Locally Inner Automorphisms of CC-Groups*

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1. INTRODUCTION

Groups with Černikov conjugacy classes, or CC-groups, were first considered by Polovickii [9, 10] as an extension of the concept of FC-groups. A group G is said to be a CC-group if $G/C_{G}(x^{G})$ is a Černikov group for each $x \in G$. Polovickii’s basic result is that G is a CC-group if and only if the normal closure $\langle x^{G} \rangle$ of each element of G is Černikov-by-cyclic and $G/C_{G}(x^{G})$ is periodic for each $x \in G$. It follows that the periodic CC-groups are the groups which are locally (normal and Černikov) in the sense that they have a local system consisting of normal Černikov subgroups.

An automorphism $\varphi$ of a group G is said to be locally inner if, for each finite set of elements $x_{1}, \ldots, x_{n} \in G$, there is an element $g \in G$ such that $x_{i} \varphi = g^{-1}x_{i}g$, for $i = 1, \ldots, n$. The locally inner automorphisms of G clearly form a subgroup of Aut G, which we denote by Linn G. Two subgroups $H$...
and $K$ are said to be \textit{locally conjugate} in $G$ if there is an automorphism $\varphi \in \text{Linn } G$ such that $H\varphi = K$. The set of all subgroups of $G$ which are locally conjugate to $H$ is $\text{Lcl } H$, the \textit{local conjugacy class} containing $H$.

The well-known Sylow theory of FC-groups has been extended to CC-groups in [1, 6, 8], and the theory of Fitting classes and injectors has been similarly extended by Dixon [3]. As in the case of FC-groups these situations lead to local conjugacy theorems and the group of locally inner automorphisms of a CC-group clearly has the same importance as in FC-groups.

We therefore consider here some of the basic properties of the locally inner automorphisms of a CC-group $G$: in particular, determining the size of $\text{Linn } G$ and $\text{Lcl } H$ when $G$ is a CC-group with $G/\text{Z}(G)$ periodic. We also obtain some general results on the question of when the local conjugacy class containing $H$ coincides with the conjugacy class containing $H$. The discussion in Chapter 4 of [12] suggests that, for an FC-group $G$, $\text{Linn } G$ is best considered as a certain profinite completion of $\text{Inn } G$, the group of inner automorphisms of $G$. This introduces some of the elementary topological properties of profinite groups which help to give insight into the nature of the group $\text{Linn } G$. We use the same approach here although there are some additional complications.

The group of inner automorphisms of a CC-group is, of course, residually Černikov and a topological approach to residually Černikov groups has been described by Dixon [2]. If $G$ is a residually Černikov group then we define a \textit{Černikov residual system} of $G$ to be a set $\mathcal{N} = \{ N_i : i \in I \}$ of normal subgroups of $G$ such that

(i) for each $i, j \in I$, there is a $k \in I$ such that $N_k \leq N_i \cap N_j$;
(ii) $\bigcap \{ N_i : i \in I \} = 1$;
(iii) for each $i \in I$, $G/N_i$ is a Černikov group.

Given a Černikov residual system $\mathcal{N}$ of $G$, $G$ can be given a topology in which the set of cosets

$$\{ Hx : x \in G \text{ and there is an } i \in I \text{ such that } H \geq N_i \}$$

forms a closed sub-base. Dixon calls this a \textit{co-Černikov topology} and $G$ is a \textit{co-Černikov group relative to $\mathcal{N}$}. A pro-Černikov group is simply an inverse limit of Černikov groups, and the pro-Černikov groups are precisely the compact co-Černikov groups [2, Theorem 2.8]. If $G$ is a co-Černikov group relative to $\mathcal{N}$, then the pro-Černikov group $L = \lim(G/N_i)$ is the \textit{pro-Černikov completion of $G$ with respect to $\mathcal{N}$}.

One of the difficulties in extending results from cofinite groups to co-Černikov groups is that co-Černikov groups are not, in general, topological groups (see p. 68 of [2]). A second difficulty which complicates our
cardinality results is that Černikov groups have too many subgroups. In a cofinite group \( G/N \) is finite and so there are only finitely many subgroups containing \( N \), but in a co-Černikov group \( G/N \) is a Černikov group and there may be uncountably many subgroups containing \( N \). A specific example of this is provided by the direct product \( G = A \times B \) of two quasicyclic \( p \)-groups

\[
A = \langle a_1, a_2, \ldots : (a_1)^p = 1, (a_{n+1})^p = a_n \rangle,
\]

\[
B = \langle b_1, b_2, \ldots : (b_1)^p = 1, (b_{n+1})^p = b_n \rangle.
\]

For any sequence \( \xi = (\xi_1, \xi_2, \ldots) \) of 0's and 1's,

\[
G_{\xi} = \langle a_1(b_1)^{\xi_1}, a_2(b_2)^{\xi_2}, a_3(b_3)^{\xi_3} (b_2)^{\xi_2} (b_1)^{\xi_1}, \ldots \rangle
\]

is a quasicyclic subgroup of \( G \) and different sequences give rise to different subgroups.

