d)) \) for some positive \( \alpha \). Therefore:

\[
A_2 = \begin{pmatrix} -a + im & -i\alpha(t(p - q) + v(c - d)) \\ -i\alpha(t(p - q) + v(c - d)) & in \end{pmatrix}
\]

and

\[
B_2 = \begin{pmatrix} -c + ip & t + iv \\ t + iv & -d + iq \end{pmatrix},
\]

where \( cd = t^2 \). Now take

\[
\beta = \frac{c^2(ct(p - q) + (c - t)(c + t)v)(t^2 + v^2)}{t^4},
\]

\[
\gamma = -p - c\frac{v}{t} \quad \text{and} \quad \delta = \frac{c^3(t^2 + v^2)}{t^2\alpha} > 0.
\]

A simple computation gives us

\[
\beta(B_2 + i\gamma I_2)^{-1} + \delta A_2 = \begin{pmatrix} \xi & 0 \\ 0 & \eta \end{pmatrix},
\]

where \( \xi \in \mathbb{C} \) and \( \eta \in \mathbb{R} \). We conclude that \( \text{conv}((A_2 - i\frac{\gamma}{\delta} I_2)^{-1}, B_2) \) is not stable. \( \square \)

3. Examples

We present an example to show that stability of the convex invertible cone \( \text{cic}(A, B) \) is not sufficient for the existence of a common solution to the Lyapunov equation for \( 2 \times 2 \) complex matrices \( A \) and \( B \).
Example 12. Take matrices $A \in \mathcal{M}_3$ and $B \in \mathcal{M}_2$, that do not have a common solution to the Lyapunov equation:

$$A = \begin{pmatrix} -\alpha + \text{i}m & r + \text{i}s \\ -r + \text{i}s & \text{i}n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \text{i}p & \alpha(r + \text{i}s) \\ \alpha(-r + \text{i}s) & -d + \text{i}q \end{pmatrix}. $$

Assume that $n \neq 0$, $p \neq 0$ and $\alpha \neq mp/(r^2 + s^2)$. We will show that in this case the convex invertible cone $\text{cic}(A, B)$ is stable.

It is easy to see that the matrix $M + M^*$ is negative semidefinite for every matrix $M \in \text{cic}(A, B)$.

We define the following sets:

$$\mathcal{N}_1 = \{\alpha A + \beta B + \gamma A^{-1} + \delta B^{-1}; \alpha \geq 0, \beta \geq 0, \gamma \geq 0, \delta \geq 0\}$$

and

$$\mathcal{N}_j = \{M_1 + M_2 + M_3^{-1}; M_1, M_2, M_3 \in \mathcal{N}_{j-1}\}.$$

Then $\mathcal{N}_j \subseteq \mathcal{N}_{j+1}$ and $\bigcup_{j=1}^{\infty} \mathcal{N}_j = \text{cic}(A_1, A_2)$, since $\bigcup_{j=1}^{\infty} \mathcal{N}_j$ is closed under addition, multiplication by a positive scalar and inversion.

Using induction we will show that the matrices of the form $\alpha A, \alpha B, \alpha A^{-1}$ or $\alpha B^{-1}$ for some $\alpha \geq 0$ are the only matrices in $\mathcal{N}_j$ for which the identity matrix is not a solution to the Lyapunov equation.

It is easy to see that the identity matrix is a solution to the Lyapunov equation for matrices $A + \alpha B, A^{-1} + \alpha B, A + \alpha B^{-1}, A^{-1} + \alpha B^{-1}, A + \alpha A^{-1}$ and $B + \alpha B^{-1}$ for every $\alpha > 0$. Hence the statement holds for $\mathcal{N}_1$.

Assuming that it is true for $\mathcal{N}_j$ we will prove it for $\mathcal{N}_{j+1}$. Take

$$M = M_1 + M_2 + M_3^{-1} \in \mathcal{N}_{j+1},$$

where matrices $M_1, M_2$ and $M_3$ lie in $\mathcal{N}_j$. Matrices $M_1 + M_1^*, M_2 + M_2^*$ and $M_3 + M_3^*$ are negative semidefinite. If the identity matrix is a solution to the Lyapunov equation for either $M_1$, $M_2$ or $M_3$, then it is also a solution for $M$. If not, then $M_1, M_2$ and $M_3$ are of the form $\alpha A, \alpha B, \alpha A^{-1}$ or $\alpha B^{-1}$ for some $\alpha \geq 0$ by induction hypothesis. This implies that $M$ lies in $\mathcal{N}_1$.

