Characterization of three-dimensional crack border fields in creeping solids

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1. Introduction

According to Riedel (1987), the typical elastic creep-time curve of solids consists of three stages of creeping deformation as shown by Fig. 1. Following an initial elastic strain e(t) produced at the instant of loading, the three stages occur progressively over time. In the primary stage, strain e increases with decreasing strain rate ˙e as time going. When entering the secondary stage (or steady state regime), strain increases at a constant strain rate obeying the Norton power law. While in the final tertiary stage, strain increases sharply with increasing strain rate and finally leads to fracture when strain reaches the failure value e. For most creeping solids, the stationary stage takes most of the creeping life. Therefore, our study in this work will concentrate to the important secondary stage.

With reference to polar coordinates, r and θ, centered at the crack tip, the crack tip fields have been described by Riedel and Rice (1980) using a single parameter C(t) in two-dimensional (2D) ideal plane stress and plane strain states as follows:

\[ \sigma_q = \frac{C(t)}{Blr} \bar{\sigma}_q(\theta, n), \]

\[ \bar{\epsilon}_q = B \left[ \frac{C(t)}{Blr} \right]^{\frac{n}{n-1}} \bar{\epsilon}_q(\theta, n), \]

\[ \bar{u}_q = Br \left[ \frac{C(t)}{Blr} \right]^{\frac{n}{n-1}} \bar{u}_q(\theta, n), \]

where B is the creep parameter and n is the creep exponent of the solids in the Norton power law (or power law creeping).

\[ \dot{e} = \frac{\sigma}{E} + Br^n, \]

where E is Young’s modulus.

The above solution has been widely recognized as the RR field solution, in which the dimensionless constant \( \dot{\epsilon}_n \) and angular functions \( \bar{\sigma}_q, \bar{\epsilon}_q \) and \( \bar{u}_q \) depend only on the creep exponent n in plane stress or plane strain state, but quite different in the two limited stress states. These functions can only be solved out through complicated numerical process and the results have been tabulated by Shih (1983). The C(t)-integral is path-independent within the creep zone, defined as the region where the equivalent creep strain \( \dot{e} = \left( \frac{1}{2} \sigma^0_{eq} \right)^{1/2} \) exceeds the equivalent elastic strain \( \dot{e}^e = \left( \frac{1}{2} \sigma^0_{eq} \right)^{1/2} \) (Riedel and Rice, 1980).
Riedel and Rice (1980) and Ohji et al. (1980) obtained the relationship of the $C(t)$-integral and $J$-integral under small scale creep conditions as

$$C(t) = \frac{J}{(n + 1)t}. \quad (1.5)$$

Under extensive creep conditions, the $C(t)$-integral approaches a constant $C$,$^1$

$$C(t) \to C. \quad (1.6)$$

While Ehlers and Riedel (1981) suggested the following formula for $C(t)$ between small scale creep and extensive creep,

$$C(t) = C \left( \frac{t_2}{t} + 1 \right), \quad (1.7)$$

where $t_2$ is the transition time, which can be obtained by replacing $C(t)$ in Eq. (1.5) by $C$ and rearranging the expression as

$$t_2 = \frac{J}{(n + 1)C}. \quad (1.8)$$

The RR solution in a semi-infinite elastic-power law creeping solid with an edge crack under plane strain condition was investigated by Li et al. (1988) employing a rate-tangent modulus finite element method (Peirce et al., 1984), and it was concluded that the RR fields dominate only about one-fifth the extent of the creep zone under small scale creep conditions. The analysis was extended to extensive creep condition for a single edge-cracked specimen under both plane stress and plane strain conditions by Yang et al. (1996), which shows that the stress-based RR-dominance zone occupies a fraction of the creep zone in the plane stress state and a very narrow strip zone in the plane strain state. These results show that the RR solution must be improved to consider the constraint effects caused by the three-dimensional (3D) geometry of components and loading mode, etc. Nguyen et al. (2000a,b) proposed a three-term description $C - \sigma_2 - \sigma_\infty$, in which the parameters $A_2$ and $\sigma_\infty$ account for the constraint effect imposed by the specific geometry and loading configuration. Laiarirandrasana and Kabiri (2006) introduced a parameter $Q'$ to take into account of the in-plane constraint effect in creeping solids by analogy to the $Q$ parameter, a hydrostatic stress parameter to represent all higher order terms of the series expansion for elastic–plastic cracks (O'Dowd and Shih, 1991; O'Dowd and Shih, 1992). Their results lead to negative $Q'$ values, means that the numerical opening stress is lower than that of the RR solution (plane strain). Concerning laboratory tests, they concluded that a creep crack growth prediction via $da/dt$ versus $C$ curve should be conservative. However, the mentioned two and three parameter approaches are mainly developed for consideration of the in-plane constraint, caused by in-plane geometry, size and loading configuration. But actual components have finite thicknesses and the stress state is 3D near a real crack border. To have a more realistic description for 3D crack border fields, the effect of out-of-plane constraint as revealed by Guo (1999a,b, 1995) for elastic–plastic solids should also be explored in creeping solids. However, the out-of-plane constraint effect on the crack border fields in power law creeping solids has not yet been reported.

In this paper, we develop a theoretical basis for the crack border fields under Mode I creep conditions by introducing Guo's conception of out-of-plane constraint. Guo (1999a) introduced the governing equation including the out-of-plane constraint factor $T_z$ is obtained theoretically and solved numerically. Further theoretical investigation on the $C(t)$-integral and crack tip opening displacement (CTOD) shows that they are correlated with $T_z$. Based on these findings, a two-parameter $C(t) - T_z$ description for the 3D crack border fields is proposed under small scale creep conditions, which is analogous to small scale yielding conditions in elastic–plastic solids; while under extensive creep conditions, it is necessary to introduce a third parameter $Q'$, which is analogous to $Q$ introduced in elastic–plastic solids under large scale yielding conditions.