Despite these complications we prove in Theorem 2.3 that, for a CC-group \( G \), \( \text{Linn} G \) is a pro-Černikov completion of \( \text{Inn} G \). When \( G/Z \) is periodic, where \( Z = Z(G) \), we are able to make use of this characterization to prove that, in general, \( |\text{Linn} G| = 2^{[G/Z]} \) and that \( |\text{Lcl} H| = 2^{[H/\text{H}G]} \). Theorems 4.5 and 5.4 give the precise conditions under which these results hold. Example 4.1 shows that these results fail if we allow \( G/Z \) to be non-periodic. In Section 6 we consider the question of when \( \text{Lcl} H \) coincides with \( \text{Cl} H \), obtaining a complete answer if \( G \) is a residually Černikov CC-group. Given a subgroup \( H \) of a CC-group \( G \), this result can be applied to the subgroup \( HZ/Z \) of \( G/Z \) and could be used to consider groups in which, say, the Sylow subgroups are conjugate. However, we do not consider these specific examples here and the restrictions on the structure of \( G \) which follows from conjugacy of particular subgroups of \( G \) may be found in [6–8].

We also remark that some of the results used here give some information about questions raised in [4]. It was pointed out there that one difficulty in working with (periodic) CC-groups is that one may have a Černikov factor group \( G/N \) but no Černikov subgroup \( H \) such that \( G = HN \). However, if \( C/N = Z(G/N) \) then Lemma 4.3 shows that there is a Černikov subgroup \( H \) such that \( G = HC \) and this partial covering result (due to M. González) is sufficient for our purposes here and has also prove useful in other contexts (see [5]).

In [4] it was also remarked that the minimal condition on the interval \( [G/N_G(H)] \) (that is, the minimal condition on the set of subgroups \( K \) such that \( N_G(H) \leq K \leq G \)) seemed difficult to work with. The discussion in Section 5 indicates the significance of this condition.
Locally inner automorphisms, by definition, coincide with inner automorphisms on finite subsets of $G$. Our first lemma shows that they also coincide with inner automorphisms on certain larger normal subgroups of $G$.

**Lemma 2.1.** Let $H$ be a normal subgroup of a group $G$ such that $G/C_G(H)$ is a Černikov group. If $\varphi \in \text{Linn}(G)$, then there is an element $g \in G$ such that $h\varphi = g^{-1}hg$, for all $h \in H$.

**Proof.** Since $G/C_G(H)$ satisfies the minimal condition, there is a finite subset $X$ of $H$ such that $C_G(X) = C_G(H)$. Since $X$ is finite, there is an element $g \in G$ such that $x\varphi = g^{-1}xg$, for all $x \in X$. Now let $h \in H$ and consider the finite set $Y = X \cup \{h\}$. There is an element $a \in G$ such that $y\varphi = a^{-1}ya$, for all $y \in Y$. In particular $x\varphi = a^{-1}xa = g^{-1}xg$, for all $x \in X$, and so $ag^{-1} \in C_G(X) = C_G(H)$. Hence $h\varphi = a^{-1}ha = g^{-1}hg$.

**Corollary 2.2.** Let $H$ be a normal subgroup of a group $G$ and let $\varphi \in \text{Linn}(G)$. If either

(a) $H = \langle X^G \rangle$, where $X$ is a finite set, and $G$ is a CC-group, or

(b) $H$ is Černikov and $G/Z$ is periodic,

then there is an element $g \in G$ such that $h\varphi = g^{-1}hg$, for all $h \in H$.

**Proof.** By [11, Theorem 3.29] or by definition of CC-group, $G/C_G(H)$ is Černikov, and we apply (2.1).

The conclusion of the above result is, of course, false for a normal Černikov subgroup of an arbitrary CC-group. Let $H = A \times B$ be the direct product of two quasicyclic $p$-groups and let $\varepsilon : A \to B$ be an isomorphism. Then $H$ has an automorphism $\gamma$ which fixes each element of $B$ and maps each element $a \in A$ to $a(\varepsilon a)$. Let $G$ be the split extension of $H$ by $\langle \gamma \rangle$; then $G$ is a CC-group. If $A = \langle a_1, a_2, \ldots : (a_1)^p = 1, (a_{n+1})^p = a_n \rangle$, then $G$ has a locally inner automorphism which coincides with $\gamma^{1+p+\cdots+p^n}$ on $\langle a_{n+1} \rangle$. This clearly does not coincide with an inner automorphism on $H$.

Now let $\Sigma = \{ F_i : i \in I \}$ be the local system of the CC-group $G$ consisting of all normal closures $F_i = \langle X_i^G \rangle$ of finite sets $X_i$ of elements of $G$. We let $A = \text{Inn} G$ and, for each $i \in I$, $C_i = C_A(F_i) = \{ x \in A : x\varphi = x, \text{ for all } x \in F_i \}$. If $Z$ is the centre of $G$, then $A \cong G/Z$ and, under this isomorphism $C_A(F_i)$ corresponds to $C_G(F_i)/Z$.

