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Sampled-data synchronization of coupled harmonic oscillators with controller failure and communication delays

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Abstract In this letter, a distributed protocol for sampled-data synchronization of coupled harmonic oscillators with controller failure and communication delays is proposed, and a brief procedure of convergence analysis for such algorithm over undirected connected graphs is provided. Furthermore, a simple yet generic criterion is also presented to guarantee synchronized oscillatory motions in coupled harmonic oscillators. Subsequently, the simulation results are worked out to demonstrate the efficiency and feasibility of the theoretical results. [doi:10.1063/2.1306302]

Keywords sampled-data synchronization, coupled harmonic oscillators, controller failure, communication delays

In the past few years, the synchronization of coupled harmonic oscillators has become a rather significant topic in both theoretical research and practical applications, which originates from two elementary reasons.[1,2] The first reason is that coupled harmonic oscillators is usually viewed as a fundamental model for the study of coordination problems of networked multi-agent systems, which are simply different from general second-order consensus problems due to its intrinsic dynamics properties. The other one is that it has a wide range of engineering applications, especially in multi-agent networks involving repetitive movements including cooperative patrol, exploration, mapping, sampling or surveillance.[3] As a consequence, a large amount of synchronization protocols (or algorithms) have recently been presented for coupled harmonic oscillators from various perspectives.[1–8]

In a recent paper, Ren[1] investigated the synchronization problem of n coupled harmonic oscillators connected by the dampers, where the dynamics of each oscillator is given by

$$\begin{aligned} \dot{r}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= -\alpha r_i(t) + u_i(t), \end{aligned} \tag{1}$$

where $r_i(t)$, $v_i(t) \in R$ $(i = 1, 2, \cdots, n,)$ are the position and velocity of the i-th oscillator at time t respectively, $\alpha > 0$ is the frequency of the oscillator, and the distributed control input is given by

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}(v_i(t) - v_j(t)),\ i = 1, 2, \cdots, n, \tag{2}$$

where a_{ij} characterizes the interaction between oscillators i and j (i.e., $a_{ij} > 0$ if oscillator i can obtain the velocity of oscillator j at time t, and $a_{ij} = 0$ otherwise). Accordingly, specifying $n = 2$ in Eq. (2), then system (1) with protocol (2) can be used to describe a basic model of two objects with unit mass connected by a damper as shown in Fig. 1. Later on, Su et al.[2] considered the same issue in a dynamic proximity network without any connectivity assumption. Ballard et al.[3] further focused on this issue in the framework of discrete-time setting, and a distributed protocol is proposed to implement synchronized motion coordination of multiple mobile robots. In addition, Cheng et al.[5] addressed the finite-time and infinite-time synchronization of networked harmonic oscillators with the external disturbance. Very recently, Zhang et al.[6,8] also taken into account the distributed synchronization problem of coupled harmonic oscillators under local instantaneous or impulsive interactions and sampled-data information with control inputs missing.[7]

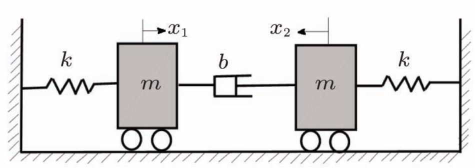

Fig. 1. Two objects of mass m connected by a damper.

With the technological appeal of digital implementations, the hybrid control technique with the form of impulsive sampled-data setting has recently drawn a lot of attention compared with general continuous control schemes. It is often regarded as an effective control strategy dealing with dynamical systems missing control inputs. In real control systems, this phenomenon is

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frequently encountered because of a wide variety of environmental factors such as actuator failures or temporal controller, network-induced packet losses, intermittent unavailability of controllers, external abrupt disturbance, etc.[9] On the other hand, it should take into account the effects of communication delays between agents, since they naturally arise from the consequence of data transmission and/or packet drop because of the limitation of the network resource. They will inevitably influence the control performance of the networked control systems (NCSs), and even cause the control systems instable.[10] However, to our knowledge, up to now just few works concerned with coupled harmonic oscillators' sampled-data synchronization with controller failure and communication delays.