The analogy between power law creeping and power law plasticity suggests that the constraints in the two kinds of solids should be analogous. As have been widely accepted, two typical configurations have been selected to represent geometries which maintain or lose in-plane constraint. Single-edge cracked (SEC) specimens exhibit positive $T$-stresses and in consequence develop full constrained crack border fields (Sham, 1991), while centre-cracked tension (CCT) specimens exhibit negative $T$-stresses and hence develop unconstrained crack border fields (Du and Hancock, 1991). Our finite element analyses for the two typical crack and loading configurations with finite thickness confirm the efficiency of two-parameter $C(t) - T_z$ description and three-parameter $C(t) - T_z - Q'$ description for the tensile stress on the ligament ahead of the cracks under small and extensive creep conditions, respectively.

With the powerful numerical analysis methodology and computer ability, stress and strain calculations and the evaluation for the dominated parameters for three-dimensional creeping cracks become practically realistic. So the two- and three-parameter description developed here should have wide engineering applications.

2. Singular structure of the field

2.1. Coordinate system

For a through-the-thickness straight crack, the coordinate system is established as shown in Fig. 2(a). The origin point $O$ is located at the center of the specimen for convenience. A thin sheet element lying in the $x$–$y$ plane is taken as the object of study, and the loading configuration of the element for mode-I cracks is illustrated in Fig. 2(b). $T_z$ is defined as Guo (1999a):

$$T_z = \sigma_{33}/(\sigma_{11} + \sigma_{22}), \quad (2.1)$$

where and hereafter the subscripts 1, 2 and 3 stand for the Cartesian coordinate components $x$, $y$ and $z$ or cylindrical coordinate components $r$, $\theta$ and $z$, respectively, with $z$ axis along the direction tangential to the crack front line.

Fig. 2. (a) Definition of the coordinate system and (b) the sheet element in $x$–$y$ lane.
Similar to power law plasticity solids (Guo, 1993a), we find that the distribution of $T_z$ near a mode I crack in power law creeping solids can be featured as:

1. Generally, $T_z = T_z(r,0,z)$, although the variation of $T_z$ with $\theta$ is somewhat slight.
2. When $r \to 0$, $T_z \to 0.5$; while $r \to \infty$, $T_z \to 0$.
3. In the interior of the cracked body, $T_z$ is higher and its change in $z$ direction is slight, while when the free surface is approached, $T_z$ decreases rapidly and at the free surface $T_z = 0$. What’s more, when $r \to 0$, the differentiation of $T_z$ with respect to $z$, $\partial T_z/\partial z$, is great near the free surface. Therefore, the differentiation of $T_z$ at the corner point, where the crack front penetrates the free surface, must be great, even may be unlimited.

The material is taken to be elastic-nonlinear viscoaccord according to Norton’s power law creeping relation-equation (1.4). Under multi-axial stress states, the expression of stress–strain behavior is

$$\dot{\varepsilon}_{ij} = \frac{1}{E} S_{ij} + 1 - 2\nu \sigma_{ij} \delta_{ij} + \frac{3}{2} \sigma_{ij}^{\sigma-1} S_{ij},$$

(2.2)

where $\dot{\varepsilon}_{ij}$ is the creeping strain rate, $S_{ij} = \sigma_{ij} - \sigma_{ij} \delta_{ij}/3$ is the deviatoric stress, $\sigma_{ij}$ is the Cauchy stress, $\sigma = (S_{ij} S_{ij})^{1/2}$ is the equivalent stress, $\nu$ is Poisson’s ratio, $E$ and $n$ are the power-law creeping parameters of the solids, and $\delta_{ij}$ is the Kronecker delta symbol. A dot over a quantity denotes the time differentiation. Generally, $E$, $\nu$, $B$ and $n$ are obtained experimentally from uniaxial tests at temperatures interested for a specific material.

As $n > 1$, the elastic strain rate near the crack border can be neglected because of the singularity, which leads to high stresses and strain rates. Then, in the vicinity of crack border, the Norton power law creeping relation-equation (2.2) can be simplified as

$$\dot{\varepsilon}_{ij} = \frac{3}{2} B \sigma_{ij}^{\sigma-1} S_{ij},$$

(2.3)

2.2. Basic equations

For a 3D isotropic continuum without body force, the equilibrium equations are written as

$$\sigma_{ij,i} = 0, \quad (i,j = 1,2,3).$$

(2.4)

The relationship between the infinitesimal strain tensor $\tilde{\varepsilon}$ and the displacement vector $\tilde{u}$ in ratio form is

$$\dot{\varepsilon}_{ij} = \frac{1}{2} (\tilde{u}_{ij,i} + \tilde{u}_{ji,i}).$$

(2.5)

The corresponding compatibility equations are given as

$$e_{ijkl} e_{i} \dot{\varepsilon}_{kl} = 0, \quad (i,j,k,l,m,n = 1,2,3),$$

(2.6)

where $e_{ijkl} = \frac{1}{2} ((i-j)(j-k)(k-l)$.

When the Maxwell stress functions $\Phi_{ij}$ are introduced, the stress tensor satisfying the equilibrium equations can be expressed as

$$\sigma_{ij} = e_{ijkl} e_{i} \Phi_{kl}.$$

(2.7)

2.3. Basic hypotheses

As analyzed by Guo (1993a), we can propose the same hypotheses as follows:

Hypotheses 1.

$$\sigma_{zz}(r,0,z) = T_z(\sigma_{xx} + \sigma_{yy}),$$

(2.8)

$$T_z = T_z(r,0,z) = T(z) \left[ 1 + \sum_{i=1}^{m} g_i(0,z)r^i \right],$$

(2.9)

where $T_z \to T(z)$ when $r \to 0$, and $0 \leq T_z \leq 0.5$.

