CHARACTERIZATION OF SELF-COMPLEMENTARY GRAPHS WITH 2-FACTORS*

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Let G be a self-complementary graph (s.c.) and π its degree sequence. Then G has a 2-factor if and only if π - 2 is graphic. This is achieved by obtaining a structure theorem regarding s.c. graphs without a 2-factor. Another interesting corollary of the structure theorem is that if G is a s.c. graph of order p ≥ 8 with minimum degree at least p/4, then G has a 2-factor and the result is the best possible.

0. Introduction

Clapham [1] proved that every self-complementary graph (abbreviated s.c. graph) has a hamiltonian chain. It has been shown by Rao [10] that every s.c. graph of order p ≥ 8 has an l-cycle for every integer l, 3 ≤ l ≤ p - 2.

A k-factor of a graph G is a spanning subgraph of G which is regular of degree k. Clapham's result [1] implies that every s.c. graph of even order has a 1-factor. The aim of this paper is to characterize s.c. graphs having a 2-factor.

Let G be a s.c. graph of order p and σ be a permutation of the vertices which maps G onto its complement G̅. Such a permutation is referred to as a complementing permutation of G. (For properties of s.c. graphs and complementing permutations, see [1, 2, 3, 10, 11, 13, 14].) Let σ = σ_1 σ_2 · · · σ_k be the decomposition of the permutation σ into disjoint cycles. It is known that the length of σ_i is a multiple of 4 for every i except possibly one i_0 (say) and the exceptional one has length 1 (the latter can occur only in the case p = 4N + 1). Let σ, have length p_i = 4n_i, 1 ≤ i ≤ k, i ≠ i_0 (possibly). Let

σ_i = (a_{i1}, a_{i2}, ..., a_{ip_i}), i ≠ i_0.

We may assume that (a_{i1}, a_{i2}) ∈ E(G) (for if not, (a_{i1}, a_{i4}) ∈ E(G) and we can relabel the vertices appropriately), and this implies that (a_{i1}, a_{i1-2}) ∈ E(G) for all odd j. We call the vertices a_{i1}, a_{i3}, ..., a_{ip}, the odd vertices of σ, and denote the set by A_i; the vertices a_{i2}, a_{i4}, ..., a_{ip}, are the even vertices of σ, and we denote the set by B_i, 1 ≤ i ≤ k, i ≠ i_0. The vertices of A_i ∪ B_i are the vertices of σ, 1 ≤ i ≤ k, i ≠ i_0.

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We label the vertices such that in each cycle consecutive odd vertices are joined by an edge. Define a directed graph $D(\sigma)$ whose vertex set is the set of all cycles of $\sigma$ and the cycles $\sigma_i$ and $\sigma_j$ ($i \neq j$) are joined by an arc $(\sigma_i, \sigma_j)$ if there is an edge in $G$ from some even vertex of $\sigma_i$ to some odd vertex of $\sigma_j$, if $i, j \neq i_0$; if $i = i_0$, then $(\sigma, \sigma_i)$ is an arc of $D(\sigma)$ if the unique vertex of $\sigma_i$ is joined to some odd vertex of $\sigma_i$; if $j = i_0$, then $(\sigma_i, \sigma_j)$ is an arc of $D(\sigma)$ if some even vertex of $\sigma_i$ is joined to some even vertex of $\sigma_j$; and $\sigma_i$ is joined to every even vertex of $\sigma_j$ and to no odd vertex of $\sigma_j$. It is shown in [1] that $D(\sigma)$ is a complete directed graph. Further, if $(\sigma_i, \sigma_j)$ with $i, j \neq i_0$ is an arc of $D(\sigma)$ then every even vertex of $\sigma_i$ is joined to some odd vertex of $\sigma_j$; and every odd vertex of $\sigma_i$ is joined to some even vertex of $\sigma_j$; if $i = i_0$, then $\sigma_i$ is joined to every odd vertex of $\sigma_j$ and to no even vertex of $\sigma_j$; if $j = i_0$, then $\sigma_i$ is joined to every even vertex of $\sigma_i$ and to no odd vertex of $\sigma_i$.

We make use of the following lemma repeatedly in our discussion.

