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An adaptive discontinuous finite volume method for elliptic problems

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ABSTRACT

An adaptive discontinuous finite volume method is developed and analyzed in this paper. We prove that the adaptive procedure achieves guaranteed error reduction in a mesh-dependent energy norm and has a linear convergence rate. Numerical results are also presented to illustrate the theoretical analysis.

1. Introduction

Adaptive procedures combined with finite element methods or finite volume methods have become important tools for scientific computing and engineering applications; see [1–4] and the references therein. These adaptive procedures usually rely on *a posteriori* type error estimates of residuals [3–6] or quantities of interest [7]. Convergence of adaptive finite element methods for elliptic problems has been investigated for continuous finite elements in [3] and for discontinuous finite elements in [2,8]. For adaptive finite volume methods, the results in [9] by Lazarov and Tomov represent noticeable early work on diffusion and convection–diffusion–reaction equations in three dimensions, in which continuous trial functions are used. A recent work on convergence of an adaptive continuous finite volume method for elliptic problems can be found in [10].

The discontinuous finite volume method developed in [11] for second order elliptic boundary value problems incorporates the ideas of the discontinuous Galerkin finite element methods and the control or dual volumes. The discontinuous finite volume method can be applied to elliptic interface problems and Darcy's flows [12]. It has been observed that the discontinuous finite volume method has easier implementation than the traditional node-oriented or cell-oriented (continuous) finite volume methods and offers local conservation on sub-triangles [12].

As a continuation of our work on *a posterior* error estimation for the discontinuous finite volume method, this paper establishes convergence of an adaptive procedure for the discontinuous finite volume method for second order elliptic problems. The residual type *a posteriori* estimator in [6] will be used as an indicator for adaptive mesh refinements. Our analysis of the adaptive discontinuous finite volume method in this paper is similar to those for adaptive discontinuous finite element methods in [2,8].

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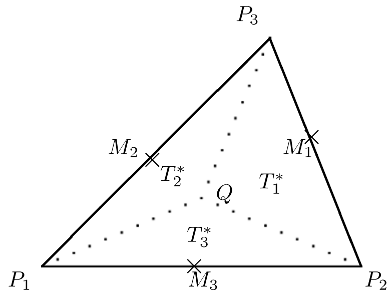

Fig. 2.1. A triangular element along with its dual volumes.

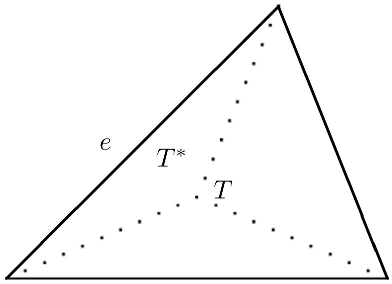

Fig. 2.2. A triangular element with edge e.

For ease of presentation, we consider the following model homogeneous Dirichlet boundary value problem

$$\mathcal{L}u := -\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain. However, our adaptive discontinuous finite volume method and convergence analysis apply to more general boundary value problems, as shown in the numerical results in Section 5.

We will use the standard definitions [13,11] for the Sobolev spaces $H^s(D)$ and their associated inner products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and seminorms $|\cdot|_{s,D}$, $s \geq 0$. The space $H^0(D)$ coincides with $L^2(D)$, in which case the norm and inner product are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. When $D = \Omega$, we drop the subscript D.

The rest of this paper is organized as follows. In Section 2, a discontinuous finite volume method is introduced. In Section 3, an *a posteriori* error estimator is presented. Convergence of the adaptive procedure is derived in Section 4. Numerical results are presented in Section 5 to illustrate the error analysis. The paper is concluded with some remarks in Section 6.

2. A discontinuous finite volume method

Let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω. Each triangular element $T \in \mathcal{T}_h$ is divided into three sub-triangles by connecting the barycenter to the three vertices of the triangle, as shown in Fig. 2.1. All these sub-triangles form a dual partition of $\mathcal{T}_h$, which is denoted as $\mathcal{T}_h^*$.

We define a finite dimensional space of piecewise linear trial functions on $\mathcal{T}_h$ as

$$V_h = \{v \in L^2(\Omega) : v|_T \in P_1(T),\ \forall T \in \mathcal{T}_h\},$$

and a finite dimensional space

$$W_h = \{q \in L^2(\Omega) : q|_{T^*} \in P_0(T^*),\ \forall T^* \in \mathcal{T}_h^*\}$$

for piecewise constant test functions on the dual partition $\mathcal{T}_h^*$.