Hypotheses 2.

$$\sigma_q(r,0,z) = f_{ij} \tilde{A}_{ij}(0,T_z),$$

(2.10)

where $f_{ij}(z)$ are functions of $T_z$, and so are dimensionless.

2.4. The singularity of stresses

If the differentiations of $T_z$ are limited, then we can have the following derivation.

Let

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z.$$

(2.11)

Then combining the last three equations of Eq. (2.7) and hypothesis 2 gives

$$\sigma_{yy} = -\partial^2 \Phi_{133} / \partial x \partial y = r f_{13}^{(2)} \tilde{A}_{133}(0,T_z),$$

(2.12)

$$\sigma_{zz} = -\partial^2 \Phi_{133} / \partial y \partial z = r f_{13}^{(2)} \tilde{A}_{133}(0,T_z),$$

(2.13)

$$\sigma_{xx} = -\partial^2 \Phi_{133} / \partial x \partial z = r f_{13}^{(2)} \tilde{A}_{133}(0,T_z).$$

(2.14)

Considering Eqs. (2.11) and (2.12) can be expanded as

$$r f_{13}^{(2)} \tilde{A}_{133}(0,T_z) = -\partial^2 \Phi_{133} / \partial x \partial z = r f_{13}^{(2)} \tilde{A}_{133}(0,T_z),$$

(2.15)

Comparing both sides of Eq. (2.15), it can be seen that the dominant term of $\Phi_{133}$ has the form of

$$\Phi_{133} = r f_{13}^{(2)} \tilde{A}_{133}(0,T_z).$$

(2.16)

Similarly, it can be obtained from Eq. (2.13),

$$r f_{13}^{(2)} \tilde{A}_{133}(0,T_z) = -\partial^2 \Phi_{133} / \partial y \partial z,$$

(2.17)

For the convenience of analysis, let

$$\Phi_{11} = r f_{11}^{(2)} \tilde{A}_{111}(0,T_z).$$

(2.18)

Then

$$\partial^2 \Phi_{11} / \partial z \partial r = f_{11}^{(2)} \partial^2 \Phi_{11} / \partial z \partial r + f_{11}^{(2)} \partial^2 \Phi_{11} / \partial r \partial z \ln r,$$

(2.19)

$$\partial^2 \Phi_{11} / \partial y \partial z = f_{11}^{(2)} \partial^2 \Phi_{11} / \partial y \partial z + f_{11}^{(2)} \partial^2 \Phi_{11} / \partial y \partial z \ln r.$$

(2.20)
Substituting Eqs. (2.19) and (2.20) into Eq. (2.17) gives
\[
\rho^{(2)} A_{yz}(\theta, T_z) = -f_{11}(z)r f_{11}(z) \frac{\partial \Phi_{11}}{\partial z} \sin \theta - rf_{11}(z) \frac{\partial^2 \Phi_{11}}{\partial z^2} \sin \theta \\
\times \ln r \frac{\partial \Phi_{11}}{\partial \theta} - \frac{\partial^2 \Phi_{11}}{\partial z^2} - f_{11}(z) \frac{\partial \Phi_{11}}{\partial z} \sin \theta - \frac{\partial^2 \Phi_{11}}{\partial z^2} \sin \theta \\
\times \cos \theta - f_{11}(z) \frac{\partial \Phi_{11}}{\partial z} \sin \theta - \frac{\partial^2 \Phi_{11}}{\partial z^2} \sin \theta \\
\times \ln r \left[ f_{11}(z) \Phi_{11} \sin \theta + \frac{\partial \Phi_{11}}{\partial \theta} \cos \theta \right],
\]
(2.21)
As showing by Guo (1993a), in the range of \(0 \leq T_z \leq 0.5\), the following inequality is tenable,
\[
f_{11}(z) > 1.
\]
(2.22)
Then there exist the following limits:
\[
\lim_{r \to 0} \frac{\rho^{(2)}(z)}{\ln r} = - \lim_{r \to 0} \frac{f_{11}(z)}{f_{11}(z)} \sim -f_{11}(z),
\]
(2.23)
\[
\lim_{r \to 0} \frac{\rho^{(2)}(z)}{\ln r} = - \lim_{r \to 0} \frac{f_{11}(z)}{f_{11}(z)} - 1 \sim -f_{11}(z). \quad (2.24)
\]
With Eqs. (2.23) and (2.24), Eq. (2.21) can be simplified near the crack border as
\[
\rho^{(2)} A_{yz}(\theta, T_z) = \rho^{(2)} f_{11}(z) \Phi_{11} + 0, \quad f_{11}(z) - 1, \quad (2.25)
\]
where the symbol 0(x) means that when \(x \to 0\), \(0(x) \to \) constant or zero, \(\Phi_{11}(\theta, f_{11}(z))\) is function of \(\theta, f_{11}(z)\) and \(\Phi_{11}\). Therefore,
\[
f_{22}(z) = f_{11}(z) - 1.
\]
(2.26)
Then
\[
\Phi_{11} = \rho^{(2)} f_{11}(z) - 1. \quad (2.27)
\]
Similarly, it can be obtained from Eq. (2.14),
\[
\Phi_{22} = \rho^{(2)} f_{11}(z) - 1. \quad (2.28)
\]
Substituting Eqs. (2.16), (2.27) and (2.28) into the first three equations of Eq. (2.7) gives
\[
\sigma_{xx} = \rho^{(2)} f_{11}(z) \left( \frac{\partial^2 \Phi_{22}(\theta, T_z)}{\partial z^2} \right) - f_{22}(z) - 1, \quad (2.29)
\]
\[
\sigma_{yy} = \rho^{(2)} f_{11}(z) \left( \frac{\partial^2 \Phi_{22}(\theta, T_z)}{\partial z^2} \right) - f_{22}(z) - 1, \quad (2.30)
\]
\[
\sigma_{zz} = \rho^{(2)} f_{11}(z) \left( \frac{\partial^2 \Phi_{22}(\theta, T_z)}{\partial z^2} \right) - f_{22}(z) - 1, \quad (2.31)
\]
where \(\Phi_{22}(\theta, \Phi_{22})\) are functions of \(\theta, \Phi_{22}\). According to hypothesis 1, the order of singularity of \(\sigma_{zz}\) is the same as \(\sigma_{xx}\) and \(\sigma_{yy}\), so it can be obtained from Eqs. (2.29) 2.30 2.31 that \(f_{22}(z) \leq f_{22}(z) - 1, \quad f_{22}(z) \leq f_{22}(z) - 1, \quad f_{22}(z) = f_{22}(z) - 1, \quad f_{22}(z) = f_{22}(z) - 2. \quad (2.32)
\]
From hypotheses 2, we can obtain that \(f(z) = 0\) is function of \(T(z)\).