With the aforementioned background, in the present letter, the major interests are the sampled-data synchronization of multi-agent systems (1) with controller failure and communication delays, here the consensus protocol is given by

$$u_i(t)=\begin{cases}-\mu\sum\limits_{j=1}^{n}a_{ij}\left(v_i(t_{k-1})-v_j(t_{k-1}-\tau)\right),\\ \quad t_{k-1}\leqslant t<\omega_{k-1},\\ 0,\quad \omega_{k-1}\leqslant t<t_k,\end{cases}$$
$$k\in\mathbf{N},\quad i=1,2,\cdots,n, \tag{3}$$

where $\mu>0$ is the control parameter assigning partially coupling strength between oscillators, a_{ij} characterizes the interaction between oscillators i and j (i.e., $a_{ij}>0$ if the i-th oscillator can acquire the velocity of oscillator j; $a_{ij}=0$, otherwise), and $\tau>0$ denotes the communication delays at the sampling time moments t_k. The sampling time sequence $\{t_k\}_{k\geqslant 0}$ is a strictly increasing sequence i.e., $0=t_0<t_1<\cdots<t_{k-1}<t_k<\cdots(k\in\mathbf{N})$ with $\lim\limits_{k\to\infty}t_k=+\infty$, and $\{\omega_k\}_{k\geqslant 0}$, an indicator time sequence, denotes the data missing status satisfying $t_{k-1}<\omega_{k-1}\leqslant t_k(k\in\mathbf{N})$. The sketch map of the sampled-data control with input control missing is shown in Fig. 2.

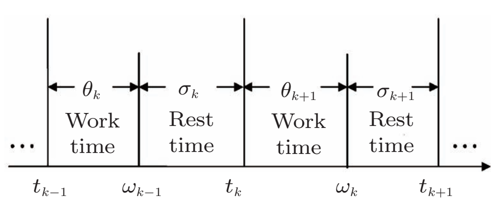

Fig. 2. Sketch map of the sampled-data control.

In this letter, the following definitions and notations will be used. $\mathbf{N}=\{1,2,\ldots\}$ is the natural numbers set, $\mathbf{R}=(-\infty,+\infty)$ denotes the real numbers set, $\mathbf{0}_n\in\mathbf{R}^n$ is the zero vector, $\mathbf{1}_n\in\mathbf{R}^n$ stands for the vector with all ones. For $\boldsymbol{v}\in\mathbf{R}^n$, $\boldsymbol{v}^{\mathrm{T}}$ denotes its transpose. $\mathbf{R}^{n\times n}$ is the set of $n\times n$ real matrices. $\boldsymbol{O}_n\in\mathbf{R}^{n\times n}$ is the zero matrix, and $\boldsymbol{I}_n\in\mathbf{R}^{n\times n}$ is the identity matrix.

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ denote a weighted undirected graph with a set of nodes $\mathcal{V}=\{1,2,\ldots,n\}(n\geqslant 2)$, a set of edges $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$. The corresponding weighted adjacency matrix $\boldsymbol{A}=[a_{ij}]\in\mathbf{R}^{n\times n}$ is defined as $a_{ij}=a_{ji}>0$ if (i,j) and $(j,i)\in\mathcal{E}$, otherwise $a_{ij}=a_{ji}=0$ for all $i\neq j$, and $a_{ii}=0$ for all $i\in\mathcal{V}$. If an undirected path contained all the nodes of the graph exists, the undirected graph is connected. We can define the Laplacian matrix of a undirected graph $\mathcal{G}$ as $\boldsymbol{L}=[l_{ij}]\in\mathbf{R}^{n\times n}$, $l_{ii}=\sum\limits_{j=1}^{n}a_{ij}$ and $l_{ij}=-a_{ij}$, $i\neq j$. It is known that for a connected graph $\mathcal{G}$, $\boldsymbol{L}$ has a zero eigenvalue and all the others are positive: $0=\lambda_{\min}(\boldsymbol{L})\overset{\mathrm{def}}{=}\lambda_1<\lambda_2\leqslant\cdots\leqslant\lambda_n\overset{\mathrm{def}}{=}\lambda_{\max}(\boldsymbol{L})$.

In what follows, convergence analysis is provided for system (1) with the synchronization protocol (3) over a fixed network topology $\mathcal{G}$. For a given partition $\pi=\{t_k\}_{k\geqslant 0}\bigcup\{\omega_k\}_{k\geqslant 0}$ of $[0,+\infty)$, let $\sigma_k=t_k-\omega_{k-1}$ and $\theta_k=\omega_{k-1}-t_{k-1}(k\in\mathbf{N})$ denote the duration of the k-th "rest time" and "work time" respectively, and define $\beta=\inf\{\theta_k,\sigma_k|k\in\mathbf{N}\}$.