**Lemma 0.1.** (Clapharil [1]; compare Rao [10]). Let $\sigma_i, \ldots, \sigma_n$ be a path in $D(\sigma)$ where all $i, j \neq i_0$, $1 \leq j \leq \theta$, then $G$ has a chain containing the vertices of all $\sigma_i$, $1 \leq j \leq \theta$, and no vertex outside, in which two consecutive odd vertices of $\sigma_i$ appear consecutively and whose end vertices are consecutive even vertices of $\sigma_i$.

The condensation $D^*$ of a directed graph $D$ has for its vertices the strong components of $D$, and two vertices $\alpha, \beta$ of $D^*$ are joined by an arc $\alpha \rightarrow \beta$ if for some $a \in V(\alpha)$ and $b \in V(\beta)$, $(a, b)$ is an arc of $D$.

As always, $K = K_n$ denotes the complete graph of order $n$, $K^e = K^e_n$ denotes the empty graph, i.e. the graph with no edges. Similarly, $K = K_{m,n}$ denotes the complete bipartite graph with two independent sets having $m$ and $n$ vertices, respectively. $K^e = K^e_{m,n}$ denotes the empty graph on $n + m$ vertices.

If $X$ and $Y$ are sets of vertices of $G$, $G[X, Y]$ denotes the subgraph of $G$ generated by $X, Y$, i.e. the graph with $X \cup Y$ as its vertices, which includes exactly those edges of $G$, having one end vertex in $X$ and the other in $Y$. We write $G[X]$ for $G[X, X]$.

1. The structure of s.c. graphs without 2-factors: the case $p = 4N$.

**Lemma 1.1.** Let $G$ be a s.c. graph of order $p = 4N$ ($> 4$) and $\sigma$, a complementing permutation of $G$. Suppose the digraph $D(\sigma)$ is strongly connected. Then $G$ has a 2-factor.

**Proof.** First suppose $n_i > 1$, for every $i$, $1 \leq i \leq k$. Then $G[A_n, B_i]$ is a regular graph of regularity $n_i$, $1 \leq i \leq k$. Hence $G[A_n, B_i]$ has $r$-factor, for every $r$, $1 \leq r \leq n_i$, (see Harary [4, p. 85]), in particular, since $n_i > 1$, it has a 2-factor, $1 \leq i \leq k$. Therefore, $G$ has a 2-factor. Thus we may take that some cycle of $\sigma$ is of length 4. Since $D(\sigma)$ is a strongly connected complete digraph, by Camion's theorem [4, p. 207], it has a hamiltonian circuit, $(\sigma_1, \ldots, \sigma_k)$ (say), where $n_1 = 1$. Now let $\mu_1$ be a hamiltonian chain of $G$ given by Lemma 0.1 in which the vertices
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$a_{1,1}, a_{1,3}$ appear consecutively and whose end vertices are $a_{k,r}, a_{k,s}$ with $j$ even, where as always the suffixes are to be taken modulo the length of the cycle of $\sigma$ in which they appear. Since $(\sigma_{u}, \sigma_{v})$ is an arc of $D(\sigma)$, there exists an odd $i$ such that $e_{1} = (a_{1,r}, a_{1,s}) \in E(G)$. Then $e_{2} = (a_{k,1}, a_{1,k}) \in E(G)$. Note that $i = 1$ or 3 and $n_{1} = 1$. Now

$$\mu = \mu_{1} - (a_{1,1}, a_{1,3}) + e_{1} + e_{2}$$

is a 2-factor of $G$. This completes the proof.

**Theorem 1.2.** Let $G$ be a s.c. graph of order $p = 4N (> 4)$. Then $G$ does not have a 2-factor if and only if $V(G)$ can be partitioned into two sets $V_{1}, V_{2}$ of order $4N_{1}, 4N_{2}$ (say) respectively where $N_{1} + N_{2} = N$ such that the following conditions hold.

1. $H_{i} = G[V_{i}]$ is a s.c. graph, $i = 1, 2$.
2. Let $L$ be the set of all vertices of $H_{2}$ whose degree in $H_{2}$ is at least $2N_{2}$; and $R = V_{1} - L$. Then $G[L] = K$ and $G[R] = K^c$.