**Theorem 2.3.** With the notation above, $\mathcal{C} = \{ C_i : i \in I \}$ is a Černikov residual system of $A = \text{Inn} G$ and $L = \text{Linn} G$ is the pro-Černikov of $A$ with
respect to \( \mathcal{G} \). Moreover, if \( \pi_i : L \to A/C_i \) is the natural homomorphism, then \( K_i = \operatorname{Ker} \pi_i = C_i(F_i) = \{ \varphi \in L : x\varphi = x, \text{ for all } x \in F_i \} \).

**Proof.** Since \( G \) is a CC-group, \( A/C_i \cong G/C_G(F_i) \) is a Černikov group, for each \( i \in I \). It is also clear that \( \bigcap \{ C_i : i \in I \} = 1 \). If \( i, j \in I \), then there is a \( k \in I \) such that \( F_iF_j \leq F_k \) (simply take \( X_k = X_i \cup X_j \)) and hence \( C_k \leq C_i \cap C_j \). Thus \( \mathcal{G} \) is a Černikov residual system of \( A \).

If \( \varphi \in L \) and \( i \in I \), then by Corollary 2.2, there is an inner automorphism \( \alpha_i \in A \) such that \( \varphi|_{F_i} = \alpha_i|_{F_i} \). Therefore \( \varphi(\alpha_i)^{-1} \in C_L(F_i) = K_i \) and so \( L = AK_i \). Hence \( L/K_i \cong A/C_i \) and we have a natural homomorphism \( \pi_i : L \to A/C_i \) with \( \operatorname{Ker} \pi_i = K_i \). The homomorphisms \( \pi_i \) induce a homomorphism \( \pi : L \to \prod \{ A/C_i : i \in I \} \). Clearly \( \varphi'' = (\alpha_i C_i) \) and so we have that \( \operatorname{Im} \pi \leq \lim(A/C_i) \). As in Theorem 4.12 of [12], it is easily checked that \( \pi \) is an isomorphism from \( L \) to \( \lim(A/C_i) \).

The topological description of \( \text{Linn } G \) as a pro-Černikov group and the fact that such a group is a compact space enable us to give a method of constructing locally inner automorphisms of \( G \) from inner automorphisms acting on the subgroups of the standard local system \( \Sigma = \{ F_i : i \in I \} \). Our later results will also require properties of the specific residual Černikov system \( \mathcal{G} = \{ C_i : i \in I \} \) of \( A \) used in forming the completion \( \text{Linn } G \). Note that \( \{ K_i : i \in I \} \) is a residual Černikov system of \( \text{Linn } G \), which we shall refer to as the standard residual Černikov system of \( \text{Linn } G \).

**Lemma 2.4.** Let \( G \) be a CC-group and let \( \Sigma = \{ F_i : i \in I \} \) be the standard local system of \( G \). For each \( i \in I \), let \( \alpha_i \) be a non-empty closed subset of the group of automorphisms induced in \( F_i \) by \( G \) (i.e., closed in the topology on \( A/C_i \)) such that, whenever \( F_i \supseteq F_j \), each element of \( \alpha_i \) induces in \( F_j \) an automorphism from \( \alpha_i \).

Then there is a locally inner automorphism \( \varphi \) of \( G \) such that \( \varphi \) induces in each \( F_i \) an automorphism from \( \alpha_i \).

**Proof.** For each \( i \in I \), let \( K_i = C_i(F_i) \) so that \( \{ K_i : i \in I \} \) is the standard residual Černikov system of \( L = \text{Linn } G \). Let \( \theta \) be the natural isomorphism from \( A/C_i \) to the group of automorphism induced in \( F_i \) by \( G \) and let \( \mathcal{B}_i = \{ \varphi \in L : \varphi|_{F_i} \in \alpha_i \} \). Then \( \alpha_i \theta^{-1} \pi_i^{-1} = \mathcal{B}_i \) is a closed subset of \( L \). If \( F_i \supseteq F_j \), then \( \mathcal{B}_i \supseteq \mathcal{B}_j \) and since the \( F_i \) form a local system of \( G \), it follows that \( \{ \mathcal{B}_i : i \in I \} \) has the finite intersection property. Since \( L \) is compact, \( \bigcap \{ \mathcal{B}_i : i \in I \} \neq \emptyset \) and so there is a locally inner automorphism \( \varphi \in \bigcap \{ \mathcal{B}_i : i \in I \} \). This \( \varphi \) is the required automorphism.

The above lemma can be used to prove local conjugacy theorems as in [1, 3]. We make use of it here to see that locally inner automorphisms of subgroups or factor groups of a CC-group \( G \) are induced by locally inner automorphisms of \( G \).
THEOREM 2.5. Let $G$ be a CC-group.

(i) If $H \leq G$ and $\varphi \in \text{Linn } H$, then there is a $\theta \in \text{Linn } G$ such that $\theta|_H = \varphi$.

(ii) If $N \leq G$ and $\varphi \in \text{Linn } G/N$, then there is a $\theta \in \text{Linn } G$ such that $\theta$ induces $\varphi$ in $G/N$.