Theorem 1 Supposing the undirected graph $\mathcal{G}$ is connected, the following assumptions hold for all $k\in\mathbf{N}$.

(i) $0<\mu\leqslant\dfrac{\sqrt{\alpha}}{4}\min\left\{\dfrac{1}{\lambda_n},\dfrac{1}{a}\right\}$ with $a_{ii}=a$ for all i,

(ii) $0<\tau\leqslant\beta\leqslant\theta_k+\sigma_k\leqslant\dfrac{\pi}{3\sqrt{\alpha}}$.

Then all the states $[r_i(t),v_i(t)]^{\mathrm{T}}$ $(i=1,2,\cdots,n)$ of the system (1) with protocol (3) will converge globally asymptotically to the synchronization state $[\gamma(t),\nu(t)]^{\mathrm{T}}$, i.e., $\lim\limits_{t\to+\infty}|r_i(t)-\gamma(t)|=0$, $\lim\limits_{t\to+\infty}|v_i(t)-\nu(t)|=0$, where

$$\begin{bmatrix}\gamma(t)\\ \nu(t)\end{bmatrix}=\frac{1}{n}\begin{bmatrix}\mathbf{1}_n^{\mathrm{T}}\boldsymbol{r}_0\cos\sqrt{\alpha}t+\mathbf{1}_n^{\mathrm{T}}\boldsymbol{v}_0\dfrac{1}{\sqrt{\alpha}}\sin\sqrt{\alpha}t\\ -\mathbf{1}_n^{\mathrm{T}}\boldsymbol{r}_0\sqrt{\alpha}\sin\sqrt{\alpha}t+\mathbf{1}_n^{\mathrm{T}}\boldsymbol{v}_0\cos\sqrt{\alpha}t\end{bmatrix}, \tag{4}$$

having the initial velocity and position $[\boldsymbol{r}_0^{\mathrm{T}},\boldsymbol{v}_0^{\mathrm{T}}]^{\mathrm{T}}$.

Brief Proof Let $\boldsymbol{r}(t)=[r_1(t),r_2(t),\cdots,r_n(t)]^{\mathrm{T}}\in\mathbf{R}^n$, $\boldsymbol{v}(t)=[v_1(t),v_2(t),\cdots,v_n(t)]^{\mathrm{T}}\in\mathbf{R}^n$. Using the proposed protocol (3), one can write the dynamics of sampled-data coupled harmonic oscillators (1) as

$$\begin{bmatrix}\dot{\boldsymbol{r}}(t)\\ \dot{\boldsymbol{v}}(t)\end{bmatrix}=\begin{cases}\begin{bmatrix}\boldsymbol{O}_n & \boldsymbol{I}_n\\ -\alpha\boldsymbol{I}_n & \boldsymbol{O}_n\end{bmatrix}\begin{bmatrix}\boldsymbol{r}(t)\\ \boldsymbol{v}(t)\end{bmatrix}-\\ \begin{bmatrix}\mathbf{0}_n\\ \mu\boldsymbol{L}\boldsymbol{v}(t_{k-1}-\tau)\end{bmatrix}-\\ \begin{bmatrix}\mathbf{0}_n\\ \mu\boldsymbol{A}(\boldsymbol{v}(t_{k-1})-\boldsymbol{v}(t_{k-1}-\tau))\end{bmatrix},\\ t_{k-1}\leqslant t<\omega_{k-1},\\ \begin{bmatrix}\boldsymbol{O}_n & \boldsymbol{I}_n\\ -\alpha\boldsymbol{I}_n & \boldsymbol{O}_n\end{bmatrix}\begin{bmatrix}\boldsymbol{r}(t)\\ \boldsymbol{v}(t)\end{bmatrix},\\ \omega_{k-1}\leqslant t<t_k.\end{cases} \tag{5}$$

where $\boldsymbol{A} = \mathrm{diag}(a_{11}, a_{22}, \cdots, a_{nn}) \in \mathbf{R}^{n\times n}$. Obviously, in order to realize complete synchronization of the coupled system (5), the assumption $a_{11} = a_{22} = \cdots = a_{nn} = a$ must be imposed.