Let $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega)$. Define a mapping $\gamma : V(h) \to W_h$ as

$$\gamma v|_{T^*} = \frac{1}{h_e}\int_e v|_{T^*} \mathrm{d}s, \quad T^* \in \mathcal{T}_h^*.$$

See Fig. 2.2.

Let e be an interior edge common to elements T_1 and T_2 in $\mathcal{T}_h$, and $\mathbf{n}_1$ and $\mathbf{n}_2$ be the unit normal vectors on e exterior to T_1 and T_2, respectively. For a scalar q or a vector $\mathbf{w}$, we define respectively its average $\{\cdot\}$ on e and jump $[\cdot]$ across e as

$$\{q\} = \frac{1}{2}(q|_{\partial T_1} + q|_{\partial T_2}), \qquad [q] = q|_{\partial T_1}\mathbf{n}_1 + q|_{\partial T_2}\mathbf{n}_2,$$

$$\{\mathbf{w}\} = \frac{1}{2}(\mathbf{w}|_{\partial T_1} + \mathbf{w}|_{\partial T_2}), \qquad [\mathbf{w}] = \mathbf{w}|_{\partial T_1}\cdot\mathbf{n}_1 + \mathbf{w}|_{\partial T_2}\cdot\mathbf{n}_2.$$

Note that the jump of a vector is a scalar, whereas the jump of a scalar is a vector. If e is an edge on the boundary of Ω, we define

$$\{q\} = q, \qquad [\mathbf{w}] = \mathbf{w} \cdot \mathbf{n}.$$

The quantities $[q]$ and $\{\mathbf{w}\}$ on boundary edges are defined analogously.

For $D \subset \Omega$, let $\mathcal{E}_h(D)$ be the set of edge in D and $\mathcal{E}_h = \mathcal{E}_h(\Omega)$. Denote $\mathcal{E}_h^0 := \mathcal{E}_h \setminus \partial\Omega$, the collection of all interior edges of $\mathcal{T}_h$. For convenience, we also define

$$(v, w)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \int_T v \cdot w \, \mathrm{d}x, \qquad (v, w)_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \int_e v \cdot w \, \mathrm{d}s.$$

The piecewise gradient operator ∇_h on $\mathcal{T}_h$ is defined as

$$(\nabla_h v)|_T = \nabla(v|_T), \quad \forall T \in \mathcal{T}_h.$$

Testing Eq. (1.1) by γv for $v \in V_h$ gives

$$(\mathcal{L}u, \gamma v)_{\mathcal{T}_h^*} = (f, \gamma v).$$

Integrating by parts and using the fact that γv is a constant on each $T^* \in \mathcal{T}_h^*$, we obtain

$$\begin{aligned}(\mathcal{L}u, \gamma v)_{\mathcal{T}_h^*} &= -\sum_{T^* \in \mathcal{T}_h^*} \int_{T^*} \Delta u \gamma v \, \mathrm{d}x = -\sum_{T^* \in \mathcal{T}_h^*} \int_{\partial T^*} \nabla u \cdot \mathbf{n} \gamma v \, \mathrm{d}s \\ &= \left(-\sum_{T^* \in \mathcal{T}_h^*} \int_{\partial T^*} \nabla u \cdot \mathbf{n} \gamma v \, \mathrm{d}s + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \nabla u \cdot \mathbf{n} \gamma v \, \mathrm{d}s \right) - \sum_{T \in \mathcal{T}_h} \int_{\partial T} \nabla u \cdot \mathbf{n} \gamma v \, \mathrm{d}s,\end{aligned}$$

where we have added and subtracted the last term to bring in the effect of the primal triangulation $\mathcal{T}_h$.

Next we define a bilinear form on $V(h) \times V(h)$ as

$$a(u, v) = -\sum_{T^* \in \mathcal{T}_h^*} \int_{\partial T^*} \nabla u \cdot \mathbf{n} \gamma v \, \mathrm{d}s + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \nabla u \cdot \mathbf{n} \gamma v \, \mathrm{d}s.$$

Utilizing the facts that $[\nabla u] = 0$ and

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \nabla u \cdot \mathbf{n} \gamma v \, \mathrm{d}s = \sum_{e \in \mathcal{E}_h} \int_e [\gamma v] \cdot \{\nabla u\} \mathrm{d}s + \sum_{e \in \mathcal{E}_h^0} \int_e \{\gamma v\} [\nabla u \cdot \mathbf{n}] \mathrm{d}s, \tag{2.1}$$

we have

$$(\mathcal{L}u, \gamma v)_{\mathcal{T}_h^*} = a(u, v) - (\{\nabla u\}, [\gamma v])_{\mathcal{E}_h} = (f, \gamma v). \tag{2.2}$$