Substituting Eq. (2.32) into Eqs. (2.16), (2.27) and (2.28) leads to
\[
\rho^{(2)} \Phi_{22}(\theta, T_z). \quad (2.33)
\]
Then we can conclude from Eqs. 2.29 2.30 2.31 that the singular terms of \(\sigma_{xx}\) and \(\sigma_{yy}\) are only related to \(\Phi_{22}\). Therefore, in the asymptotic field at 3D crack borders, Eq. (2.7) can be simplified as:
\[
\sigma_{xx} = \rho^{(2)} \Phi_{22}(\theta, T_z),
\]
(2.34)
\[
\sigma_{yy} = \rho^{(2)} \Phi_{22}(\theta, T_z),
\]
(2.35)
\[
\sigma_{zz} = \rho^{(2)} \Phi_{22}(\theta, T_z),
\]
(2.36)
Considering Eqs. (2.22), (2.26) and (2.32), the stresses \(\sigma_{yz}\) and \(\sigma_{xx}\) are of the order of unity, and so can be ignored in the asymptotic analysis. While the in-plane stresses \(\sigma_{xx}, \sigma_{yy}, \sigma_{yy}\) and \(\sigma_{zz}\) are singular with the order of singularity \(f(z) - 2\), which is a function of \(T(z)\). Therefore, the singular stresses can be determined by the stress function \(\Phi_{22}\) alone.

2.5. The singularity of strain rates

Substituting Eqs. (2.34)–(2.39) into Eq. (2.3) can yield the dominant term of strain rates
\[
\dot{\varepsilon}_{ij} = \rho^{(2)} \dot{f}(z) - 2 \varepsilon_{ij}(\theta, T_z), \quad (i, j = x, y)
\]
(2.40)
\[
\dot{\varepsilon}_{xy} = \rho^{(2)} \dot{f}(z) - 2 \dot{\varepsilon}_{xy}(\theta, T_z),
\]
(2.41)
\[
\dot{\varepsilon}_{zz} = \rho^{(2)} \dot{f}(z) - 2 \dot{\varepsilon}_{zz}(\theta, T_z),
\]
(2.42)
Obviously, the in-plane strain rates \(\dot{\varepsilon}_{xx}, \dot{\varepsilon}_{yy}, \dot{\varepsilon}_{yy}\) have the same singularity of the order of \(n(f(z) - 2\), while the strain rates \(\dot{\varepsilon}_{yy}\) and \(\dot{\varepsilon}_{yy}\) are still of the order of unity. As \(f(z)\) is function of \(T(z)\), the singularity obtained here for 3D crack border strain rates keeps changing with \(T(z)\), and the effect of \(T(z)\) on \(\dot{\varepsilon}_{xx}\) is much stronger. At the corner point, \(T_z \to 0\), \(\dot{\varepsilon}_{zz}\) has the same singularity as in-plane strain rates, and it means large out-of-plane deformation will occur, which has been proved by experiments (Masaaki et al., 1991).

2.6. The compatibility of strain rates

As the singular stress fields near the creeping crack can be described by one of the Maxwell stress functions \(\Phi_{22}\), the following compatibility equation can be used to determine this stress function
\[
\frac{\partial^2 \dot{\varepsilon}_{xx}}{\partial x^2} + \frac{\partial^2 \dot{\varepsilon}_{yy}}{\partial x^2} - \frac{\partial^2 \dot{\varepsilon}_{yy}}{\partial xy} = 0.
\]
(2.43)
dominated by one stress function \( \Phi = \Phi_{33} \) with the triaxial stress constraint \( T_z \) being considered.

### 3. Asymptotic solution for the field

#### 3.1. Analysis method

It has been proven above that all of the in-plane stresses \( (\sigma_{xx}, \sigma_{xy}, \sigma_{yy}) \) and \( \sigma_{zz} \) have the same stress function, which can be assumed to have the form of

\[
\Phi = K f(z) \Phi(0, T_z).
\]

where \( f(z) \) is a function of \( T_z \) and \( K \) is the amplitude coefficient.

For the convenience of analysis, let \( (\cdot)' = \frac{\partial}{\partial r}, (\cdot)'' = \frac{\partial^2}{\partial r^2} \).