Proof. First we prove the sufficiency. Since $H_{2}$ is a s.c. graph of order $4N_{2}$, we have $|L| = |R| = 2N_{2}$. Now if $N_{2} = 1$, then by (2), $G$ has a vertex of degree 1 and therefore $G$ does not have a 2-factor. Thus we may take that $N_{2} > 1$. If now $G$ has a 2-factor, then by (1) and (2), note that, because of $|L| = |R|$, a 2-factor of $G$ cannot contain any edge connecting $H_{1}$ with $H_{2}$, it follows that $H_{1}$ also has a 2-factor, contradicting (3).

To prove the necessity, let $G$ be a s.c. graph of order $p = 4N (> 4)$ without a 2-factor, and $\sigma$, a complementing permutation of $G$. By Lemma 1.1, $D(\sigma)$ is not strongly connected. Since $D(\sigma)$ is a complete digraph, the condensation of $D(\sigma)$ is a nontrivial transitive tournament (Harary et al. [5, p. 298]). Let $C_{1}, \ldots, C_{s}$ be the strong components of $D(\sigma)$ arranged in such an order that every even vertex of all $\sigma_{u} \in V(C_{1})$ is adjacent in $G$ to all odd vertices of every $\sigma_{v} \in V(C_{1})$, $1 < i < j < s$, where $s > 2$. Define $V_{1}$ to be the set of all vertices of the cycles of $\sigma$ in $\bigcup_{i=1}^{s} V(C_{i}) = W_{1}$ (say); and $V_{2}$ to be the set of all vertices of the cycles of $\sigma$ in $V(C_{1}) = W_{2}$ (say). We show that $V_{1}, V_{2}$ satisfy the conditions (0) through (3) of the statement of the theorem. Clearly $G[V_{1}] = H_{1}$ is a s.c. graph of order $4N$, (say), $i = 1, 2$; with $N_{1} + N_{2} = N$. We first prove three assertions (a), (b) and (c) below and then complete the proof.

(a) $(u_{i}, v_{j}) \notin E(G)$, whenever $u_{i} \in W_{1}$, $v_{j} \in W_{2}$ and $i, j$ even.

Suppose $e_{1} = (u_{i}, v_{j}) \in E(G)$, with $u, v, i, j$ as above. Then $e_{2} = (u_{i+1}, v_{j+1}) \in E(G)$. Let $\sigma_{u} \in V(C_{l})$ where $1 \leq l \leq s - 1$. Note that there is a $\sigma_{u} - \sigma_{u}$ path in $D(\sigma)$, containing all the vertices of $\bigcup_{i=1}^{l} V(C_{i})$ and none of $\bigcup_{i=l+1}^{s} V(C_{i})$, where $\sigma_{u} \in V(C_{l})$. Now obtain, by Lemma 0.1, a chain $\mu_{1}$ in $G$ by combining the cycles in this $\sigma_{u} - \sigma_{u}$ path, for which $a_{u,i}, a_{u,i+2}$ are end vertices.
Similarly, obtain a chain $\mu_2$ in $G$ by combining the cycles of $\sigma$ in $\bigcup_{i=1}^{n_v} V(C_i)$ in which two consecutive odd vertices of a cycle of $\sigma$ in $V(C_{n_v+1})$ appear consecutively and whose end vertices are the consecutive even vertices $a_{v,n} a_{v,n+2}$. Now a hamiltonian cycle $\mu$ in $G$ may be obtained by defining

$$\mu = \mu_1 + e_1 + e_2 + \mu_2,$$

and this is a contradiction.

(b) $(a_{v,n}, a_{w,j}) \notin E(G)$, whenever $\sigma_v, \sigma_w \in W_1$, $v \neq w$ and $i, j$ even.