Since $[u] = \mathbf{0}$, we can add a penalty term to the above equation and still maintain consistency of the method:

$$a(u, v) - (\{\nabla u\}, [\gamma v])_{\mathcal{E}_h} + \alpha (h_e^{-1}[u], [v])_{\mathcal{E}_h} = (f, \gamma v).$$

Then we define

$$A_h(u, v) = a(u, v) - (\{\nabla u\}, [\gamma v])_{\mathcal{E}_h} + \alpha (h_e^{-1}[u], [v])_{\mathcal{E}_h}. \tag{2.3}$$

Now our discontinuous finite volume method can be formulated as Seek $u_h \in V_h$ such that

$$A_h(u_h, v) = (f, \gamma v) \quad \forall v \in V_h. \tag{2.4}$$

The formulation (2.4) is consistent, i.e., the true solution u satisfies

$$A_h(u, v) = (f, \gamma v) \quad \forall v \in V_h. \tag{2.5}$$

Subtracting (2.4) from (2.5), we obtain the Galerkin orthogonality

$$A_h(u - u_h, v) = 0, \quad \forall v \in V_h. \tag{2.6}$$

For $v, w \in V(h)$, it has been proved in [11] that

$$a(v, w) = (\nabla_h v, \nabla_h w) + \sum_{T \in \mathcal{T}_h} \int_{\partial T} \nabla v \cdot \mathbf{n}(\gamma w - w) \mathrm{d}s + \sum_{T \in \mathcal{T}_h} (\Delta v, w - \gamma w)_T. \tag{2.7}$$

Furthermore, we define a mesh-dependent norm

$$|||v|||^2 = |v|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} h_e \|\{\nabla v\}\|_e^2 + \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[v]\|_e^2.$$

The following *a priori* error estimate has been established in [11].

Theorem 2.1. *Let u and u_h be respectively the solutions of (1.1) and (2.4). Then*

$$|||u - u_h||| \le C \inf_{v \in V_h} |||u - v|||,$$

where C is a constant independent of the mesh size h. □

3. An *a posteriori* error estimator

First, we make an assumption as in [8] that f is a piecewise constant, since the data oscillation is essentially a higher order term. Techniques for handling data oscillations can also be found in [2].

It is clear that

$$(f, v - \gamma v)_{\mathcal{T}_h} = 0. \tag{3.1}$$

We define

$$\eta_h^2 = \sum_{T\in\mathcal{T}_h} \eta_T^2 + \sum_{e\in\mathcal{E}_h} \eta_e^2 + \sum_{e\in\mathcal{E}_h} \eta_{e,1}^2, \tag{3.2}$$

where

$$\eta_T = h_T \|f\|_T, \quad \forall T \in \mathcal{T}_h,$$

and

$$\eta_e = h_e^{1/2} \|[\nabla u_h]\|_e, \qquad \eta_{e,1} = h_e^{-1/2} \|[u_h]\|_e, \quad \forall e \in \mathcal{E}_h.$$

Let T be an element with edge e. It is well known [14] that there exists a constant C such that for any function $g \in H^1(T)$,

$$\|g\|_e^2 \le C(h_T^{-1}\|g\|_T^2 + h_T\|\nabla g\|_T^2). \tag{3.3}$$

We denote $\mathcal{T}_e = T_1 \cup T_2$ with T_1, T_2 in $\mathcal{T}_h$ and $T_1 \cap T_2 = e$. The following two theorems established in [6] provide an upper bound and a lower bound for the error.

Theorem 3.1. *Let u and u_h be respectively the solutions of (1.1) and (2.4). Then there exists a positive constant C such that*

$$\|\nabla_h(u - u_h)\|^2 \le C\eta_h^2. \quad \square \tag{3.4}$$

Theorem 3.2. *There exists a constant $C > 0$ such that*

$$h_T^2 \|f\|_T^2 \le C\|\nabla(u - u_h)\|_T^2, \quad \forall T \in \mathcal{T}_h, \tag{3.5}$$

and

$$h_e\|[\nabla u_h]\|_e^2 \le C(h_T^2\|f\|_{\mathcal{T}_e}^2 + \|\nabla_h(u - u_h)\|_{\mathcal{T}_e}^2), \quad \forall e \in \mathcal{E}_h. \quad \square \tag{3.6}$$

Now we cite a result in [8] about approximating a discontinuous piecewise polynomial in V_h by a continuous piecewise polynomial.