Based on the theory of general linear ordinary differential equations, the analytical solution of Eq. (5) with $\boldsymbol{L} = \boldsymbol{O}_n$ can be written as

$$\begin{bmatrix} \boldsymbol{r}(t) \\ \boldsymbol{v}(t) \end{bmatrix} = \begin{bmatrix} \gamma(t)\mathbf{1}_n \\ \nu(t)\mathbf{1}_n \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{r}}(t) \\ \tilde{\boldsymbol{v}}(t) \end{bmatrix}, \tag{6}$$

where $[\gamma(t)\mathbf{1}_n^{\mathrm{T}}, \nu(t)\mathbf{1}_n^{\mathrm{T}}]^{\mathrm{T}}$, the synchronization state is a solution of the homogeneous equation of Eq. (5) with the initial value $[\gamma(0)\mathbf{1}_n^{\mathrm{T}}, \nu(0)\mathbf{1}_n^{\mathrm{T}}]^{\mathrm{T}} = (\boldsymbol{I}_2 \otimes \boldsymbol{P})[\boldsymbol{r}_0^{\mathrm{T}}, \boldsymbol{v}_0^{\mathrm{T}}]^{\mathrm{T}}$, with $\boldsymbol{P} = \mathbf{1}_n \cdot \mathbf{1}_n^{\mathrm{T}}/n$, and

$$\begin{bmatrix} \tilde{\boldsymbol{r}}(t) \\ \tilde{\boldsymbol{v}}(t) \end{bmatrix} = \begin{cases} (\boldsymbol{\Gamma}(t, t_{k-1})\boldsymbol{\Phi}(t_{k-1} - \tau, t_{k-1}) - \boldsymbol{\Gamma}(t, t_{k-1}))\cdot \\ \quad \prod\limits_{j=1}^{k-1} \boldsymbol{N}_j \boldsymbol{I} \otimes \boldsymbol{P} \begin{bmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{bmatrix}, \quad t_{k-1} \leqslant t < \omega_{k-1}, \\ (\boldsymbol{\Phi}(t, \omega_{k-1})\boldsymbol{\Gamma}(\omega_{k-1}, t_{k-1})\boldsymbol{\Phi}(t_{k-1} - \tau, t_{k-1}) - \\ \quad \boldsymbol{\Phi}(t, \omega_{k-1})\boldsymbol{\Gamma}(\omega_{k-1}, t_{k-1}))\cdot \\ \quad \prod\limits_{j=1}^{k-1} \boldsymbol{N}_j \boldsymbol{I} \otimes \boldsymbol{P} \begin{bmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{bmatrix}, \quad \omega_{k-1} \leqslant t < t_k, \end{cases} \tag{7}$$

where the Cauchy matrix of the homogeneous equation of Eq. (5) for t, $t' \in [t_{k-1}, t_k)$ is given by

$$\boldsymbol{\Phi}(t, t') = \begin{bmatrix} \cos\sqrt{\alpha}(t - t') & \dfrac{1}{\sqrt{\alpha}}\sin\sqrt{\alpha}(t - t') \\ -\sqrt{\alpha}\sin\sqrt{\alpha}(t - t') & \cos\sqrt{\alpha}(t - t') \end{bmatrix} \otimes \boldsymbol{I}_n,$$

and

$$\boldsymbol{N}_j = \boldsymbol{\Phi}(t_j, \omega_{j-1})\boldsymbol{\Gamma}(\omega_{j-1}, t_{j-1})\boldsymbol{\Phi}(t_{k-1} - \tau, t_{k-1}) - \boldsymbol{\Phi}(t_j, \omega_{j-1})\boldsymbol{\Gamma}(\omega_{j-1}, t_{j-1}),$$

and

$$\boldsymbol{\Gamma}(t, t_{k-1}) = \begin{bmatrix} \boldsymbol{O}_n & \dfrac{\mu(\cos\sqrt{\alpha}(t - t_{k-1}) - 1)}{\alpha}\boldsymbol{A} \\ \boldsymbol{O}_n & -\dfrac{\mu\sin\sqrt{\alpha}(t - t_{k-1})}{\sqrt{\alpha}}\boldsymbol{A} \end{bmatrix}.$$