Suppose $e_1 = (a_{v,n}, a_{w,j}) \in E(G)$, where $v, w, i, j$ are as above. Then $e_2 = (a_{w,j+1}, a_{w,j+2}) \notin E(G)$. Let $\rho_1, \ldots, \rho_r$ be a hamiltonian circuit (the case $r = 2$ is also included) in $C_i$ with $\rho_1 = \sigma_v$ and $\rho_l = \sigma_w$, $2 \leq l \leq r$. Let $\mu_1$ be a hamiltonian chain in $\rho_1$ in which two consecutive odd vertices of $\rho_1$ say $a_{w,j}, a_{w,j+2}$ ($\alpha$ odd) appear consecutively and whose end vertices are $a_{v,n}, a_{w,j+2}$. Obtain a chain $\mu_2$, by combining the cycles $\rho_2, \ldots, \rho_l$, in which two consecutive odd vertices of $\rho_2, b_{v,j+2}, b_{v,j+4}$ (say) appear consecutively and whose end vertices are $a_{v,n}, a_{w,j+2}$. Let $\mu$, be a hamiltonian chain in $H_1$ whose end vertices are consecutive even vertices of some cycle $\sigma_v$ (say) of $\sigma$ in $V(C_{n_v+1})$, $a_{w,n}, a_{w,n+2}$ (say). We now consider two cases.

Case (i) $l = 1$. Then

$$\mu^* = \mu_3 + (a_{v,n}, b_{v,j}) + (a_{v,n+2}, b_{v,j+2}) + \mu_2 - (b_{v,j}, b_{v,j+2}) + e_1 + e_2 + \mu_1,$$

is a hamiltonian cycle in $G$. Thus we may take

Case (ii) $r \geq 1$. Since $(\rho_v, \rho_l)$ is an arc of $D(\sigma)$ and $\alpha$ is odd, $(a_{v,n}, b_{v,j}) \in E(G)$ for some even $l$. Now let $\mu_4$ be a chain obtained by combining the cycles $\rho_1, \ldots, \rho_r$ of $\sigma$ in which two consecutive odd vertices of $\rho_1$ appear consecutively and whose end vertices are the consecutive even vertices $b_{v,n}, b_{v,n+2}$ of $\rho$. Then $\mu$ and $\mu^*$ of case (i) may be combined by defining

$$\mu = \mu^* + \mu_4 + (a_{v,n}, b_{v,j}) + (a_{v,n+2}, b_{v,j+2}) - (a_{v,n}, a_{v,n+2}).$$

Now $\mu$ is a hamiltonian cycle of $G$, a contradiction.

(c) $(a_{v,n}, a_{w,i}) \notin E(G)$, where $\sigma_v \in W_2$ and $i, j$ even.

This is clearly true if $n_v = 1$. So we may take that $n_v \geq 1$. Now it is enough to show that $(a_{v,n}, a_{v,j}) \notin E(G)$, whenever $j$ is even, $4 \leq j \leq 4n_v$. First let $j \neq 2n_v + 2$. Then $G[B_v]$ has a 2-factor $\mu_0$ (say). Let $\mu_1$ be the cycle $(a_{v,n+1}, a_{v,n+3}, \ldots, a_{v,4n_v-1})$. Let $(\rho_1, \ldots, \rho_r)$ be a hamiltonian circuit in $C$, with $\rho_1 = \sigma_v$. Obtain a hamiltonian chain $\mu_2$ in $H_1$ whose end vertices are consecutive even vertices of a cycle $\sigma_v$ in $V(C_{n_v+1})$. Now if $r = 1$, then $\mu_2, \mu_1, \mu_0$ can be combined to yield a 2-factor of $G$. Thus we may take that $r > 1$. Since $(\rho_v, \sigma_v)$ is an arc of $D(\sigma)$, $(a_{v,n}, b_{v,i}) \in E(G)$ for some even $l$. Now let $\mu_3$ be a chain obtained by combining the cycles $\rho_2, \ldots, \rho_r$ in which two consecutive odd vertices of $\rho_2$ appear consecutively and whose end vertices are $b_{v,n}, b_{v,n+2}$. Then $\mu_3, \mu_2, \mu_1$ and $\mu_0$ can be combined suitably to get a 2-factor of $G$.

Thus we may take $j = 2n_v + 2$. Then
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\[ F = \{(a_{i,n}, a_{i,1}), (a_{i,2,1}, a_{i,2,2}), \text{ i odd, } 1 \leq i \leq 4n\} \]
\[ + \{(a_{i,n}, a_{i,1,2n}), \text{ i even, } 1 \leq i \leq 4n\} \]

is a 2-factor of \( G[A_1 \cup B_1] \). This \( F \) and the chains \( \mu_2, \mu_3 \) described above may be combined to yield a 2-factor of \( G \) itself, a contradiction.