Proof. (i) Let $\mathcal{A}_i$ be the set of automorphisms of $F_i$ induced by inner automorphisms of $G$ and which coincide with $\varphi$ on $H \cap F_i$. By Corollary 2.2, $\mathcal{A}_i \neq \emptyset$. Also, $\mathcal{A}_i$ is a coset of $C_A(H \cap F_i)/C_i$ and so is a closed set. Therefore we can apply Lemma 2.4 to obtain the result.

(ii) Now let $\mathcal{A}_i$ be the set of automorphisms of $F_i$ induced by inner automorphisms of $G$ and which induce in $F_i/N/N$ an automorphism coinciding with $\varphi$. Again, by Corollary 2.2, $\mathcal{A}_i \neq \emptyset$. If $\tilde{A} = \text{Inn}(G/N)$, $\tilde{F}_i = F_i/N/N$ and $\tilde{C}_i = C_A(\tilde{F}_i)$, then there is a natural homomorphism $\rho: A/C_i \to \tilde{A}/\tilde{C}_i$. By Lemma 2.2(iii) of [2], $\rho$ is a closed continuous map. By Corollary 2.2, there is a $\tilde{a} \in \tilde{A}$ such that $\varphi|_{\tilde{F}_i} = \tilde{a}|_{\tilde{F}_i}$ and $\mathcal{A}_i = (\tilde{a} \tilde{C}_i) \rho^{-1}$ is a closed set. Thus the condition of Lemma 2.4 are again satisfied and the result follows immediately.

There is one simple corollary of this result which will be require later and which we state without proof.

COROLLARY 2.6. Let $G$ be a CC-group.

(i) If $H \leq K \leq G$, then $\text{Lcl}_K(H) \subseteq \text{Lcl}_G(H)$.

(ii) If $N \leq G$ and $N \leq H$, then $\text{Lcl}_{G/N}(H/N) = \{ K/N : K \in \text{Lcl}_G(H) \}$; in particular $|\text{Lcl}_{G/N}(H/N)| = |\text{Lcl}_G(H)|$.

3. THE INDEX OF CERTAIN SUBGROUPS OF Linn $G$

For a profinite group it is possible to determine the index of any closed subgroup. Because of the difficulties referred to in the Introduction this is not possible in general in a pro-Černikov group but we are able to consider a subgroup $U$ of Linn $G$ which is the intersection of certain subgroups $U_\lambda$ which each contain a subgroup from the standard residual system. Before proving this result we first separate out one of the technical consequences of the compactness of Linn $G$.

LEMMA 3.1. Let $G$ be a CC-group and let $\{ K_i = C_L(F_i) : i \in I \}$ be the standard residual Černikov system of $L = \text{Linn } G$. Let $U_\lambda$, $\lambda \in \Lambda$, be subgroups of $L$ such that each $U_\lambda$ contains a subgroup $K_{\beta(\lambda)}$, for some $\beta(\lambda) \in I$.

Suppose that $\beta \in \Lambda$ and $\Gamma \subseteq \Lambda$ such that $U_\beta \supseteq \bigcap \{ U_\gamma : \gamma \in \Gamma \}$. Then there is a finite subset $\{ \gamma_1, \ldots, \gamma_k \} \subseteq \Gamma$ such that $U_\beta \supseteq U_{\gamma_1} \cap \cdots \cap U_{\gamma_k}$.
Proof. We suppose the result is false. Let \( \Omega \) be the set of all finite subsets of \( I \) and, for each \( \sigma \in \Omega \), write \( U_\sigma = \bigcap \{ U_\gamma : \gamma \in \sigma \} \). Then \( U_\beta \geq \bigcap \{ U_\gamma : \gamma \in I \} = \bigcap \{ U_\sigma : \sigma \in \Omega \} \) and, for each \( \sigma \in \Omega \), \( U_\beta \nsubseteq U_\sigma \).

Let \( V_\beta = \text{core}_\beta(U_\beta) \geq K_{\eta_\beta} \), and consider \( V = \bigcap \{ U_\sigma V_\beta : \sigma \in \Omega \} \). Since \( L/V_\beta \) satisfies the minimal condition, there is a finite set \( \{ \sigma_1, \ldots, \sigma_n \} \subseteq \Omega \) such that

\[
V = U_{\sigma_1} V_\beta \cap \cdots \cap U_{\sigma_n} V_\beta \geq (U_{\sigma_1} \cap \cdots \cap U_{\sigma_n}) V_\beta = U_{\sigma_0} V_\beta,
\]

where \( \sigma_0 = \sigma_1 \cup \cdots \cup \sigma_n \in \Omega \).

Let \( \Omega_0 = \{ \sigma \in \Omega : \sigma_0 \subseteq \sigma \} \); then \( U_\beta \geq \bigcap \{ U_\sigma : \sigma \in \Omega \} = \bigcap \{ U_\sigma : \sigma \in \Omega_0 \} \) and, for all \( \sigma \in \Omega_0 \), \( U_\sigma V_\beta = V \).