We have proved that the only matrices in $\text{cic}(A, B)$ for which the identity matrix is not a solution to the Lyapunov equation are the matrices of the form $\alpha A, \alpha B, \alpha A^{-1}$ or $\alpha B^{-1}$ for some $\alpha \geq 0$. Since matrices $A, B$ are stable, this implies that $\text{cic}(A, B)$ is stable.

In the following example we look at a special case of the previous example. We show that stability of $\text{cic}(A, B)$ does not imply the existence of a common solution to the Lyapunov equation for complex $2 \times 2$ matrices $A$ and $B$ even in the case when rank of the matrix $A - B$ is one.

Example 13. Let $m, n, r$ be real numbers such that $r$ is not equal to $0$ or $1$, $m \neq 0$, $n \neq 0$ and $r(r - 1) \neq m^2$. Let

$$R = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. $$

Consider the following matrices:

$$A = \begin{pmatrix} -1 + \text{i}m & r \\ -r & \text{i}n \end{pmatrix} \quad \text{and} \quad B = A + R.$$

Corollary 7 tells us matrices $A$ and $B$ have a common solution to the Lyapunov equation if and only if $r$ lies in the interval $(0,1)$ and Example 12 tells us that $\text{cic}(A, B)$ is stable.
4. Real case

In this section we will explain how can results for $2 \times 2$ complex matrices be related to a class of $4 \times 4$ real matrices. We will use the standard embedding of $\mathbb{C}^{n \times n}$ into $\mathbb{R}^{2n \times 2n}$.

We will write matrix $A$ in $\mathbb{C}^{n \times n}$ in the following way:

$$A = A_{\text{Re}} + i A_{\text{Im}}$$

where $A_{\text{Re}}$ and $A_{\text{Im}}$ are matrices in $\mathbb{R}^{n \times n}$. Denote $\hat{A} = A_{\text{Re}} - i A_{\text{Im}}$ and

$$\hat{\hat{A}} = \begin{pmatrix} A_{\text{Re}} & A_{\text{Im}} \\ -A_{\text{Im}} & A_{\text{Re}} \end{pmatrix}.$$ 

Since the spectrum of the matrix $\hat{\hat{A}}$ is the union of the spectra of matrices $A$ and $\overline{A}$, the matrix $\hat{\hat{A}}$ is stable if and only if the matrix $A$ is stable.

**Proposition 14.** There exists a common solution to the Lyapunov equation $P \in \mathbb{C}^{n \times n}$ for matrices $A$ and $B$ if and only if there exists a common solution $\hat{P} \in \mathbb{R}^{2n \times 2n}$ for matrices $\hat{A}$ and $\hat{B}$.

**Proof.** Let

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} I_n & iI_n \\ iI_n & I_n \end{pmatrix}.$$ 

Observe that:

$$\tilde{A} = T^* \hat{A} T = \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} \quad \text{and} \quad \tilde{B} = T^* \hat{B} T = \begin{pmatrix} B & 0 \\ 0 & \overline{B} \end{pmatrix}.$$ 

Assume that $A$ and $B$ have a common solution to the Lyapunov equation $P \in \mathbb{C}^{n \times n}$. Then

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & \overline{P} \end{pmatrix}$$

is a common solution to the Lyapunov equation for $\tilde{A}$ and $\tilde{B}$. Lemma 9 tells us that

$$\hat{\tilde{P}} = T \tilde{P} T^* = \begin{pmatrix} P_{\text{Re}} & P_{\text{Im}} \\ -P_{\text{Im}} & P_{\text{Re}} \end{pmatrix} \in \mathbb{R}^{2n \times 2n}$$

is a common solution to the Lyapunov equation for matrices $\hat{A}$ and $\hat{B}$.