Now we are ready to prove the necessity of conditions (1) through (3). Let \( A^*, B^* \) be the sets of the odd vertices or even vertices of the cycles of \( \sigma \) in \( V(C_j) \), respectively. Since \( H_1, H_2 \) are s.c. graphs and \( \sigma(A^*) = B^*, \sigma(B^*) = A^*, \) and \( C_j \) being the bottom most strong component of the complete digraph \( D(\sigma) \), it follows, by assertion (a), that \( G[V, A^* = K \) and \( G[V, B^* = K'. \) By assertions (b) and (c), \( G[B^* = K', \) hence \( G[A^* = K. \) Now it is clear that \( L = A^* \) and \( R = B^* \).

Thus by what has been proved above it follows that conditions (0), (1) and (2) are satisfied. If \( N > 1 \) and \( H_1 \) has a 2-factor, then since \( C_j \) is a strong component, it follows, by Lemma 1.1, that the s.c. graph \( H_2 \) has a 2-factor. This in turn implies that \( G \) also has a 2-factor, contradicting the hypothesis. This completes the proof of the theorem.

The following remark and Lemma will be used in Section 2.

Remark 1.3. The 2-factors obtained in the proofs of assertions (b) and (c) have two consecutive odd vertices of a cycle of \( \sigma \) in \( V(C_j) \) appearing consecutively in them.

Lemma 1.4. Let \( G \) be a s.c. graph of order \( 4N \), and \( \sigma \), a complementing permutation of \( G \). Suppose \( D(\sigma) \) is strongly connected. Then \( G \) has a 2-factor in which two consecutive odd vertices of a cycle of \( \sigma \) appear consecutively, if and only if \( G[A] \neq K \), where \( A \) is the set of all odd vertices of \( \sigma \).

Proof. The proof is similar to the proof of assertions (b) and (c) of Theorem 1.2.

2. The structure of s.c. graphs without 2-factors: the case \( p = 4N + 1 \)

Lemma 2.1. Let \( G \) be a s.c. graph of order \( 4N + 1 \), and \( \sigma \) a complementing permutation of \( G \). Suppose the unique fixed point \( \sigma_0 \) of \( \sigma \) belongs to the bottom strong component \( C \), (say) of \( D(\sigma) \) (the case \( D(\sigma) \) is strong is not excluded). Then \( G \) has a hamiltonian cycle.

Proof. Let \( \rho_1, \ldots, \rho_r \), with \( \rho_1 = \sigma_0 \), be a hamiltonian circuit in \( C \), \( (r = 1 \) is possible). Note that \( \sigma - \sigma_0 \) is a complementing permutation of the s.c. graph \( G - \sigma_0 \) which is of even order. By Lemma 0.1, there exists a hamiltonian chain \( \mu_1 \) (say) in \( G - \sigma_0 \) whose end vertices are consecutive even vertices of \( \rho_{r-1} \) if \( r = 2 \), or consecutive even vertices of a cycle of \( \sigma \) in \( V(C_{r-1}) \) if \( r = 1 \). Since \( \sigma_0 \) is joined to all even vertices of \( \rho_{r-1} \) if \( r = 2 \), and also to all even vertices of every cycle of \( \sigma \) in \( V(C_{r-1}) \), the vertex \( \sigma_0 \).
can be incorporated at the ends of the hamiltonian chain \( \mu_1 \) to get a hamiltonian cycle in \( G \). This completes the proof.

**Theorem 2.2.** Let \( G \) be a s.c. graph of order \( 4N + 1 \). Then \( G \) does not have a 2-factor if and only if \( G \) can be partitioned into two sets \( V_1, V_2 \) of order \( 4N_1 + 1, 4N_2 \) respectively where \( N_1 + N_2 = N \) and \( N_1 \geq 0, N_2 \geq 1 \), such that the conditions (0) through (3) of the statement of Theorem 1.2 hold.