In particular, \( V \nsubseteq U_\beta \). Let \( \{ S_\alpha/V_\beta : \alpha < 2^\omega \} \) be the ascending socle series of the Černikov group \( L/V_\beta \). There is a least \( \alpha \) such that \( V \cap S_\alpha \nsubseteq U_\beta \). Clearly \( \alpha \) is not a limit ordinal and so \( \alpha - 1 \) exists and \( V \cap S_{\alpha - 1} \subseteq U_\beta \). Thus, for each \( \sigma \in \Omega_0 \), \( U_\sigma \cap S_{\alpha - 1} \subseteq U_\beta \) and \( U_\sigma \cap S_\alpha \nsubseteq U_\beta \), since \( V \cap S_\alpha = U_\sigma V_\beta \cap S_\alpha = (U_\sigma \cap S_\alpha) V_\beta \).

Now \( S_{\alpha - 1} \) is finite and so \( |U_\sigma \cap S_\alpha : U_\sigma \cap S_\alpha \cap U_\beta| \leq |U_\sigma \cap S_\alpha : U_\sigma \cap S_{\alpha - 1}| \) is also finite. Also, \( U_\sigma \cap S_\alpha \cap U_\beta \geq \bigcap \{ K_{\eta_\gamma} : \gamma \in \sigma \} \cap K_{\eta_\beta} = C_\lambda(X^\sigma) \), where \( X \) is the finite set given by \( X = \{ X_{\eta_\alpha : \gamma \in \sigma} \cup X_{\eta_\beta} \} \), and so \( U_\sigma \cap S_\alpha \cap U_\beta \) is a closed subgroup of \( L \). Hence \( (U_\sigma \cap S_\alpha) - U_\beta \) is a union of finitely many cosets of the closed subgroup \( U_\sigma \cap S_\alpha \cap U_\beta \) and so is a non-empty closed subset of \( I \). If \( \sigma_1, \ldots, \sigma_m \in \Omega_0 \), then \( \bigcap \{ (U_\sigma \cap S_\alpha) - U_\beta : i = 1, \ldots, m \} = (U_\sigma \cap S_\alpha) - U_\beta \), where \( \sigma = \sigma_1 \cup \cdots \cup \sigma_m \in \Omega_0 \), and so the family of closed sets \( \{ (U_\sigma \cap S_\alpha) - U_\beta : \sigma \in \Omega_0 \} \) satisfies the finite intersection property. Since \( L \) is compact, \( \bigcap \{ (U_\sigma \cap S_\alpha) - U_\beta : \sigma \in \Omega_0 \} \neq \emptyset \). But this implies that \( \bigcap \{ U_\sigma : \sigma \in \Omega_0 \} \nsubseteq U_\beta \), contrary to hypothesis.

We now prove our most general result on the index of certain subgroups of \( \text{Linn} \, G \). It will be seen that our hypotheses are chosen to avoid the complications caused by Černikov groups having uncountably many subgroups. However, we shall see in the following sections that these hypotheses are satisfied by the subgroups in which we are interested.

**Theorem 3.2.** Let \( G \) be a CC-group and let \( \{ K_i = C_i(F_i) : i \in I \} \) be the standard residual Černikov system of \( L = \text{Linn} \, G \). Let \( U_\lambda, \lambda \in \Lambda \), be distinct subgroups of \( L \) such that

(a) each \( U_\lambda \) contains a subgroup \( K_{i(\lambda)} \), for some \( i(\lambda) \in I \);
(b) each finite intersection \( U_{\lambda_1} \cap \cdots \cap U_{\lambda_m} \) is contained in only countably many \( U_\mu, \mu \in \Lambda \);
(c) \( U = \bigcap \{ U_\lambda : \lambda \in \Lambda \} \) does not contain any \( K_i, i \in I \).

Then \( |L : U| = 2^4 \).
Proof. The set \( A \) may be well-ordered and so we may assume that \( A = \{ \lambda : \lambda < \rho \} \), where \( \rho \) is the least ordinal of cardinality \(|A|\). We define a second well-ordering on \( A \) by defining a bijection \( \varphi \) from a certain set of ordinals onto the set \( \{ \lambda : \lambda < \rho \} \) as follows. Let \( \varphi(1) \) be any element of \( A \) and then suppose that we have defined \( \varphi(\beta) \) for each \( \beta < \alpha \). Let \( A(\alpha) = \{ \lambda \in A : U_\lambda \supseteq \bigcap_{\beta_1 \leq \beta \leq \alpha} \bigcap_{\beta_1 \leq \beta \leq \alpha} U_{\varphi(\beta_1) \cap \cdots \cap U_{\varphi(\beta_\alpha)}}, \) for some \( \beta_1, \ldots, \beta_\alpha < \alpha \).

(I) If \( A(\alpha) \) is not contained in \( \{ \varphi(\beta) : \beta < \alpha \} \), then define \( \varphi(\alpha) \) to be the smallest ordinal \( \lambda \in A(\alpha) - \{ \varphi(\beta) : \beta < \alpha \} \).

(II) If \( A(\alpha) \subseteq \{ \varphi(\beta) : \beta < \alpha \} \), then define \( \varphi(\alpha) \) to be the smallest \( \lambda \notin \{ \varphi(\beta) : \beta < \alpha \} \).

