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Fitting Subgroups and Profinite Completions of Polycyclic Groups

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For any group G, the profinite topology on G is the uniform topology on G for which a neighborhood basis of the identity is given by subgroups of finite index in G. The profinite completion $\hat{G}$ of G is the completion of G in this topology. If G is a polycyclic-by-finite ($\mathscr{PF}$-) group and if H is a subgroup of G, then the induced topology on H is the profinite topology on H so that $\hat{H}$ may be considered to be a subgroup of $\hat{G}$. If H is normal in G, then $\hat{H}$ is normal in $\hat{G}$ and $(G/H)\hat{}$ is isomorphic to $\hat{G}/\hat{H}$. The purpose of this paper is to prove the following generalization of [5, Lemma 2], in which the corresponding result was shown for nilpotent-by-finite groups.

THEOREM 1. *Let G be a $\mathscr{PF}$-group and let N be the maximal normal nilpotent (Fitting) subgroup of G. Then $\hat{N}$ is the Fitting subgroup of $\hat{G}$.*

For a given group G, let $\mathscr{F}(G)$ denote the set of isomorphism classes of finite homomorphic images of G. We say groups G and H have isomorphic finite quotients if $\mathscr{F}(G) = \mathscr{F}(H)$. The following proposition was proven in [5].

PROPOSITION 2. *If G and H are $\mathscr{PF}$-groups, then G and H have isomorphic finite quotients if and only if $\hat{G}$ is isomorphic to $\hat{H}$.*

As an immediate consequence of Theorem 1 and Proposition 2, we have:

THEOREM 3. *If two $\mathscr{PF}$-groups have isomorphic finite quotients, then their respective Fitting subgroups must have the same finite quotients.*

Preliminaries. We first quote some as yet unpublished work of E. Formanek [2]. Let R be the ring of algebraic integers in a number field F. A subgroup G of the group $GL(n, R)$ of invertible matrices with entries in R and determinant in R is said to have the congruence subgroup property (CSP) if each subgroup H of finite index in G contains a congruence subgroup

$$G(m) = \{g \in G \mid g \equiv 1 \bmod m\}$$

for some integer m. Thus G has CSP if and only if the profinite topology coincides with the congruence topology (the topology generated by taking the congruence subgroups as a neighborhood base for the identity). If we let $T(n, R)$ denote the subgroup of $GL(n, R)$ consisting of upper triangular matrices, $D(n, R)$ the subgroup of diagonal matrices and $N(n, R)$ the subgroup of upper unitriangular matrices (matrices with 1 on the diagonal), we have

THEOREM 4. [2, Theorem 1]. *Let G be a subgroup of $T(n, R)$. Then G has* CSP.

THEOREM 5. [2, Theorem 2] *Let G be an abelian subgroup of $GL(n, R)$. Then G has* CSP.

Theorem 4 if obtained by first proving the result for subgroups of $N(n, R)$ and $D(n, R)$ and then combining the two. CSP for subgroups of $D(n, R)$ is a consequence of an arithmetic theorem of Chevalley [1]. Theorem 5 is obtained by upper triangulating G (see Lemma 7 below).

LEMMA 6. (essentially [2, Lemma 10]) *Let M be an abelian subgroup of automorphisms of a finitely generated free abelian group A. Suppose N is a subgroup of M and g is an element of M but not of N. Then there is an integer m such that $g \not\equiv h$* mod *m for any h in N.*

Proof. M and thus M/N are finitely generated and consequently residually finite. Since $gN \neq 1$ in M/N, gN is not in $(M/N)^a$ for some integer a. Thus g is not in M^aN. Since M has CSP by Theorem 5, M^a contains $M(m)$ for some m. This means that g is not in $M(m)N$ or $g \not\equiv h$ mod m as required.