To prove the other implication, we suppose that $P_1 \in \mathbb{R}^{2n \times 2n}$ is a common solution to the Lyapunov equation for matrices $\hat{A}$ and $\hat{B}$. Then

$$\hat{P}_1 = T^* P_1 T = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

is a common solution to the Lyapunov equation for matrices $\tilde{A}$ and $\tilde{B}$. Now it is easy to see that $P_{11}$ is a common solution to the Lyapunov equation for matrices $A$ and $B$. $\square$

We are ready to state the conditions for the existence of a common solution to the Lyapunov equation for real $4 \times 4$ matrices that correspond to $2 \times 2$ complex matrices. Note that matrix $J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ corresponds to the matrix $iI_2$ in the standard embedding.

**Theorem 15.** Let $A_1$, $A_2$, $B_1$ and $B_2$ be real $2 \times 2$ matrices such that the matrices:

$$A = \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix}$$
are stable. Then $A$ and $B$ have a common solution to the Lyapunov equation if and only if the following conditions are satisfied:

\begin{align*}
(D1) & \text{ The convex cone } \text{conv}((A + \alpha J), B) \text{ is stable for all } \alpha \in \mathbb{R}. \\
(D2) & \text{ The convex cone } \text{conv}((A + \alpha J)^{-1}, B) \text{ is stable for all } \alpha \in \mathbb{R}.
\end{align*}

**Proof.** By Proposition 14 matrices $A$ and $B$ have a common solution to the Lyapunov equation if and only matrices $A_1 + iA_2$ and $B_1 + iB_2$ have. Therefore we have to show that conditions $(A1)$ and $(A2)$ hold for matrices $A_1 + iA_2$ and $B_1 + iB_2$ if and only if the conditions $(D1)$ and $(D2)$ hold for matrices $A$ and $B$.

Let $T$ be the matrix (8) defined in the proof of Proposition 14. Since

$$T^*(A + \alpha J)T = \begin{pmatrix} A_1 + i(A_2 + \alpha I_2) & 0 \\ 0 & A_1 - i(A_2 + \alpha I_2) \end{pmatrix}$$

and

$$T^*(A + \alpha J)^{-1}T = \begin{pmatrix} (A_1 + i(A_2 + \alpha I_2))^{-1} & 0 \\ 0 & (A_1 - i(A_2 + \alpha I_2))^{-1} \end{pmatrix}$$

the conditions are clearly equivalent. □

**Corollary 16.** Let $A_1, A_2, B_1$ and $B_2$ be real $2 \times 2$ matrices such that the matrices:

\begin{align*}
A &= \begin{pmatrix} A_1 & A_2 \\ -A_2 & A_1 \end{pmatrix} \text{ and } B &= \begin{pmatrix} B_1 & B_2 \\ -B_2 & B_1 \end{pmatrix}
\end{align*}

are stable. Then $A$ and $B$ have a common solution to the Lyapunov equation if and only if the following conditions are satisfied:

\begin{align*}
(E1) & \text{ The convex cone } \text{conv}((A + i\alpha I_4), B) \text{ is stable for all } \alpha \in \mathbb{R}. \\
(E2) & \text{ The convex cone } \text{conv}((A + i\alpha I_4)^{-1}, B) \text{ is stable for all } \alpha \in \mathbb{R}.
\end{align*}

**Proof.** We observe that

$$T^*(iI_{2n})T = iI_{2n}$$

for the matrix $T$ defined in (8). The rest of the proof is similar to the proof of Theorem 15. □

**Remark 17.** In the previous section we have seen that for real $2 \times 2$ matrices $A$ and $B$, stability of the convex cones $\text{conv}(A, B)$ and $\text{conv}(A^{-1}, B)$ implies the stability of convex cones $\text{conv}(A + i\alpha I_2, B + i\beta I_2)$ and $\text{conv}((A + i\alpha I_2)^{-1}, B + i\beta I_2)$ for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$. We see that this is not true for real $4 \times 4$ matrices. Therefore stability of these cones is a necessary condition for the existence of a common solution to the Lyapunov equation for matrices $A$ and $B$ that is stronger than the conditions that the convex cones $\text{conv}(A, B)$ and $\text{conv}(A^{-1}, B)$ are stable.

We can use Example 13 to show that the stability of $\text{cic}(A, B)$ is not sufficient for the existence of a common solution to the Lyapunov equation for real $4 \times 4$ matrices. In particular, this is not true even in the case when rank of the matrix $A - B$ is two.