(That is, after introducing a new \( U_\lambda \) we then include the set all the \( U_\lambda \) which contain a finite intersection of listed subgroups.)

We can continue to define the \( \varphi(\alpha) \) until the set \( \{ \lambda : \lambda < \rho \} \) is exhausted so that \( \varphi \) is defined on a set \( J \) of ordinals such that \(|J| = |A|\).

We say that \( \alpha \in J \) is closed if \( A(\alpha) \subseteq \{ \varphi(\beta) : \beta < \alpha \} \).

If \( \alpha \in J \) is finite, then by condition (a) there are only countably many \( U_\lambda \)'s containing \( \bigcap_{\beta < \alpha} \{ U_{\varphi(\beta)} : \beta < \alpha \} \) and so the succeeding closed ordinal \( \lambda \in J \) is countable. Also, using (a), \( \bigcap_{\beta < \alpha} \{ U_{\varphi(\beta)} : \beta < \alpha \} \supseteq \bigcap_{\beta < \alpha} K_{\beta < \alpha} \geq C_L X(G) \), where \( X = \bigcup_{\beta < \alpha} X(\beta) \) is a finite subset of \( G \). Therefore, by condition (c), there is a \( \lambda \in A \) such that \( \lambda \notin \{ \varphi(\beta) : \beta < \alpha \} \).

If \( \alpha \in J \) is infinite, then by condition (a) there are only \( |\alpha| \) distinct \( U_\lambda \)'s containing finite intersections \( \bigcap_{\beta < \alpha} \{ U_{\varphi(\beta)} \cap \cdots \cap U_{\varphi(\beta)}(\beta_1, \ldots, \beta_\alpha < \alpha) \} \) and so \( |\lambda| = |\alpha| \). In both cases it follows that \( J \) contains \(|A| \) closed ordinals. We let \( J = \{ \alpha : \alpha < \sigma \} \). (In fact, if \( A \) is uncountable, then \( \sigma = \rho \) and if \( A \) is countable then \( \sigma = \omega^2 \).)

We show that there are \( 2^{|A|} \) distinct cosets of \( U \) in \( L \) by forming \( 2^{|A|} \) sequences of cosets \( \{ U_{\varphi(\alpha)} x_\alpha : \alpha < \sigma \} \) such that each finite subset \( \{ U_{\varphi(\alpha)} x_\alpha, \ldots, U_{\varphi(\alpha)} x_\alpha \} \) has a non-empty intersection. Since the \( U_{\varphi(\alpha)} x_\alpha \) are closed subsets of the compact space \( L \), it follows that there is an \( x \in \bigcap_{\alpha < \sigma} \{ U_{\varphi(\alpha)} x_\alpha : \alpha < \sigma \} \) and so \( Ux = \bigcap_{\alpha < \sigma} \{ U_{\varphi(\alpha)} x_\alpha : \alpha < \sigma \} \) is a coset of \( U \) in \( L \). Distinct sequences of cosets give rise to distinct cosets of \( U \).

We say that the sequence \( \{ U_{\varphi(\beta)} x_\beta : \beta < \alpha \} \) is coherent of length \( \alpha \) if the intersection \( \bigcap_{\beta < \alpha} \{ U_{\varphi(\beta)} x_\beta : \beta < \alpha \} \neq \emptyset \). We show that each coherent sequence of length \( \alpha \) can be extended to one of length \( \alpha + 1 \) and that if \( \alpha \) is closed the sequence can be extended in two different ways. (Note that because of the finite intersection property there is no problem in extending through limit ordinals.)

(A) Suppose that \( \{ U_{\varphi(\beta)} x_\beta : \beta < \alpha \} \) is coherent. Then there is an element \( x \in \bigcap_{\beta < \alpha} \{ U_{\varphi(\beta)} x_\beta : \beta < \alpha \} \). We put \( x_\alpha = x \) so that \( x \in \bigcap_{\beta < \alpha} \{ U_{\varphi(\beta)} x_\beta : \beta \leq \alpha \} \) and so the sequence \( \{ U_{\varphi(\beta)} x_\beta : \beta \leq \alpha \} \) is coherent of length \( \alpha + 1 \).

(B) Suppose that \( \{ U_{\varphi(\beta)} x_\beta : \beta < \alpha \} \) is coherent and \( \alpha \) is closed. Again
there is an element \( x \in \bigcap \{ U_{\alpha(\beta)}x_{\alpha} : \beta < \alpha \} \). But also, by Lemma 3.1, \( U_{\alpha(\beta)} \nsubseteq \bigcap \{ U_{\alpha(\beta)} : \beta < \alpha \} \) and so there is a \( y \in \left[ \bigcap \{ U_{\alpha(\beta)}x_{\alpha} : \beta < \alpha \} \right] - U_{\alpha(\beta)} \) and the sequence can be extended by putting \( x_{\gamma} = x \) or \( x_{\gamma} = yx \). The cosets \( U_{\alpha(\beta)}x_1 \) and \( U_{\alpha(\beta)}y_1 \) are distinct and so this completes our construction of the \( 2^{144} \) sequences of length \( \sigma \).