LEMMA 7. *Suppose M is a finitely generated abelian subgroup of $GL(n, Z)$ and that F is an algebraic number field which contains the eigenvalues of a set of generators of M. Let R be the ring of algebraic integers in F. Then M is conjugate over F to a subgroup M' of $T(n, R)$ such that each element of M' commutes with its diagonal part.*

Proof. Consider M as acting on F^n. Since the elements of M commute, any generalized eigenspace of an element x of M (set of vectors v such that $(x - \alpha I)^n v = 0$ for a particular eigenvalue α) is invariant under M. We may thus decompose F^n as a direct sum of subspaces V_i, invariant under M, on which each element of M has a single eigenvalue. Choose a basis of each

V_i in which each element of M has an upper triangular matrix. Let d be the least common denominator of entries in elements of a generating set of M in this new basis. By conjugating by the diagonal matrix

$$\mathrm{diag}(1, d, d^2, \ldots, d^{n-1})$$

we may insure that the elements of M in the new basis have integer entries. Let the group of matrices in the new basis be M'. On each V_i the diagonal part of any element of M' is a scalar matrix. Thus each element of M' commutes with its diagonal part.

Recall that an automorphism α of an abelian group A is unipotent if α can be written as $1 + \eta$ with η nilpotent. If B is an α-invariant subgroup of A, then α is unipotent on A if and only if the automorphisms induced on B and A/B are unipotent. If A is torsion-free, every nontrivial unipotent automorphism has infinite order. The group generated by an element x and a normal abelian subgroup A is nilpotent if and only if x induces a unipotent automorphism on A.

Now let G be a $\mathscr{PF}$-group and let $N = \mathrm{Fitt}\, G$. By Mal'cev [4] there is a free abelian normal subgroup K/N of finite index in G/N. Let I/N' be the periodic part of N/N'.

PROPOSITION 8. *No element of K, not in N, induces a unipotent automorphism of N/I.*

Proof. Note that $\mathrm{Fitt}\, K = \mathrm{Fitt}\, G = N$. Suppose x in K induces a unipotent automorphism of N/I. Since I/N' is finite, some power x^m of x centralizes I/N'. It follows that x^m induces a unipotent automorphism of N/N'. Let $L = gp\{x^m, N\}$. L/N' is nilpotent since x^m induces a unipotent automorphism of N/N'. N is a normal nilpotent subgroup of L so, by [3, Theorem 7], L is also nilpotent. Since L is also normal in K, L is contained in $\mathrm{Fitt}\, K = N$. Thus x^m is in N. Since K/N is torsion-free, we must have x in N as well.

PROPOSITION 9. *No element of $\hat{K}$, not in $\hat{N}$, induces a unipotent automorphism of $\hat{N}/\hat{I}$.*

Proof. K acts on N/I by conjugation. By Proposition 8, K/N acts as a free abelian group K^* of automorphisms of the free abelian group N/I with no nontrivial unipotent elements. K^* may be considered to be a subgroup of $\mathrm{GL}(n, Z)$. The group of automorphisms of $\hat{N}/\hat{I}$ induced by $\hat{K}/\hat{N}$ is the completion $(K^*)^\wedge$ of K^* in the congruence topology (as a subgroup of $\mathrm{GL}(n, Z)$). If x in $\hat{K}$ induces a unipotent automorphism of $\hat{N}/\hat{I}$, the corresponding element in $(K^*)^\wedge$ must be unipotent. Thus it suffices to show that no nontrivial element of $(K^*)^\wedge$ is unipotent.

By Lemma 7, K^* is conjugate in $\mathrm{GL}(n, F)$ to a subgroup $\bar{K}$ of $T(n, R)$.

The congruence topology in $\mathrm{GL}(n, R)$ is unchanged by conjugation by elements of $GL(n, F)$. Thus $(K^*)^\wedge$ is conjugate to $(\bar{K})^\wedge$ as well. Since unipotents are also preserved by conjugation, it is enough to show that no nontrivial element of $(\bar{K})^*$ is unipotent.

Consider the map d: $T(n, R) \to D(n, R)$ which erases off diagonal elements. Since by Proposition 8, no nontrivial element of $\bar{K}$ is unipotent, d: $\bar{K} \to d(\bar{K})$ is an isomorphism. Thus

(A) If x is an element of $\bar{K}$, $d(x)$ is a kth power in $d(\bar{K})$ if and only if x is a kth power in $\bar{K}$.