Lemma 3.3. *Let $\mathcal{T}_h$ be a conforming triangular mesh. Then for any $v \in V_h$, there exists $v_I \in V_h \cap H_0^1(\Omega)$ satisfying*

$$\sum_{T\in\mathcal{T}_h} \|\nabla(v - v_I)\|_T^2 \le C \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[v]\|_e^2, \tag{3.7}$$

$$\sum_{T\in\mathcal{T}_h} \|v - v_I\|_T^2 \le C \sum_{e\in\mathcal{E}_h} h_e\|[v]\|_e^2, \tag{3.8}$$

where C is independent of the mesh size h. $\square$

Since $[\nabla u] = 0$ and $[\nabla u_h]$ is a constant, the definition of $A_h(\cdot,\cdot)$ along with (2.7) and (3.1) imply that for any $v \in V(h)$,

$$\begin{aligned} A_h(e_h, v) &= (\nabla e_h, \nabla v)_{\mathcal{T}_h} + ([\gamma v - v], \{\nabla e_h\})_{\mathcal{E}_h} + (\{\gamma v - v\}, [\nabla e_h])_{\mathcal{E}_h} \\ &\quad - (f, v - \gamma v) - ([\gamma v], \{\nabla e_h\})_{\mathcal{E}_h} - \alpha(h^{-1}[u_h], [v])_{\mathcal{E}_h} \\ &= (\nabla e_h, \nabla v)_{\mathcal{T}_h} - ([v], \{\nabla e_h\})_{\mathcal{E}_h} - \alpha(h^{-1}[u_h], [v])_{\mathcal{E}_h}. \end{aligned} \tag{3.9}$$

For $v \in V_h$, we have

$$(\nabla e_h, \nabla v)_{\mathcal{T}_h} - ([v], \{\nabla e_h\})_{\mathcal{E}_h} - \alpha(h^{-1}[u_h], [v])_{\mathcal{E}_h} = 0. \tag{3.10}$$

Lemma 3.4. *Let u_h be the solution of (2.4). Then we have*

$$\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2 \le \frac{C}{\alpha^2} \sum_{T\in\mathcal{T}_h} \|\nabla e_h\|_T^2. \tag{3.11}$$

Proof. Substituting v by $u_h - u_I$ in (3.10) gives

$$\alpha \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2 = (\nabla e_h, \nabla(u_h - u_I))_{\mathcal{T}_h} - (\{\nabla e_h\}, [u_h - u_I])_{\mathcal{E}_h}.$$

Applying Lemma 3.3 and integration by parts, we rewrite the above equation as

$$\begin{aligned}\alpha \sum_{e\in\mathcal{E}_h} h^{-1}\|[u_h]\|_e^2 &= -(\Delta e_h, u_h - u_I)_{\mathcal{T}_h} + (\{\nabla e_h\}, [u_h - u_I])_{\mathcal{E}_h} + ([\nabla e_h], \{u_h - u_I\})_{\mathcal{E}_h} - (\{\nabla e_h\}, [u_h - u_I])_{\mathcal{E}_h}\\ &= -(\Delta e_h, u_h - u_I)_{\mathcal{T}_h} - ([\nabla u_h], \{u_h - u_I\})_{\mathcal{E}_h}\\ &\le \left(\frac{C}{\alpha}\left(\sum_{T\in\mathcal{T}_h}\eta_T^2 + \sum_{e\in\mathcal{E}_h}\eta_e^2\right)\right)^{\frac{1}{2}}\left(\alpha\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2\right)^{1/2}.\end{aligned}$$

Combining the above inequality with Theorem 3.2 gives (3.11). □

Lemma 3.5. *There exists a constant C independent of h such that*

$$A_h(e_h, e_h) \ge C\|\nabla_h e_h\|^2, \tag{3.12}$$

$$A_h(e_h, e_h) \le C(\|\nabla_h e_h\|^2 + \alpha\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2). \tag{3.13}$$

Proof. It follows from (3.10) that

$$\begin{aligned}([u_h], \{\nabla e_h\})_{\mathcal{E}_h} &= (\nabla e_h, \nabla(u_h - u_I))_{\mathcal{T}_h} - \alpha(h^{-1}[u_h], [u_h])_{\mathcal{E}_h}\\ &\le \frac{1}{2}\|\nabla_h e_h\|^2 + (\alpha + C)\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2.\end{aligned}$$