**Proof.** The proof of the sufficiency is exactly similar to the proof of the sufficiency of Theorem 1.2.

To prove the necessity, let \( G \) be a s.c. graph of order \( 4N + 1 \) without a 2-factor and \( \sigma \) a complementing permutation of \( G \) and \( \sigma_0 \) the unique fixed point of \( \sigma \). By Lemma 2.1, \( \sigma_0 \notin V(C_i) \), the bottom strong component of \( D(\sigma) \). Now define \( H_1, H_2 \) as in the proof of Theorem 1.2. Since the vertex \( \sigma_0 \notin V(H_1) = V_1 \), it follows that \( H_1 \) is a s.c. graph of odd order, \( 4N_1 + 1 \) say. Let \( H_2 \) be of order \( 4N_2 \), then \( N_1 + N_2 = N \). If \( N_1 = 0 \), then we assert that \( G[A] = K \) where \( A \) is the set of all odd vertices of \( \sigma - \sigma_0 \) which is a complementing permutation of \( G - \sigma_0 \). Suppose \( G[A] \neq K \). Note that \( D(\sigma - \sigma_0) \) is strongly connected. Hence by Lemma 1.4, \( G - \sigma_0 \) has a 2-factor \( F \) in which two consecutive odd vertices of some cycle \( \sigma_m \) ( \( \neq \sigma_0 \) ) of \( \sigma \) appear consecutively. Then \( \sigma_0 \) may be incorporated in between these two odd vertices of \( F \) to get a 2-factor of \( G \), contradicting the hypothesis. Thus \( G[A] = K \). Since \( \sigma(A) = B \), we have \( G[B] = K^c \). Further, since \( G[\sigma_0, A] = K \), we have \( G[\sigma_0, B] = K^c \). Thus \( G \) satisfies the properties (0) through (3) with \( V_1 = \{\sigma_0\} \), \( L = A, R = B \) and \( V_2 = A \cup B \). Therefore, henceforth we may take that \( N_1 \geq 1 \).

We now prove the three assertions (a), (b) and (c) of Theorem 1.2. Suppose (a) does not hold with \( \sigma_0 \in V(C_m) \) and \( \sigma_0 \in V(C_i) \). Let \( \sigma_0 \in V(C_i) \), \( 1 \leq l, m \leq s - 1 \). We consider three subcases according as \( l < m \), \( l = m \), or \( l > m \).

Case (i) \( l < m \). Then in \( D(\sigma - \sigma_0), C_{m+1} \) is the immediate successor of \( C_m \) and \( C_i \) is the bottom strong component. Then, as in the proof of assertion (a) of Theorem 1.2, we obtain a 2-factor \( F_0 \) of \( G - \sigma_0 \) in which two consecutive odd vertices of some cycle of \( \sigma \) in \( V(C_{m+1}) \) appear consecutively. Now \( \sigma_0 \) may be incorporated in between these odd vertices of \( F_0 \) to get a 2-factor of \( G \), a contradiction.

Case (ii) \( l = m \). Let \( \rho_1, \ldots, \rho_r \) be a hamiltonian circuit in \( C_m \) with \( \rho_1 = \sigma_0 \) and \( \rho_r = \sigma_0, 2 \leq t \leq r \). If \( t = 2 \), then as in case (i) we get a 2-factor of \( G \). Thus we may take that \( 2 < t \leq r \). Now \( G_1 = G - \bigcup_{i=2}^{r} \rho_i \) is a s.c. graph with \( \sigma - \bigcup_{i=2}^{r} \rho_i \) as a complementing permutation. As in the proof of (a) of Theorem 1.2 it can be shown that \( G \) has a 2-factor \( F_1 \) (say). Now \( \rho_2, \ldots, \rho_r \) can be combined to get a cycle \( F_2 \) (note \( \rho_r = \sigma_0 \)). Then \( F_1 + F_2 \) is a 2-factor of \( G \), a contradiction.