It is easy to obtain that the eigenvalues of $\boldsymbol{N}_j$ are given by $\lambda_1^{(i)}(\boldsymbol{N}_j) = 0$ and

$$\lambda_2^{(i)}(\boldsymbol{N}_j) = -\frac{4\mu a}{\sqrt{\alpha}}\sin\sqrt{\alpha}\left(\sigma_j + \frac{\theta_j}{2} - \frac{\tau}{2}\right)\cdot \sin\sqrt{\alpha}\frac{\theta_j}{2}\sin\sqrt{\alpha}\tau, \quad i = 1, 2, \cdots, n. \tag{8}$$

Through introducing the synchronization error

$$\begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{s}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}(t) \\ \boldsymbol{v}(t) \end{bmatrix} - \begin{bmatrix} \gamma(t)\mathbf{1}_n \\ \nu(t)\mathbf{1}_n \end{bmatrix} - \begin{bmatrix} \tilde{\boldsymbol{r}}(t) \\ \tilde{\boldsymbol{v}}(t) \end{bmatrix}, \tag{9}$$

we can get the error dynamical systems as

$$\begin{bmatrix} \dot{\boldsymbol{e}}(t) \\ \dot{\boldsymbol{s}}(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} \boldsymbol{O}_n & \boldsymbol{I}_n \\ -\alpha\boldsymbol{I}_n & \boldsymbol{O}_n \end{bmatrix}\begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{s}(t) \end{bmatrix} - \\ \quad \begin{bmatrix} \mathbf{0}_n \\ \mu\boldsymbol{L}\boldsymbol{s}(t_{k-1} - \tau) \end{bmatrix} - \\ \quad \begin{bmatrix} \mathbf{0}_n \\ \mu\boldsymbol{A}(\boldsymbol{s}(t_{k-1}) - \boldsymbol{s}(t_{k-1} - \tau)) \end{bmatrix}, \\ \quad t_{k-1} \leqslant t < \omega_{k-1}, \\ \begin{bmatrix} \boldsymbol{O}_n & \boldsymbol{I}_n \\ -\alpha\boldsymbol{I}_n & \boldsymbol{O}_n \end{bmatrix}\begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{s}(t) \end{bmatrix}, \\ \quad \omega_{k-1} \leqslant t < t_k, \end{cases} \tag{10}$$

with the initial condition below

$$\begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{s}(t) \end{bmatrix} = [\boldsymbol{I}_2 \otimes (\boldsymbol{I}_n - \boldsymbol{P})]\begin{bmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{bmatrix}, -\tau \leqslant t \leqslant t_0.$$

Note that $\boldsymbol{\Phi}(t, t')$ and $\boldsymbol{I}_2 \otimes (\boldsymbol{I}_n - \boldsymbol{P})$ commute, $\boldsymbol{PL} = \boldsymbol{LP} = \boldsymbol{O}_n$ and $\boldsymbol{P}^2 = \boldsymbol{P}$, then for $k \geqslant 1$, the analytical solution of Eq. (10) can be described as

$$\begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{s}(t) \end{bmatrix} = \begin{cases} \big(\boldsymbol{\Phi}(t, t_{k-1}) + \boldsymbol{\Lambda}(t, t_{k-1})\boldsymbol{\Phi}(t_{k-1} - \tau, \omega_{k-1}) + \\ \quad \boldsymbol{\Gamma}(t, t_{k-1})\boldsymbol{\Phi}(t_{k-1} - \tau, t_{k-1}) - \boldsymbol{\Gamma}(t, t_{k-1})\big)\cdot \\ \quad \prod\limits_{j=1}^{k-1} \boldsymbol{H}_j \begin{bmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{bmatrix}, \quad t_{k-1} \leqslant t < \omega_{k-1}, \\ \big(\boldsymbol{\Phi}(t, t_{k-1}) + \boldsymbol{\Phi}(t, \omega_{k-1})\boldsymbol{\Lambda}(\omega_{k-1}, t_{k-1})\cdot \\ \quad \boldsymbol{\Phi}(t_{k-1} - \tau, t_{k-1}) + \boldsymbol{\Phi}(t, \omega_{k-1})\cdot \\ \quad \boldsymbol{\Gamma}(\omega_{k-1}, t_{k-1})\boldsymbol{\Phi}(t_{k-1} - \tau, t_{k-1}) - \\ \quad \boldsymbol{\Phi}(t, \omega_{k-1})\boldsymbol{\Gamma}(\omega_{k-1}, t_{k-1})\big)\cdot \\ \quad \prod\limits_{j=1}^{k-1} \boldsymbol{H}_j \begin{bmatrix} \boldsymbol{r}_0 \\ \boldsymbol{v}_0 \end{bmatrix}, \quad \omega_{k-1} \leqslant t < t_k, \end{cases} \tag{11}$$