By using Lemma 7, we have arranged that

(B) If x is an element of $\bar{K}$, $d(x)$ commutes with x and with $n(x) = x - d(x)$. Thus for any positive integer m, $x^{mn!}$ is congruent mod m to $d(x)^{mn!}$ (expand $(d(x) + n(x))^{mn!}$ by the Binomial Theorem and use $n(x)^n = 0$).

By Theorem 4,

(C) If D is a subgroup of $D(n, R)$, then for each positive integer m, there is a positive integer $\theta(m)$ such that if g is in D and $g \equiv 1 \bmod \theta(m)$, then g is an mth power in D.

Now suppose that α is a unipotent element of $(\bar{K})^\wedge$ and let a_1, a_2,... be a sequence of elements of $\bar{K}$ converging to α, such that

(D) $a_i \equiv \alpha \bmod i!$ for each i, and

(E) $a_i \equiv a_j \bmod i!$ for each j greater than i.

Since α is unipotent, $d(\alpha) = 1$ so that $d(a_i) \equiv 1 \bmod i!$ for each i. For any integer m, let $k = m \cdot \theta(m!n!)$, where θ is the function given by (C) for the subgroup $d(\bar{K})$ of $D(n, R)$. By (D), $d(a_k) \equiv 1 \bmod \theta(m!n!)$ so by (C), $d(a_k)$ is an $m!n!$ power in $d(\bar{K})$. By (A), a_k is an $m!n!$ power in $\bar{K}$. By (B), $a_k \equiv d(a_k) \bmod m!$. But $d(a_k) \equiv 1 \bmod k!$ so $d(a_k) \equiv 1 \bmod m!$ and $a_k \equiv 1 \bmod m!$. By (E) $a_m \equiv a_k \bmod m!$, so that $a_m \equiv 1 \bmod m!$. Thus for each m, $\alpha \equiv a_m \equiv 1 \bmod m!$ so $\alpha = 1$. This shows that the only unipotent element of $(\bar{K})^\wedge$ is the identity, as required.

Proof of Theorem 1. Let H be the centralizer of N/I in G. By Proposition 8, no element of $K\backslash N$ can centralize N/I so that $H \cap K = N$ and H/N must be finite.

LEMMA 10. $\hat{H} = C_{\hat{G}}(\hat{N}/\hat{I})$.

Proof. Suppose x in $\hat{G}$ centralized $\hat{N}/\hat{I}$. We may write $x = gk$ with g in G and k in $\hat{K}$. The automorphism g^* induced by g on $\hat{N}/\hat{I}$ must be the same as that induced by k^{-1}. Thus g^* is in $(K^*)^\wedge$, the group of automorphisms induced on $\hat{N}/\hat{I}$ by elements of $\hat{K}$. This implies that g^* commutes with all elements of the group K^* of automorphisms induced by K on N/I. Now g^* must be in K^* for otherwise, letting $M = gp\{g^*, K^*\}$, $N = K^*$, $A = N/I$, we would contradict Lemma 6. Thus there is an element k' of K such that gk' centralizes N/I. Therefore $gk' = h$ is in H and x may be written $x = hk''$ with k'' in $\hat{K}$. Since x and h centralize $\hat{N}/\hat{I}$, k'' must also centralize $\hat{N}/\hat{I}$. Now Proposition 9 implies that k'' must be in $\hat{N} \subset \hat{H}$, so that $\hat{x}$ is in $\hat{H}$.

Now let $M = \text{Fitt}(\hat{G})$. Since every element of M must induce a unipotent automorphism of $\hat{N}/\hat{I}$, Proposition 9 implies that $M \cap \hat{K}$ equals $\hat{N}$. This implies that $M/\hat{N}$ is finite. Since $\hat{N}/\hat{I}$ is torsion-free, no nontrivial unipotent automorphism of $\hat{N}/\hat{I}$ can have finite order, so M must centralize $\hat{N}/\hat{I}$. By Lemma 10, M must be contained in $\hat{H}$. Since H is nilpotent-by-finite, we may apply [5, Lemma 2] to conclude that M is contained in $\text{Fitt}(\hat{H}) = \hat{N}$. This completes the proof of Theorem 1.

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