Using the above inequality and (3.11), we obtain

$$\begin{aligned}A_h(e_h, e_h) &= \|\nabla_h e_h\|^2 + \alpha\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2 - ([u_h], \{\nabla e_h\})_{\mathcal{E}_h}\\ &\ge \frac{1}{2}\|\nabla_h e_h\|^2 - C\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2\\ &\ge \frac{1}{2}\|\nabla_h e_h\|^2 - \frac{C}{\alpha^2}\|\nabla_h e_h\|^2\\ &\ge C\|\nabla_h e_h\|^2.\end{aligned}$$

Similarly, we can prove (3.13). □

4. Convergence of the adaptive procedure

We adopt the marking strategy in [2]: for a given parameter $\theta \in (0, 1)$, we mark subsets $\mathcal{M}_T \subset \mathcal{T}_H$ and $\mathcal{M}_E \subset \mathcal{E}_H$ such that

$$\sum_{T\in\mathcal{M}_T}\eta_T^2 \ge \theta\sum_{T\in\mathcal{T}_H}\eta_T^2, \tag{4.1}$$

$$\sum_{E\in\mathcal{M}_E}\eta_E^2 \ge \theta\sum_{E\in\mathcal{E}_H}\eta_E^2. \tag{4.2}$$

The refinement strategy in [2] that does not require the interior node property is used to refine $\mathcal{M}_T$ and $\mathcal{M}_E$. Any $T \in \mathcal{M}_T$ will be refined by bisecting the longest edge, whereas the two triangles sharing any $E \in \mathcal{M}_E$ will be refined by bisection. Let $\mathcal{T}_h$ be a refined mesh obtained in such a way from $\mathcal{T}_H$.

As follows, Lemmas 4.1 and 4.2 respectively state how the errors related to the to-be-refined elements and edges can be controlled.

Lemma 4.1. *The following holds*

$$\sum_{T\in\mathcal{M}_T}\eta_T^2 \le C\|\nabla_h(u_h - u_H)\|^2 + C\alpha\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2. \tag{4.3}$$

Proof. Let $T \in \mathcal{T}_H$ be a triangle refined as $T = T_1 \cup T_2, T_1, T_2 \in \mathcal{T}_h$. As in [2], let $\phi_h \in V_h$ be a Crouzeix–Raviart type linear shape function such that $\phi_h|_{T'} = 0$ for $T' \in \mathcal{T}_h \setminus \{T\}$ and

$$\sum_{i=1}^{2} h_T^2 \|f\|_{T_i}^2 = \sum_{i=1}^{2} (f, \phi_h)_{T_i}, \tag{4.4}$$

$$\|\phi_h\|_{T_i}^2 \le C h_T^4 \|f\|_{T_i}^2, \quad i = 1, 2, \tag{4.5}$$

$$\int_E \{\phi_h\} \mathrm{d}s = 0, \quad E \in \partial T. \tag{4.6}$$

Applying the fact $\phi_h \in V_h$ and (4.4)–(4.6), we obtain

$$\begin{aligned} A_h(u_h, \phi_h) &= \sum_{i=1}^{2} (\nabla u_h, \nabla \phi_h)_{T_i} + \alpha \sum_{e \in \mathcal{E}_h(T)} h_e^{-1} ([u_h], [\phi_h])_e \\ &= \sum_{i=1}^{2} (f, \phi_h)_{T_i}. \end{aligned} \tag{4.7}$$

Using (4.4), (4.6), (4.7), and integration by parts, we have

$$\begin{aligned} \sum_{i=1}^{2} h^2 \|f\|_{T_i}^2 &= \sum_{i=1}^{2} (f, \phi_h)_{T_i} + \sum_{i=1}^{2} (\Delta u_H, \phi_h)_{T_i} \\ &= \sum_{i=1}^{2} (f, \phi_h)_{T_i} - \sum_{i=1}^{2} (\nabla u_H, \nabla \phi_h)_{T_i} \\ &= \sum_{i=1}^{2} (\nabla (u_h - u_H), \nabla \phi_h)_{T_i} + \alpha \sum_{e \in \mathcal{E}_h(T)} h_e^{-1} ([u_h], [\phi_h])_e \\ &\le C \left(\left(\sum_{i=1}^{2} \|\nabla (u_h - u_H)\|_{T_i}^2 \right)^{\frac{1}{2}} + \left(\alpha \sum_{e \in \mathcal{E}_h(T)} h_e^{-1} \|[u_h]\|_e^2 \right)^{1/2} \right) \left(\sum_{i=1}^{2} h_T^2 \|f\|_{T_i}^2 \right)^{1/2}. \end{aligned}$$