where $\boldsymbol{H}_j = \boldsymbol{M}_j + \boldsymbol{N}_j(\boldsymbol{I}_2 \otimes \boldsymbol{P})$ with

$$\boldsymbol{M}_j = \boldsymbol{\Phi}(t_j, t_{j-1})[\boldsymbol{I}_2 \otimes (\boldsymbol{I}_n - \boldsymbol{P})] + \boldsymbol{\Phi}(t_j, \omega_{j-1})\cdot \boldsymbol{\Lambda}(\omega_{j-1}, t_{j-1})\boldsymbol{\Phi}(t_{j-1} - \tau, t_{j-1}).$$

and

$$\boldsymbol{\Lambda}(t, t_{k-1}) = \begin{bmatrix} \boldsymbol{O}_n & \dfrac{\mu(\cos\sqrt{\alpha}(t - t_{k-1}) - 1)}{\alpha}\boldsymbol{L} \\ \boldsymbol{O}_n & -\dfrac{\mu\sin\sqrt{\alpha}(t - t_{k-1})}{\sqrt{\alpha}}\boldsymbol{L} \end{bmatrix}.$$

Let $\boldsymbol{L} = \boldsymbol{UJU}^{\mathrm{T}}$ be the eigenvector decomposition of $\boldsymbol{L}$, in which $\boldsymbol{U} \in \mathbf{R}^{n\times n}$ is a normal orthogonal matrix, and $\boldsymbol{J} = \mathrm{diag}(0, \lambda_2, \cdots, \lambda_n)$. Note that the matrices $\boldsymbol{H}_j$ and $(\boldsymbol{I}_2 \otimes \boldsymbol{U}^{\mathrm{T}})\boldsymbol{H}_j(\boldsymbol{I}_2 \otimes \boldsymbol{U})$ have the same eigenvalues. Note that the matrices $\boldsymbol{H}_j$ and $(\boldsymbol{I}_2 \otimes \boldsymbol{U}^{\mathrm{T}})\boldsymbol{H}_j(\boldsymbol{I}_2 \otimes \boldsymbol{U})$ have the same eigenvalues. Since $(\boldsymbol{I}_n - \boldsymbol{P})\boldsymbol{L} = \boldsymbol{L}(\boldsymbol{I}_n - \boldsymbol{P}) = \boldsymbol{L}$, it follows that $\boldsymbol{U}^{\mathrm{T}}(\boldsymbol{I}_n - \boldsymbol{P})\boldsymbol{U} = \mathrm{diag}(0, 1, \cdots, 1)$.

Therefore, by a finite sequence of same elementary operations with respect to the interchange of rows and columns of $(\boldsymbol{I}_2 \otimes \boldsymbol{U}^{\mathrm{T}})\boldsymbol{H}_j(\boldsymbol{I}_2 \otimes \boldsymbol{U})$, respectively, the eigenvalues of $\boldsymbol{H}_j$ are given by $\lambda_{\pm}^{(1)}(\boldsymbol{M}_j) = 0$ and

$$\lambda_{\pm}^{(i)}(\boldsymbol{H}_j) = \mathcal{K} \pm \sqrt{\mathcal{K}^2 - \mathcal{L}} \tag{12}$$

where

$$\mathcal{K} = \cos\sqrt{\alpha}(\theta_j + \sigma_j) + \frac{\mu a}{\sqrt{\alpha}} \cdot \sin\sqrt{\alpha}\frac{\theta_j}{2}\cos\sqrt{\alpha}\left(\sigma_j + \frac{\theta_j}{2}\right) - \frac{\mu(\lambda_i + a)}{\sqrt{\alpha}} \cdot \sin\sqrt{\alpha}\frac{\theta_j}{2}\cos\sqrt{\alpha}\left(\frac{\theta_j}{2} + \sigma_j - \tau\right),$$

and

$$\mathcal{L} = 1 + \frac{\mu a}{\sqrt{\alpha}}\sin\sqrt{\alpha}\theta_j - \frac{2\mu(\lambda_i + a)}{\sqrt{\alpha}} \cdot \sin\sqrt{\alpha}\frac{\theta_j}{2}\cos\sqrt{\alpha}\left(\frac{\theta_j}{2} + \tau\right).$$