**Example 18.** Let $m, n, r, s$ be real numbers such that $r$ is not equal to 0 or 1, $m \neq 0$, $n \neq 0$ and $r(r - 1) \neq m^2$. Let
\[ R = \begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 1 & -1 
\end{pmatrix}. \]

Consider the following matrices:
\[ \hat{A} = \begin{pmatrix}
-1 & r & m & 0 \\
-r & 0 & 0 & n \\
m & 0 & -1 & r \\
0 & n & -r & 0 
\end{pmatrix} \quad \text{and} \quad \hat{B} = \hat{A} + \hat{R}. \]

The matrices \( \hat{A} \) and \( \hat{B} \) do not have a common solution to the Lyapunov equation if and only if \( r \) does not lie in the interval \((0, 1)\). However, cic(\( \hat{A}, \hat{B} \)) is stable for every \( r \in \mathbb{R} \).

Next theorem shows how we can use the results in this section to study the existence of a common solution to the Lyapunov equation for a class of real matrices of size \( n \times n \).

**Theorem 19.** Let \( A \) and \( B \) be real \( n \times n \) matrices, such that the matrices \( A^k \) and \( B^l \) commute with both \( A \) and \( B \) for some \( k, l \in \{1, 2, 3, 4\} \). Then matrices \( A \) and \( B \) have a common solution to the Lyapunov equation if and only if the following conditions are satisfied:

(A1) The convex cone \( \text{conv}((A + i \alpha I_n), B) \) is stable for all \( \alpha \in \mathbb{R} \).

(A2) The convex cone \( \text{conv}((A + i \alpha I_n)^{-1}, B) \) is stable for all \( \alpha \in \mathbb{R} \).

**Proof.** If \( k = 1 \) or \( l = 1 \) then matrices \( A \) and \( B \) commute. We observe that for a stable matrix \( A \) the set of matrices that commute with \( A^2 \) is the same as the set of matrices that commute with \( A \). Thus matrices \( A \) and \( B \) commute if \( k = 2 \) or \( l = 2 \). Commuting matrices have a common solution to the Lyapunov equation.

Now we consider the case when \( k, l \in \{3, 4\} \). Assume that conditions (A1) and (A2) hold. Let \( \mathcal{S} \) be a simple component of the algebra over \( \mathbb{C} \) generated by matrices \( A \) and \( B \) and let \( A_{\mathcal{S}} \) and \( B_{\mathcal{S}} \) be the images of matrices \( A \) and \( B \) in \( \mathcal{S} \). To prove our statement it suffices to show that matrices \( A_{\mathcal{S}} \) and \( B_{\mathcal{S}} \) have a common solution to the Lyapunov equation.

Matrices \( A^k_{\mathcal{S}} \) and \( B^l_{\mathcal{S}} \) are central in \( \mathcal{S} \). Hence \( A^k_{\mathcal{S}} = \alpha I \) and \( B^l_{\mathcal{S}} = \beta I \) for some \( \alpha \in \mathbb{C} \) and \( \beta \in \mathbb{C} \). It follows that minimal polynomial of \( A \) divides polynomial \( q(x) = x^k - \alpha \). For \( k = 3 \) or \( k = 4 \) at most two \( k \)th roots of \( \alpha \) have negative real part, hence the minimal polynomial of the matrix \( A \) is linear or quadratic. The same argument tells us that the matrix \( B_{\mathcal{S}} \) has a linear or a quadratic minimal polynomial. Laffey [17] proved that this implies that \( \mathcal{S} \) is isomorphic to \( \mathbb{C} \) or \( \mathbb{C}^{2 \times 2} \).

If \( \mathcal{S} \) is isomorphic to \( \mathbb{C} \), then the matrices \( A_{\mathcal{S}} \) and \( B_{\mathcal{S}} \) commute. Thus they have a common solution to the Lyapunov equation. Observe that matrices \( A_{\mathcal{S}} \) and \( B_{\mathcal{S}} \) satisfy conditions (A1) and (A2). Therefore we can use Corollary 16 to prove the statement when \( \mathcal{S} \) is isomorphic to \( \mathbb{C}^{2 \times 2} \). \( \square \)

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