Case (iii) \( l > m \). Now \( C_i, C_{m+1}, \ldots, C_r \) may be combined to get a \( F_1 \) in \( G \) (note \( \sigma_0 \in C_i \)). Also \( C_m, C_{m+1}, \ldots, C_r \) may be combined, as in the proof of (a) of Theorem 1.2, to get a 2-factor \( F_2 \) of the corresponding graph. But then \( F_1 + F_2 \) is a 2-factor of \( G \), a contradiction.
In case assertions (b) or (c) of Theorem 1.2 are not valid, we get, by Remark 1.3, a 2-factor $F_n$ of $G - \sigma_n$ in which two consecutive odd vertices of a cycle of $\sigma$ in the bottom most strong component of $D(\sigma - \sigma_n)$ (which is $C_s$) appear consecutively. Then $\sigma_n$ may be incorporated in between these odd vertices of $F_n$ to get a 2-factor of $G$, contradicting the hypothesis that $G$ does not have a 2-factor. Thus the assertions of (a), (b) and (c) in Theorem 1.2 are valid in the case $p = 4N + 1$ also. Now as in the proof of Theorem 1.2, it can be shown that $H_1, H_2$ satisfy the conditions (0) through (3) of Theorem 1.2. This completes the proof of Theorem 2.2.

3. Characterization of s.c. graphs with 2-factors

In this section we prove the following:

Theorem 3.1. Let $G$ be a s.c. graph of order $p$, and $\pi = (d_1, \ldots, d_p)$ be its degree sequence. Then $G$ has a 2-factor if and only if $\pi - 2 = (d_1 - 2, \ldots, d_p - 2)$ is graphic.

We use the following three theorems

Theorem 3.2. (Kundu [8], Kleitman, Wang [6]). Let $\pi$ and $\pi - k$ be both graphic. Then there is a realization of the former which has one of the latter as a subgraph.

Theorem 3.3. (Koren [7], Compare Rao, Rao [9, p. 187–188]). Let $\pi = (d_1, \ldots, d_p)$ be a graphic nonincreasing sequence. Let $\delta(j, \pi) = j(j-1) + \Sigma'_{i=1} \min(d_i, j) - \Sigma'_{i=1} d_i$. Suppose $\delta(j, \pi) = 0$ for some $j$, $1 \leq j < p$. If $d_{i+1} > j$, let $r = r(j)$ be an index such that $d_r \geq j \geq d_{i+1}$. If $d_{i+1} \leq j$, let $r = j$. For any realization $H = H(u_1, \ldots, u_p)$ of $\pi$ with degree of $u_i = d_i$, $1 \leq i \leq p$, define

$$S = \{u_1, \ldots, u_i\}, \quad T = \{u_{r+1}, \ldots, u_p\}, \quad U = \{u_{r+1}, \ldots, u_i\}.$$ 

Then

1. $H[S] = K$,
2. $H(T) = K'$. If $U \neq \emptyset$,

then

3. $H[S, U] = K$,
and


Theorem 3.4. (Koren [7]). Suppose $H(u_1, \ldots, u_p)$ realizes $\pi$, $S = \{u_1, \ldots, u_i\}$, $p > j \geq 1$, $T = \{u_{r+1}, \ldots, u_p\}$, $(r \geq j)$, $U = \{u_{r+1}, \ldots, u_i\}$ and conditions (1), (2) hold for $S$ and $T$, and if $U \neq \emptyset$, then conditions (3), (4) hold as well. Then $\delta(j, \pi) = 0$.

Proof of Theorem 3.1. The proof is by induction on $p$. For $p = 4$, the result is vacuously true. Assume the result for all values less than $p$ and let $G$ be a s.c. graph with degree sequence $\pi = (d_1, \ldots, d_p)$ such that $\pi - 2$ is also graphic. Suppose $G$ does not have a 2-factor. Then by Theorems 1.2 and 2.2, $V(G)$ can be partitioned
into two sets $V_1, V_2$ of order $4N + \delta, 4N_2$ (where $\delta = 0$ or $1$ according as $p$ is $4N$ or $4N + 1$ respectively) such that the conditions (0) through (3) of Theorem 1.2 hold. Put

$$S = \{u_1, \ldots, u_{2N}\},$$

$$U = \{u_{2N+1}, \ldots, u_{\theta - 1}\},$$

$$T = \{u_\theta, \ldots, u_p\},$$

where $\theta = 2N_2 + 4N + 1 + \delta$.