Summing the above inequality over all $T \in \mathcal{M}_T$ gives (4.3). □

Lemma 4.2. *The following holds with C being a constant independent of the meshes*

$$\sum_{E \in \mathcal{M}_E} \eta_E^2 \le C \|\nabla_h (u_h - u_H)\|^2 + C\alpha \sum_{E \in \mathcal{E}_H} h_E^{-1} \|[u_H]\|_E^2 + C\alpha \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[u_h]\|_e^2. \tag{4.8}$$

Proof. Let $E \in \mathcal{M}_E$ be an edge with $E = T_1 \cap T_2, T_1, T_2 \in \mathcal{T}_H$. Let $\phi_H \in V_H$ be a Crouzeix–Raviart type shape function such that $\phi_H|_T = 0$ for any $T \in \mathcal{T}_h \setminus \{\mathcal{T}_E\}$ and

$$h_E \|[\nabla u_H]\|_E^2 = ([\nabla u_H], \{\phi_H\})_E, \tag{4.9}$$

$$\|\phi_H\|_{T_i}^2 \le C h_E^{\frac{3}{2}} \|[\nabla u_H]\|_E, \quad i = 1, 2, \tag{4.10}$$

$$\int_E \{\phi_H\} \mathrm{d}s = 0. \tag{4.11}$$

Integration by parts yields

$$0 = -(\Delta u_H, \phi_H)_{\mathcal{T}_E} = (\nabla u_H, \nabla \phi_H)_{\mathcal{T}_E} - ([\nabla u_H], \{\phi_H\})_E. \tag{4.12}$$

Since $\phi_H \in V_H$, we have

$$(\nabla u_H, \nabla \phi_H)_{\mathcal{T}_E} + \alpha \sum_{E \in \mathcal{E}_H(\mathcal{T}_E)} h_E^{-1} ([u_H], [\phi_H])_E = (f, \phi_H)_{\mathcal{T}_E}. \tag{4.13}$$

It follows from (4.9), (4.12) and (4.13) that

$$\begin{aligned} h_E \|[\nabla u_H]\|_E^2 &= ([\nabla u_H], \{\phi_H\})_E = (\nabla u_H, \nabla \phi_H)_{\mathcal{T}_E} \\ &= (f, \phi_H)_{\mathcal{T}_E} - \alpha \sum_{E \in \mathcal{E}_H(\mathcal{T}_E)} h_E^{-1} ([u_H], [\phi_H])_E. \end{aligned}$$

Applying (4.3), (4.10) and (4.11), we obtain

$$h_E\|[\nabla u_H]\|_E^2 \le C\left(\sum_{T\in\mathcal{T}_E} h_T^2\|f\|_T^2 + \sum_{E\in\mathcal{E}_H(\mathcal{T}_E)} h_E^{-1}\|[u_H]\|_E^2\right).$$

Summing the above inequality over all $E \in \mathcal{M}_E$ and applying Lemma 4.1 give (4.8) as desired. □

Since $\mathcal{T}_h$ is a refinement obtained from $\mathcal{T}_H$, it is easy to see [8] that

$$A_h(e_H, e_H) \le A_H(e_H, e_H) + \alpha \sum_{E\in\mathcal{E}_H} h_E^{-1}\|[u_H]\|_E^2. \tag{4.14}$$

It is also clear from the fact $V_H \subset V_h$ that

$$A_h(e_h, e_h) = A_h(e_H, e_H) + A_h(u_h - u_H, u_h - u_H). \tag{4.15}$$

A combination of (4.14) and (4.15) leads to the main theoretical result on error reduction stated in the theorem below.

Theorem 4.3. *There exists* $\rho \in (0, 1)$ *such that*

$$A_h(e_h, e_h) \le \rho A_H(e_H, e_H).$$

Proof. Applying (3.4) and (3.13), we obtain

$$A_H(e_H, e_H) \le C\eta_H^2. \tag{4.16}$$

Combining the coercivity of $A_h(\cdot,\cdot)$, Lemmas 4.1 and 4.2 gives

$$\begin{aligned} A_h(u_h - u_H, u_h - u_H) &\ge C\|\nabla_h(u_h - u_H)\|^2 \\ &\ge C\theta\eta_H^2 - C\alpha\sum_{E\in\mathcal{E}_H} h_E^{-1}\|[u_H]\|_E^2 - C\alpha\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2. \end{aligned} \tag{4.17}$$