By using the same arguments as in the proof of Theorem 1 in Ref. 6, and by the assumptions (i) and (ii), it is easy to confirm that for all $j \in \mathbf{N}$,

$$\rho(\boldsymbol{N}_j) \leqslant \max\left\{ |\lambda_2^{(i)}(\boldsymbol{N}_j)| \,\middle|\, \tau \leqslant \beta \leqslant \theta_j + \sigma_j \leqslant \frac{\pi}{3\sqrt{\alpha}} \right\} < 1,$$

$$\rho(\boldsymbol{H}_j) \leqslant \max\left\{ |\lambda_{\pm}^{(i)}(\boldsymbol{H}_j)| \,\middle|\, \tau \leqslant \beta \leqslant \theta_j + \sigma_j \leqslant \frac{\pi}{3\sqrt{\alpha}} \right\} < 1.$$

Consequently, it follows immediately that all the states $[r_i(t), v_i(t)]^{\mathrm{T}}$ $(i = 1, 2, \cdots, n)$ of the system (1) with protocol (3) will converge globally asymptotically to the synchronization state $[\gamma(t), \nu(t)]^{\mathrm{T}}$. This completes the proof of Theorem 1.

Theorem 1 gives a simple yet generic criterion for sampled-data synchronization of coupled harmonic oscillators with controller failure and communication delays. Theorem 1 shows that coupled harmonic oscillators can obtain synchronized oscillatory motions under network connectivity provided that the communication delays less than either the work or rest time intervals. It should be noted that Theorem 1 can be generalized to the case of switching topology, leader-following synchronization, which we will further investigate in the future work.

Some simulations will be worked out to show the validity of the proposed protocol algorithms. For network communication topology $\mathcal{G}$ with a ring coupling, we assume that $a_{ij} = a_{ji} = 1$ if oscillators i and j can communicate with each other, and $a_{ij} = a_{ji} = 0$, otherwise. Consider a team of $n = 4$ harmonic oscillators with protocol (3) where each oscillator is coupled with only two other agents.

By some computations, the biggest eigenvalue of Laplacian $\boldsymbol{L}$ is $\lambda_4(\boldsymbol{L}) = 4$. The work time and rest time are both selected as $\theta_k = \sigma_k = 0.2$, the other parameters are chosen as $\alpha = 4, \mu = 0.25, \tau = 0.05$. It is easy to verify that the conditions in Theorem 1 are satisfied. Figure 3 shows the evolution of the oscillator states with selected initial conditions as $\boldsymbol{r}_0 = [1, 2, 1.5, 2.5]$, $\boldsymbol{v}_0 = [2, 4, 6, 8]$. From Fig. 3, we can easily see that the 4 coupled harmonic oscillators synchronize their motion as time evolves.

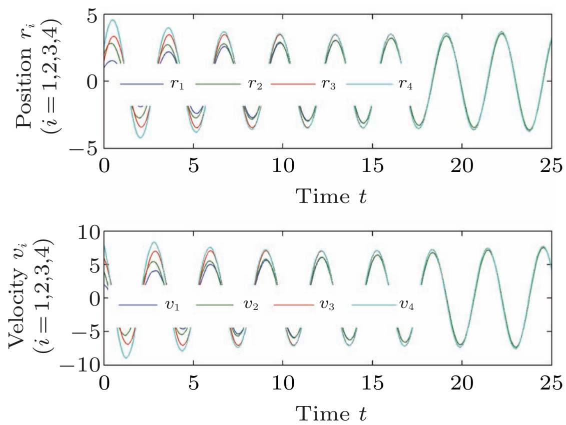

Fig. 3. Evolution of 4 coupled harmonic oscillators.

In this letter, we proposed a distributed synchronization protocol for sampled-data coupled harmonic oscillators with controller failure and communication delays. We provided a brief procedure of convergence analysis for such algorithm over undirected connected graphs. We also presented a simple yet generic criterion by which all the coupled harmonic oscillators can achieve synchronized oscillatory motions. Simulation examples have been provided to verify the effectiveness and feasibility of the theoretical results.

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