Substituting Eq. (3.1) into Eq. (2.7) leads to

\[
\begin{align*}
\sigma_{rr} &= K f(z) \Phi(0, T_z), \\
\sigma_{\theta \theta} &= K f(z) \Phi(0, T_z), \\
\sigma_{\theta \phi} &= K f(z) \Phi(0, T_z),
\end{align*}
\]

where

\[
\begin{align*}
\sigma_{rr}(0, T_z) &= f \Phi + \Phi', \\
\sigma_{\theta \theta}(0, T_z) &= f (f - 1) \Phi, \\
\sigma_{\theta \phi}(0, T_z) &= -(f - 1) \Phi'.
\end{align*}
\]

If \( T_z \) is defined as \( T_z = \frac{\sigma_{zz}}{\sigma_{rr} + \sigma_{\theta \theta}} \), then the dominant term of \( \sigma_{zz} \) is

\[
\sigma_{zz} = T_z K f(z) \Phi(0, T_z).
\]

On the basis of Eq. (2.3), Eqs. (3.2)-(3.4), the strain rates can be expressed as:

\[
\begin{align*}
\dot{\varepsilon}_{rr} &= \frac{3}{2} B K n f(z) \Phi(0, T_z), \\
\dot{\varepsilon}_{\theta \theta} &= \frac{3}{2} B K n f(z) \Phi(0, T_z), \\
\dot{\varepsilon}_{\theta \phi} &= \frac{3}{2} B K n f(z) \Phi(0, T_z).
\end{align*}
\]

Fig. 3. Analytical results of \( f(z) - 2 \) for different \( n \) plotted versus \( T_z \).

Fig. 4. (a) Schematic of the SEC specimen subjected to symmetrical remote loading. (b) Schematic of the CCT specimen subjected to symmetrical remote loading. (c) Finite element mesh for upper half, crack-tip region and half-thickness.

Fig. 5. Comparison of results obtained by the RR description for plane stress and plane strain, \( C(t) - T_z \) description and the 3D FE results for CCT specimens at (a) \( r' = 0.1 \) and (b) \( r' = 10 \).
where

\[
\ddot{\varepsilon}_{rr}^{\text{inc}}(\theta, T_z) = \ddot{\varepsilon}_e^{\text{inc}} \left[ \left( 1 - \frac{1 + T_z}{3} f \right) \Phi + \frac{2 - T_z}{3} \Phi^0 \right],
\]

\[
\ddot{\varepsilon}_{\theta\theta}(0, T_z) = \ddot{\varepsilon}_e^{\text{inc}} \left[ \left( 1 - \frac{1 + T_z}{3} f \right) \Phi + \frac{2 - T_z}{3} \Phi^0 \right],
\]

\[
\ddot{\varepsilon}_{rr}(0, T_z) = \ddot{\varepsilon}_e^{\text{inc}} (1 - f) \Phi^0.
\]

\[
\sigma_{rr}^{\text{inc}}(0, T_z) = \left[ 1 - T_z + \frac{T_z^2}{2} \right] \left[ (\ddot{\phi} + \ddot{\Phi}^0)^2 + f^2 (f - 1)^2 \ddot{\Phi}^0 \right] - (1 + 2T_z - 2T_z^2) [f(f - 1) \ddot{\Phi} + \ddot{\Phi}^0 + 3(f - 1)^2 \ddot{\Phi}^2].
\]

While \( \ddot{\varepsilon}_{rr} \) can not be determined by Eqs. (2.3) and (3.5). The main compatibility relation equation (2.44) can be expressed as

\[
\frac{1}{r} (r \ddot{\varepsilon}_{rr})^{\tau*} + \frac{1}{r^2} \ddot{\varepsilon}_{rr}^{\tau*} - \frac{2}{r^2} (r \ddot{\varepsilon}_{\theta\theta})^{\tau*} = 0,
\]

where, \( \gamma = \partial / \partial r \), \( \gamma = \partial / \partial \theta \). Substituting Eqs. (3.6)–(3.8) into Eq. (3.9), the governing equation is analogous to the elastic–plastic plane strain, \( C(1/C_{15}) \)

\[
\sigma_{rr}^{\text{inc}}(0, T_z) = \left[ 1 - T_z + \frac{T_z^2}{2} \right] \left[ (\ddot{\phi} + \ddot{\Phi}^0)^2 + f^2 (f - 1)^2 \ddot{\Phi}^0 \right] - (1 + 2T_z - 2T_z^2) [f(f - 1) \ddot{\Phi} + \ddot{\Phi}^0 + 3(f - 1)^2 \ddot{\Phi}^2].
\]

\[
\frac{1 + T_z}{3} \left\{ a_1 \left[ \ddot{\varepsilon}_e^{\text{inc}} (f^2 \ddot{\phi} + \ddot{\Phi}^0) \right] - \left[ \ddot{\varepsilon}_e^{\text{inc}} (f^2 \ddot{\phi} + \ddot{\Phi}^0) \right]^{\tau*} \right\} \\
+ \left[ \ddot{\varepsilon}_e^{\text{inc}} (f \ddot{\phi} + \ddot{\Phi}^0) \right] + a_3 \left[ \ddot{\varepsilon}_e^{\text{inc}} (f \ddot{\phi} + \ddot{\Phi}^0) \right]^{\tau*} \\
+ n(f - 2) \left[ \ddot{\varepsilon}_e^{\text{inc}} (a_i \ddot{\phi} + \ddot{\Phi}^0) \right]^{\tau*} = 0.
\]

For a stress free mode-I crack, the symmetric (about \( \theta = 0 \)) boundary and the stress-free boundary conditions are described as:

\[
\frac{\partial \sigma_{rr}}{\partial \theta}|_{\theta=\pi} = 0, \quad \sigma_{rr}|_{\theta=\pi} = 0 \Rightarrow \ddot{\Phi}^{\tau*}(0, T_z) = \ddot{\Phi}^{\tau*}(0, T_z) = 0,
\]

\[
\sigma_{\phi\phi}|_{\theta=\pi} = 0, \quad \sigma_{\phi\phi}|_{\theta=\pi} = 0 \Rightarrow \ddot{\Phi}^{\tau*}(\pi, T_z) = \ddot{\Phi}^{\tau*}(\pi, T_z) = 0.
\]