Using (3.4), (3.11), (3.12) and (4.14)–(4.17), we have

$$\begin{aligned} A_h(e_h, e_h) &= A_h(e_H, e_H) - A_h(u_h - u_H, u_h - u_H) \\ &\le A_H(e_H, e_H) + \alpha\sum_{E\in\mathcal{E}_H} h_E^{-1}\|[u_H]\|_E^2 - C\|\nabla_h(u_h - u_H)\|^2 \\ &\le A_H(e_H, e_H) - \left(C\theta - \frac{C}{\alpha}\right)\eta_H^2 + C\alpha\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[u_h]\|_e^2 \\ &\le (1 - C\theta)A_H(e_H, e_H) + \frac{C}{\alpha}A_h(e_h, e_h). \end{aligned}$$

Therefore,

$$A_h(e_h, e_h) \le \rho A_H(e_H, e_H).$$

for some constant $\rho \in (0, 1)$. □

Remark. As one can check, the above proof requires the penalty factor α to be large enough. But the theorem implies guaranteed error reduction and linear convergence of the adaptive procedure.

5. Numerical results

In this section, we validate the adaptive discontinuous finite volume method through numerical results. The algorithms and Matlab implementation in [1] have been adopted for our numerical experiments.

Example 1. We consider an elliptic boundary value problem on an L-shaped domain $\Omega = (-1, 1)^2 \setminus ([0, 1] \times [-1, 0])$ with a known exact solution

$$u(r, \theta) = r^{2/3}\sin(2\theta/3),$$

where (r, θ) are the polar coordinates. Nonhomogeneous Dirichlet boundary conditions are specified using the values of the exact solution.

Table 5.1
Numerical results for Example 1.

Level	#Elts	#NewElts	Error	η	EffIndex	RelErr	η^r
1	12		0.5714	0.2255	0.3946	0.4217	0.1664
2	16	3	0.4975	0.2172	0.4366	0.3671	0.1603
3	20	2	0.4337	0.1975	0.4554	0.3201	0.1457
4	23	3	0.3910	0.1848	0.4726	0.2885	0.1364
5	32	6	0.2982	0.1720	0.5768	0.2201	0.1269
6	43	9	0.2691	0.1580	0.5871	0.1986	0.1166
7	70	13	0.2005	0.1273	0.6349	0.1480	0.0939
8	95	14	0.1646	0.1117	0.6786	0.1215	0.0824
9	138	26	0.1396	0.0983	0.7042	0.1030	0.0725
10	188	34	0.1145	0.0864	0.7546	0.0845	0.0638
11	251	48	0.0973	0.0751	0.7718	0.0718	0.0554
12	351	67	0.0787	0.0625	0.7942	0.0581	0.0461
13	478	93	0.0670	0.0543	0.8104	0.0494	0.0401
14	653	134	0.0554	0.0464	0.8375	0.0409	0.0342
15	888	187	0.0460	0.0398	0.8652	0.0339	0.0294
16	1 211	254	0.0393	0.0344	0.8753	0.0290	0.0254
17	1 660	347	0.0327	0.0293	0.8960	0.0241	0.0216
18	2 265	488	0.0277	0.0250	0.9025	0.0204	0.0184
19	3 068	674	0.0234	0.0215	0.9188	0.0173	0.0159
20	4 178	927	0.0199	0.0184	0.9246	0.0147	0.0136
21	5 667	1275	0.0168	0.0158	0.9405	0.0124	0.0117
22	7 617	1708	0.0143	0.0136	0.9510	0.0106	0.0100
23	10,258	2351	0.0121	0.0117	0.9669	0.0089	0.0086

This is a widely used test problem on a nonconvex domain for which the exact solution does not have full elliptic regularity and hence adaptive mesh refinements are needed to resolve the corner singularity. To calibrate the adaptive discontinuous finite volume method developed in this paper, we compute errors and relative errors as follows

$$\text{Error} = |||u - u_h|||, \qquad \text{RelErr} = \frac{|||u - u_h|||}{\|\nabla u\|}, \tag{5.1}$$

where u, u_h are respectively the exact and numerical solutions. The error indicator $\eta = \eta_h$ defined in (3.2) will be computed for all meshes. For convenience, we also compute a relative error indicator

$$\eta^r = \frac{\eta}{\|\nabla u\|}.$$

More importantly the effectiveness index is calculated as

$$\text{EffIndex:} = \frac{\eta}{|||u - u_h|||}. \tag{5.2}$$

A stopping criterion RelErr $\leq$ tol based on the relative error is adopted.