4. THE GROUP OF LOCALLY INNER AUTOMORPHISMS

We have described \( \text{Linn} G \) as a pro-Černikov completion of \( \text{Inn} G \) with respect to a certain residual Černikov system. Our aim in this section is to show that the standard residual system \( \{ K_i : i \in I \} \) of \( \text{Linn} G \) satisfies the conditions for \( U_\zeta (\zeta \in \Lambda) \) in Theorem 3.2 and contains precisely \( |G/Z| \) subgroups. Unfortunately, further conditions are necessary as the following example shows that, in general, this is false.

**Example 4.1.** Let \( A \) and \( B \) be isomorphic to \( \mathbb{Z}(p^\infty) \); then \( \text{Hom}(A, B) \cong R \), the ring of \( p \)-adic integers. For each \( r \in R \) denote the corresponding homomorphism by \( \eta_r \). Then form the split extension \( G \) of \( A \oplus B \) by \( R \), where the action of \( R \) on \( A \oplus B \) is given by \( (a, b)r = (a, ar + b) \).

Then \( Z(G) = G' = B \) is Černikov and it is easy to check that \( G \) is a CC-group. If \( i \) is the identity element of \( R \), then \( C_G(i) = BR = C_G(R) \). If \( \phi \in \text{Linn} G \), then \( i\phi = a_0^{-1}ia_0 \) for some \( a_0 \in A \) and, since \( C_G(i) = C_G(R) \), it follows that \( r\phi = a_0^{-1}r\phi a_0 \), for all \( r \in R \). Also, the mapping \( a \rightarrow a^{-1}(a\phi) \) is a homomorphism from \( A \) to \( B \) so that \( a^{-1}(a\phi) = an_\phi \), for some \( r \in R \), and so \( a\phi = (a, an_\phi) = a' \). Thus \( \phi \) acts on \( G \) as conjugation by \( a_0r \) and so \( \text{Linn} G = \text{Inn} G \).

In this example, \( G/Z(G) \cong \mathbb{Z}(p^\infty) \oplus R \) is nonperiodic and it is this which causes the expected results to fail. Our remaining results in this section are aimed at obtaining the conditions of Theorem 3.2 for CC-groups \( G \) in which \( G/Z(G) \) is periodic.

**Lemma 4.2.** Let \( G \) be a CC-group and \( T/Z \) a periodic abelian normal subgroup of \( G/Z \), where \( Z = Z(G) \). Then \( T \) is contained in \( FC(G) \), the FC-centre of \( G \).

**Proof.** Let \( a \in T \) and \( a^n \in Z \). Then \( \langle a^G \rangle Z/Z \) is a normal abelian subgroup of \( G/Z \) generated by elements of order \( n \). Also, \( \langle a^G \rangle Z/Z \) is a Černikov group and so it is finite.

Therefore \( \langle a^G \rangle \) is centre-by-finite and hence finite-by-abelian. Let \( F = \langle a^G \rangle' \), a finite normal subgroup of \( G \), and consider \( \bar{G} = G/F \). In \( \bar{G} \), \( \langle \bar{a}^G \rangle \) is abelian and \( \bar{a}^n \) is central. If \( \bar{g} \in \bar{G} \), then \([\bar{a}, \bar{g}]^n = [\bar{a}^n, \bar{g}] = 1 \) and so
\[ [a, \overline{G}] \text{ is abelian of finite exponent and so is finite. Since } F \text{ is finite, it}
\text{follows that } [a, G] \text{ is finite and so } a \in FC(G).\]

Lemma 4.3 (M. González [5]). Let \( N \) be a normal subgroup of the
CC-group \( G \) such that \( G/N \) is a Černikov group. If \( C/N = \mathbb{Z}(G/N) \) then there
is a normal subgroup \( H = \langle X^G \rangle \) of \( G \) such that \( X \) is a finite subset of \( G \) and
\( G = HC \).

Proof. Let \( D/N \) be the divisible part of \( G/N \); then \( G/N \) has a finite
subgroup \( S/N \) such that \( G = SD \). There is a finite set \( X \) of elements of \( G 
\text{such that } S = \langle X \rangle N \) and hence \( G = \langle X \rangle D \). Since \( D/N \) is divisible
and \( S/N \) is finite, by Lemma 3.29.1 of [11], we have that \( D/N = ([D, S]/N/N)\)
\( C_D/N(S/N) \leq [D, S]/C/N \). Therefore \( G = \langle X \rangle D = SC[D, S] \). But \( [D, S] \leq
\langle X^G \rangle N \) and so \( G = \langle X^G \rangle C \), as required.

Lemma 4.4. Let \( G \) be a CC-group with \( G/Z \) periodic and let \( N \) be a nor-
mal subgroup of \( G \) with \( G/N \) Černikov. Then \( C_G(N)/Z \) is a Černikov group.