In the vicinity of the crack border, the stress field is governed by Eq. (3.10) together with the boundary conditions Eqs. (3.11) and (3.12). To solve this problem, firstly set \( \ddot{\Phi}(0, T_z) = 1 \), \( \ddot{\Phi}^{\tau*}(0, T_z) = x \) to normalize the homogeneous equation. Then the problem can be separated into two processes. One is initial value problem of Eq. (3.10) with

\[
\ddot{\Phi}^{\tau*}(0, T_z) = \Phi^{\tau*}(0, T_z) = 0, \quad \ddot{\Phi}(0, T_z) = 1, \quad \ddot{\Phi}^{\tau*}(0, T_z) = x.
\]

and the other is a generalized equation set

\[
\ddot{\Phi}^{\tau*}(\pi, T_z) = F(x, f) = 0, \quad \ddot{\Phi}(\pi, T_z) = G(x, f) = 0.
\]

A combination of the fourth order Runge–Kutta method and a shooting procedure is adopted to solve Eq. (3.10). In order to improve precision and accelerate convergent speed, the Richardson extrapolation method is applied to vary the step lengths in order to meet the precision during the iteration automatically.
3.2. \( C(t) \)-integral

Taking the integral path to be circular with radius \( r \) lying within the dominant singularity characteristic zone, \( C(t) \)-integral can be evaluated as follow (Landes and Begley, 1976):

\[
C(t) = \int_0^\pi (W^r \cos \theta r d\theta - \sigma_{ij} n_\theta \sigma_{ij} r d\theta), \quad (i,j = r, \theta),
\]

where \( W^r = \int_0^r dW^r = \int_0^r \sigma_{ij} d\epsilon_{ij}. \)

It can be obtained from the strain rate solution-equtions (3.6)–(3.8) and geometrical relation-(2.5) that

\[
\dot{u}_i = \frac{3}{2} BK^{m(j-2)+1} \tilde{u}_i(\theta, Tz, n), \quad (i,j = r, \theta), \tag{3.15}
\]

where

\[
\dot{u}_i = \frac{\bar{\sigma}^{n+1}_i}{n(f-2)+1}\left[ (1 + \frac{1}{3} Tz f) \tilde{\Phi} + \frac{2}{3} - Tz \tilde{\Phi}\right],
\]

\[
\ddot{u}_i = -\frac{1}{n(f-2)} \left[2\bar{\sigma}^{n+1}_i (1-f) \Phi' - \dot{\Phi}' \right].
\]

Substituting Eqs. (3.2)–(3.8) and (3.16) into Eq. (3.15), it can be obtained that

\[
C(t) = BK^{m(j-2)+1} \int_0^{Tz(n+1)} l(Tz, n), \quad (i,j = r, \theta), \tag{3.17}
\]

where

\[
l(Tz, n) = \int_0^{\pi} \left\{ \frac{n}{n+1} \sigma^{n+1}_i \cos \theta - \frac{2}{3} \sin \theta \left[ \sigma_{ij}(u_i - \ddot{u}_i) - \sigma_{ij}(\ddot{u}_i + \dot{u}_i) \right] \right\} d\theta + \frac{3}{2} \int_{\pi}^{2\pi} \left( \bar{\sigma}^{n+1}_i \tilde{u}_i + \bar{\sigma}^{n+1}_i \tilde{u}_i \right) \cos \theta d\theta.
\]

Substituting Eq. (3.17) into Eq. (3.16)

\[
\dot{u}_i = \frac{3}{2} K_0^f \sigma_i \sigma_{ij}(Tz, n)^{n(n+1)} \left( \frac{C(t)}{\sigma_f} \right)_i \dot{u}_i(\theta, Tz, n). \tag{3.18}
\]

Guo (1993a) gave the results for \( f(z) \) as follow

\[
f(z) \geq (2n + 1)/(n + 1). \tag{3.19}
\]

Only when \( Tz = 0 \) or 0.5, the equal sign in Eq. (3.19) is tenable. Then it can be found from Eq. (3.17) that only when \( Tz = 0 \) or 0.5, \( f = (2n+1)/(n+1) \) and the relation of amplitude \( K \) with \( C(t) \) will be independent of \( r \), or else \( f = (2n+1)/(n+1) \) and the relation will be closely dependent upon \( Tz \).

3.3. Characterization of crack border fields

Substituting Eq. (3.17) into Eqs. (3.2)–(3.4), (3.6)–(3.8) and (3.16) leads to

\[
\sigma_{ij} = \sigma_{ij} \left[ \frac{C(t)}{\sigma_{ij} \sigma_{ij}(Tz, n)^{n+1}} \right] \dot{u}_i(\theta, Tz), \tag{3.20}
\]

\[
\ddot{u}_i = \frac{3}{2} K_0^f \sigma_i \sigma_{ij}(Tz, n)^{n(n+1)} \left( \frac{C(t)}{\sigma_f} \right)_i \ddot{u}_i(\theta, Tz), \tag{3.21}
\]

\[
\dddot{u}_i = \frac{3}{2} K_0^f \sigma_i \sigma_{ij}(Tz, n)^{n(n+1)} \left( \frac{C(t)}{\sigma_f} \right)_i \dddot{u}_i(\theta, Tz), \tag{3.22}
\]

Fig. 8. The percentage errors with respect to the numerical results for \( C(t) - Tz \) description for SEC specimens at (a) \( t^* = 0.1 \) and (b) \( t^* = 10 \).

Fig. 9. The percentage errors with respect to the numerical results for (a) \( C(t) - Tz \) description and (b) \( C(t) - Tz - Q' \) description for CCT specimens under extensive creep conditions.
In the range of $0 < T_z < 0.5$, Fig. 3 shows that $f(z)$ varies with $T_z$ slightly, which means that the influence of $r$ on $C(t)$ is small. And further, the exponents of $C(t)$ in Eqs. (3.20), (3.21) and (3.22) are $1/(n+1)$, $n/(n+1)$ and $n/(n+1)$, respectively. Then the influence of $C(t)$ dependent on stresses is much smaller, especially for higher $n$. While that influence have obvious effect on strain rates and displacement rates. Therefore $C(t)$ in Eq. (3.20) can be approximated as a constant in creep zone.