Tabulated in Table 5.1 are our numerical results. We choose tol $= 0.01$, that is, 1% as a threshold for the relative error. We start from a regular triangular mesh that has only 12 elements. Shown in Fig. 5.1 is the adaptively refined mesh at level 19. With about 3000 triangular elements, the relative error is smaller than 2%. After 22 adaptive mesh refinements, we end up with 10,258 triangular elements and a relative error 0.89%. Error reductions in Columns 4 & 5 can be clearly observed. As meshes are refined, the effectiveness index (Column 6) clearly approaches 1. Our adaptive discontinuous finite volume method is well validated.

Shown in Fig. 5.1 is the adaptively refined mesh at Level 19. Presented in Fig. 5.2 is a log–log plot of the error of the adaptive numerical solution versus the number of nodes. The slope $-\frac{1}{2}$ indicates that the adaptive discontinuous finite volume method exhibits asymptotical optimality in nonlinear approximation [15,16,8,3], that is, the error is proportional to $N_{\text{dof}}^{-1/d}$, $d = 2$, where N_{dof} is the degree of freedom.

6. Concluding remarks

In this paper, we have developed and analyzed an adaptive discontinuous finite volume method for solving second order elliptic boundary value problems. A previously established *a posteriori* error estimator [6] has been used for adaptive mesh refinements. The efficiency of the error indicator has been verified by numerical results on a widely tested problem. Error reduction has been clearly demonstrated by numerical results. The adaptive discontinuous finite volume method is asymptotically optimal.

A residual type *a posteriori* error estimator has been used to establish an adaptive procedure for the discontinuous finite volume method. It should be interesting to explore combination of the adjoint-based *a posteriori* error estimation in [7] and the discontinuous finite volume method in this paper.

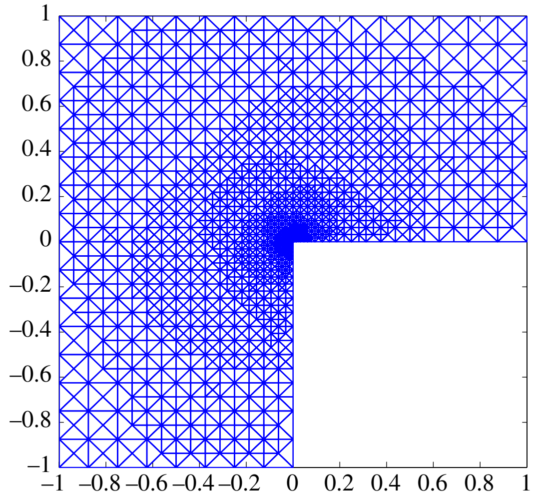

Fig. 5.1. The adaptively refined mesh at Level 19.

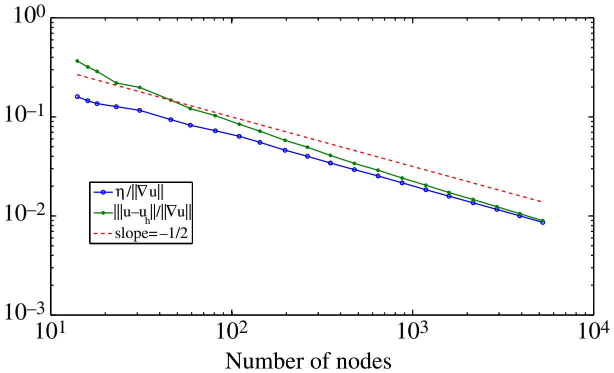

Fig. 5.2. Convergence of the error of the adaptive solution.

It should also be noticed that the discontinuous finite volume method we have analyzed in this paper is nonsymmetric, i.e., nonsymmetric interior penalty Galerkin (NIPG), and hence stable for any penalty factor $\alpha > 0$. It is well known that there are other formulations such as symmetric interior penalty Galerkin (SIPG) and incomplete interior penalty Galerkin (IIPG). Along this line, another interesting formulation is to drop both terms for the averages/jumps of trial and test functions, but to apply weak penalization in the penalty term. This approach has been investigated for finite elements [17,18]. A weakly over-penalized discontinuous finite volume method for elliptic problems has been developed in [19]. It is shown that the weakly over-penalized discontinuous finite volume method offers even easier implementations, especially in the construction of preconditioners. Establishing *a posterior* error estimators and adaptive procedures for the weakly over-penalized finite volume method is currently under our investigation and will be reported in our future work.

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