Now it is not difficult to check that for $u \in S$, $v \in U$ and $w \in T$, we have degree $u > \deg v > \deg w$ where the degree is to be taken in the graph $G$. Further, $G$ satisfies conditions (1) through (4) of Theorem 3.3. Hence, by Theorem 3.4, $\delta(2N_2, \pi) = 0$. Now by Theorem 3.2, $\pi$ has a realization $G^*$ (say) such that $G^*$ has a 2-factor. Since $\delta(2N_2, \pi) = 0$, it follows by Theorem 3.3, that $G^*$ satisfies the conditions (1) through (4) of Theorem 3.3 (with $H$ replaced by $G^*$). Since $G^*$ has 2-factor, it is evident that the graphs $G^*[U]$, $G^*[S \cup T]$ have 2-factors. By the structure of $G$ and $G^*$ it is also evident that degree sequence of $G^*[U] = \deg sequence of G[U] = \deg sequence of H_i = \pi_i$ (say); and also degree sequence of $G^*[S \cup T] = \deg sequence of G[S \cup T] = \deg sequence of H_2 = \pi_2$ (say).

Since $G^*[U], G^*[S \cup T]$ have 2-factors it follows that $\pi_i - 2$ is graphic, $i = 1, 2$.

Thus $H_i$ is a s.c. graph with degree sequence $\pi_i$ such that $\pi_i - 2$ is graphic, $i = 1, 2$. Hence by induction hypothesis, $H_i$ has a 2-factor $F_i$ (say), $i = 1, 2$. But then $F_1 + F_2$ is a 2-factor of $G$, a contradiction. This completes the proof of the theorem.

**Theorem 3.5.** Let $G$ be a s.c. graph of order $p \geq 8$ such that minimum degree of $G \geq p/4$, then $G$ has a 2-factor.

**Proof.** Suppose $G$ does not have a 2-factor. Then let $V_1, L, R$ be as in Theorems 1.2 and 2.2. It is clear that $q(H_2[L, R]) = 2N_2^2$ (where $q =$ number of edges). It follows that for some vertex $w$ of $R$, $q(H_2[L, \{w\}]) \leq N_2$. Since $G[R]$ is the empty graph, we have $q(G[L, \{w\}]) \leq N_2$. Thus minimum degree in $G \leq N_2$. Since $p = 4N_1 + 4N_2 + \delta$, it can be easily seen that $N_2 < p/4$, a contradiction to the hypothesis.

To show that the result is the best possible, we consider two cases:

Case (i) $p = 4N$. A required graph $G$ whose vertex set is $V = \{u_1, \ldots, u_p\}$ may be constructed as follows: Define $V_1 = \{u_1, u_2, \ldots, u_4\}, V_2 = V - V_1$.

$G[V_1]$ is the s.c. graph of order 4,

$L = \{u_5, \ldots, u_{2N}\}, R = V_2 - L$;


$G[L, R]$ is the disconnected graph having exactly two components each of which is regular of degree $N - 1$. Clearly, $G$ is a s.c. graph of order $4N$ in which minimum degree is $N - 1$. Further, $G$ does not have a 2-factor.
Case (ii) \( p = 4N + 1 \). A required graph \( G \) whose vertex set is \( V = \{ u_0, \ldots, u_{4N} \} \) may be constructed as follows:

\[
V_1 = \{ u_0 \}, \quad V_2 = V - V_1;
\]

\[
L = \{ u_1, \ldots, u_{2N} \}, \quad R = V_2 - L.
\]

\[
G[L] = K, \quad G[R] = K', \quad G[V_1, L] = K, \quad G[V_1, R] = K', \quad \text{and}
\]

\[
G[L, R] \quad \text{is the disconnected graph having exactly two components each of which is regular of degree} \ N. \quad \text{G is a s.c. graph in which minimum degree is} \ N \quad \text{and} \ G \quad \text{does not have a 2-factor.}
\]

4. Epilogue

The problem of characterizing s.c. graphs with \( k \)-factors seems to be much deeper. In this connection we take the risk of conjecturing the following:

**Conjecture.** Let \( G \) be a s.c. graph of order \( p \), \( \pi \) its degree sequence. Then \( G \) has a \( k \)-factor if and only if \( \pi - \varepsilon \) is graphic.

In a forthcoming paper Rao [12] we characterize, by using the techniques developed in the present paper, hamiltonian s.c. graphs. For a characterization of the degree sequences of self-complementary graphs, see Clapham and Kleitman [2].

**References**