### 3.4. Crack tip opening displacement (CTOD)

At short time after load application (small scale creep), substituting Eq. (1.5) into Eq. (3.22),

$$u_i = 3\sqrt{3} \dot{e}_0 \frac{J}{\sqrt{\sigma_0 r(T_z, n) r(n + 1) T_z}} \left(\frac{n}{n+1}\right)^{(n+1)/2} \delta_i(\theta, T_z, n).$$

(3.23)

Integral equation (3.23),

$$u_i = 3\sqrt{3} \dot{e}_0 \left(\frac{J}{\sqrt{\sigma_0 r(T_z, n) r(n + 1) T_z}} \right) \frac{n}{n+1} \delta_i(\theta, T_z, n).$$

(3.24)

Using the $90^\circ$ intercept definition for CTOD frequently adopted in numerical analyses leads to

$$\delta_\Gamma = 2u_{i_z}|_{\theta=\pi/2-s_i}.$$  

(3.25)

It can be obtained from Eqs. (3.24) and (3.25)

$$\delta_i = 2u_{i_z}|_{\theta=\pi/2-s_i} = d_\sigma \frac{J(n + 1)^{1/n} \pi^{1/n}}{\sigma_0} = d_\sigma \frac{C(t)(n + 1)^{n/(n+1)/2} \pi^{1/n}}{\sigma_0},$$

(3.26)

$$d_\sigma = d_\sigma(\dot{e}_0, T_z, n) = 3 \left(\frac{2}{3}\right)^{1/n} \frac{\dot{e}_0 \pi(T_z, n)^{n/(n+1)/2}}{\pi(T_z, n)}.$$  

(3.27)

where $\dot{e}_0$ and $\dot{t}$ are dimensionless quantities extracted from $\dot{e}_0$ and $t$, respectively.

Similarly, we can get the $\delta_i$ under extensive creep and transition period, as follows:

$$\delta_i = d_\sigma C^{(n+1)/2}$$ (extensive creep),

(3.28)

$$\delta_i = d_\sigma C^{(n+1)/2} + C^{t z^{1/n}}$$ (transition period).  

(3.29)

Evidently, Eqs. (3.26)–(3.29) show that the linear relation between CTOD and $C(t)$-integral is affected by $T_z$ and $t$.

### 4. Numerical model

To verify the efficiency of the asymptotic solution, ABAQUS 6.8 is employed to model the cases of a single-edge cracked (SEC) specimen and a centre-cracked tension (CCT) specimen under mode I condition as shown in Fig. 4(a) and (b). The dimensions of the specimens are as follows: $W/2 = 0.5, W = 25.4$ mm, $L = 114.3$ mm, thickness $h = 4$ mm. For the SEC and CCT specimens, the crack length are

![Fig. 10. The percentage errors with respect to the numerical results for (a) $C(t)$ - $T_z$ description and (b) $C(t)$ - $T_z - Q^*$ description for SEC specimens under extensive creep conditions.](image)

![Fig. 11. Comparison of results obtained by $C(t)$ - $T_z$ description and $C(t)$ - $T_z - Q^*$ description for (a) CCT specimens and (b) SEC specimens.](image)
designated \(a\) and \(2a\) and the total width as \(W\) and \(2W\) as usual. The straight crack front is located at the center of the width along the \(z\)-axis \((x, y) = 0\). Only a quarter of the plate is modeled with finite elements, since the problem has reflective symmetry with respect to the mid-plane \((z = 0)\) and the crack ligament plane \((y = 0)\). The finite element mesh is constructed with 20-node 3D brick elements. In the plane \((x-y)\) perpendicular to the crack front, the element size gradually increases with increasing radial distance from the crack border, while there are 16 elements in each circular ring surrounding the crack border. To consider the detail of large deformation and blunting of the crack border, an initial notches with root radius \(r_p = 0.001\) mm is adopted, as shown in Fig. 4(c). The identical planar mesh is repeated along the \(z\)-axis from the mid-plane \((z/h = 0)\) to the edge-plane \((z/h = 0.5)\). In order to accommodate the strong variations of field quantities with respect to the \(z\)-axis, the thickness of successive element layers is gradually reduced toward the free surface. There are 10 element layers through the half-thickness, and each layer contains 656 elements.

The material constitutive model is given by Eq. (1.4). In this work, the material properties are \(n = 5\), \(E = 154\) GPa, \(v = 0.33\), \(B = 1.348 \times 10^{-6}\) (Pa) \(^{1/2}\) h/\(r\). The initial tensile yield stress is \(\sigma_0 = 417\) MPa, and \(\dot{\varepsilon}_0 = 1.699 \times 10^{-3}\) hr\(^{-1}\). The specimen is subjected to a constant remote tensile stress for all time \((t = 0 \rightarrow \infty)\), \(P = 5080\) N.

5. Result analyses

Here, the transition time \(T_f\) is obtained by substituting \(C^*\) and \(J\) in the mid-plane into Eq. (1.8). For the convenience of analysis, let \(t^* = t/T_f\).

When \(T_f = 0.05\), \(d_e\) in Eqs. (3.26), (3.28) and (3.29) is actually the same as that in power law plasticity solids of plane stress/strain problems. Then, we define a dimensionless length \(\Lambda\) analogous to \(J/\sigma_0\) in power law plasticity solids as follows:

\[
\Lambda = r_d \sigma_0 / [C(t)(n + 1)/n]^{1/n}, \quad \text{(small scale creep)}, \quad (5.1)
\]

\[
\Lambda = r_d \sigma_0 / [C(t^*)^{1/n}], \quad \text{(extensive creep)}, \quad (5.2)
\]

\[
\Lambda = r_d \sigma_0 / [C(t)(n + 1)/n]^{1/n} + C^{1/n}, \quad \text{(transition period)}. \quad (5.3)
\]

The \(C(t)\)-integral is obtained using the inherent program in ABAQUS.

5.1. Comparisons between \(C(t) - T_f\) description and RR description \((\theta = 0\) and \(z/h = 0)\)

Figs. 5 and 6 show comparisons among 3D finite element results, RR and \(C(t) - T_f\) description for \(\sigma_{\theta 0}(\theta = 0\) and \(z/h = 0)\) in both small scale creep and extensive creep conditions. It is obvious that \(\sigma_{\theta 0}\) obtained by \(C(t) - T_f\) description (3.20) is much closer to the finite element results than the RR descriptions, and represents the transition state from RR (plane strain) at the crack border to RR (plane stress) far away from the tip.

5.2. The percentage errors with respect to the numerical results for different layers \((\theta = 0)\)

For fracture in the brittle and ductile region, we often postulate that the fracture initiation occurs when a critical tensile stress is achieved over a significant microstructural distance, which normally encompasses the fracture process zone. Ritchie et al. (1973); Ritchie and Thompson (1985) suggested that the critical distance \(r_c\) from the crack border is within the range of \(1 < r_c / \sqrt{\sigma_0} < 5\). Analogous to brittle and ductile fracture, the area of interest in creep fracture is \(1 < r_c / \Lambda < 5\), and we investigate the percentage errors between \(C(t) - T_f\) description and 3D finite element results just in the range of \(1 < r_c / \Lambda < 5\).

Figs. 7(a) and 8(a) indicate that \(C(t) - T_f\) description can provide an accurate prediction for the circumferential stress near the crack border under small scale creep conditions, even for CCT specimens which develop unconstrained crack border fields. While under extensive creep conditions, as shown in Figs. 7(b) and 8(b), it is necessary to introduce a third parameter \(Q^*\); a quantitative measure of the crack border constraint caused by geometries and loadings, even for SEC specimens which develop full constrained crack border fields, especially for edge-plane.

5.3. Comparison between \(C(t) - T_f\) description and \(C(t) - T_f - Q^*\) description under extensive creep conditions \((\theta = 0\) )

Analogous to \(J - T_f - Q^*\) theory proposed by Guo (2000), we propose the following formulation

\[
\sigma_{\theta 0} = \sigma_{\theta 0} / [C(t)(T_f, \theta)] \sigma_{\theta 0} \theta_{T_f} + Q^* \sigma_{\theta 0} \theta_{T_f}, \quad (5.4)
\]

where

\[
Q^* = \frac{(\sigma_{\theta 0})_{\theta 0} - (\sigma_{\theta 0})_{C(t), \theta 0}}{\sigma_{\theta 0}} \text{ at } \theta = 0 \text{ and } r = 2 \Lambda. \quad (5.5)
\]

According to McMeeking (1977), the influence of the finite geometry fields associated with crack border blunting on near tip stress and deformation fields is confined in a region of \(1 < r / (2 - 3)\), \(\delta_t\), and O’Dowd and Shih (1992) suggested \(Q^*\) in elastic–plastic solids to be calculated on \(r = 2/\Lambda\), just outside the zone of finite geometry fields. Then, the proposed \(Q^*\) is calculated at \(r = 2 \Lambda\). As shown in Figs. 9–11, it is quite obvious that \(C(t) - T_f - Q^*\) description is better...
than \( C(t) - T_z \) description, especially for the plane near edge plane \( (z/h = 0.454) \). Furthermore, Fig. 12 shows that \( Q' \) decreases with \( z/h \) increases.

5.4. Singularities of \( \sigma_{rz} \) and \( \sigma_{rz} \) (SEC specimens)

Here, SEC specimens are considered at \( t' = 0.1 \), and the position of model for drawing isosurfaces is shown in Fig. 13.

Isosurfaces with different value are presented in Fig. 14, where the shear yield stress \( \tau_0 = \sigma_0 / \sqrt{3} = 240 \) MPa. Both positive direction and negative direction are considered to investigate the singularities of \( \sigma_{rz} \) and \( \sigma_{rz} \), and it is obviously that the isosurfaces of \( \sigma_{rz} \) and \( \sigma_{rz} \) shrink into the corner point as the absolute value of isosurface increase. Then it can be concluded that \( \sigma_{rz} \) and \( \sigma_{rz} \) have no singularity in the interior of the cracked plate.

6. Conclusions

The main results in this study are summarized as follows:

1. The three-dimensional asymptotic solution for mode-I crack border fields is obtained theoretically for creeping solids obeying the Norton power law. It is shown that the out-of-plane shear stresses \( \sigma_{rz} \) and \( \sigma_{rz} \) can be ignored in the asymptotic analysis, which leads to the singular stresses being determined by the stress function \( \Phi_{33} \) alone. Therefore, a three-dimensional crack problem can be simplified as a quasi-planar problem in the \( x-y \) plane with the triaxial stress constraint \( T_z \). The asymptotic fields are sensitive to \( T_z \), and a two parameter \( C(t) - T_z \) description is necessary to characterize the three-dimensional crack border fields.

2. Detailed finite element analyses show that two parameter \( C(t) - T_z \) description is efficient to dominate the three-dimensional stress fields ahead of a through-thickness crack under small-scale creep conditions, while three parameter \( C(t) - T_z - Q' \) description is necessary for the three-dimensional crack border fields under extensive creep conditions. The three parameter description based on the three-dimensional constraints, including both out-of-plane constraint and in-plane constraint, can provide an advanced theoretical basis for three-dimensional creeping cracked solids.

For more general and complicated creeping solids, such as elastic-plastic creeping cracked body, the solution will be difficult to be obtained analytically. However, the out-of-plane constraint factor \( T_z \) should also be a dominating parameter as it dominates the crack border field in both plastic and creeping cracked bodies.

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References

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