Proof. Let \( Y/N = \mathbb{Z}(G/N) \) and let \( R/N \) be the divisible part of \( Y/N \). By
Lemma 4.3, there is a normal subgroup \( H = \langle X^G \rangle \) of \( G \) such that \( G = HC \)
and \( X \) is a finite set. Therefore \( G/C_G(H) \) is Černikov and \( Z = C_G(H) \cap
C_G(R) \). Hence \( C_G(R)/Z \) is Černikov.

Let \( T = R \cap C_G(N) \); then \( C_G(N)/T \) is Černikov and so it is sufficient to
prove that \( T/Z \) is Černikov. Now \( [T, G] = T \cap N \) and so \( [T, G, T] = [G, T, T] = 1 \).
By the Three Subgroup Lemma, \( [T, T, G] = 1 \) so that \( T' \leq Z \). By Lemma 4.2,
\( T \leq FC(G) \). If \( a \in T \), then \( C_G(a) \geq N \); but \( R/N \) is
divisible abelian and so \( C_G(a) \geq R \). Thus \( A \leq C_G(R) \) and so \( A/Z \) is
Černikov, as required.

Theorem 4.5. Let \( G \) be a CC-group with \( G/Z \) periodic. If \( G/Z \) is not
Černikov, then \( |\text{Linn } G| = 2^{[G/Z]} \).

Proof. We consider the standard residual Černikov system
\( \{ K_i = C_i(F_i) : i \in I \} \) of \( L = \text{Linn } G \) and show that the subgroups \( K_i 
\)satisfy the conditions of Theorem 3.2 for the subgroups \( U_i \).

Condition (a) is trivially satisfied and the condition that \( G/Z \) is not
Černikov ensures that condition (c) is satisfied. For \( \bigcap \{ K_i : i \in I \} = 1 \) and,
since \( L \) is not Černikov, we cannot have \( K_i = 1 \) for any \( i \in I \).

Recall our previous notations: \( A = \text{Inn } G, C_i = C_G(F_i) \), and in the
isomorphism \( A \cong G/Z, C_i \) corresponds to \( C_G(F_i)/Z \). Now define
\( F_i^* = C_G(K_i) = \{ g \in G : g \varphi = g, \text{ for all } \varphi \in K_i \} \). Then, making use of
Corollary 2.2, we have that \( C_G(C_i) \leq C_G(K_i) \) and therefore \( F_i^* = C_G(K_i) = C_G(C_i) = C_G(C_G(F_i)) \). By Lemma 4.4, \( F_i^*/Z \) is Černikov. Also, \( C_i(F_i^*) = K_i \)
so that \( K_i \leq K_j \) if and only if \( F_i \leq F_j \leq F_i^* \). Since \( F_i^*/Z \) contains only
countably many distinct finite subsets it follows that there are only countably many distinct subgroups $K_i$ containing $K_i$. Thus condition (b) is satisfied and so $|\text{Linn } G| = 2^{\omega_1}$.

Also, $G/Z = \langle F_i Z : i \in I \rangle = \langle F_i^* Z : i \in I \rangle$ and so there must be $|G/Z|$ distinct subgroups $F_i^*$. It follows that there are $|G/Z|$ distinct subgroups $K_i$ and the result follows.

5. The Local Conjugacy Class Containing a Subgroup

We begin by discussing the exceptional case for which our formula for $|\text{Lcl } H|$ will not apply.

The form in which the condition will be most convenient here is that $N_G(H) \supseteq C_G(F_i)$ for some $F_i = \langle X_i^G \rangle$, the normal closure of a finite set $X_i$ of elements of $G$. If we write $N_L(H)$ for $\{ \phi \in L : H \phi = H \}$ this is equivalent to $N_L(H) \supseteq K_i$ for some $i$. It is clear that if $N_G(H) \supseteq C_G(F_i)$ then the interval $[G/N_G(H)]$ satisfies the minimal condition. Conversely, since $N_G(H) = \bigcap \{ N_G(H \cap X^G) : X \text{ a finite subset of } H \}$, the minimal condition on $[G/N_G(H)]$ implies that there is a finite set $X$ such that $N_G(H) = N_G(H \cap X^G) \supseteq C_G(X^G)$ and so these two conditions are equivalent in a CC-group.

The following result gives a more interesting equivalent condition, namely that $\text{Lcl } H$ is countable. This condition appears in [1, 6, 8], where it is shown that the Sylow subgroups of a CC-group are conjugate if and only if there are only countably many of them.

**Lemma 5.1.** Let $H$ be a subgroup of the CC-group $G$. Then $\text{Lcl } H$ is countable if and only if there is a finite subset of elements of $G$ such that $N_G(H) \supseteq C_G(X^G)$.

**Proof.** If $N_G(H) \supseteq C_G(X^G)$, then $N_L(H) \supseteq K_i$ and so $|\text{Lcl } H| = |L : N_L(H)|$ is countable.

To prove the converse suppose that $N_G(H)$ does not contain any $C_G(X^G)$. We show that $\text{Lcl } H$ is uncountable by constructing a countable normal subgroup $N$ of $G$ such that $\text{Lcl}_{\mathcal{N}}(H \cap N)$ is uncountable. It then follows from Corollary 2.6 that $\text{Lcl}_{\mathcal{G}}(H \cap N)$ is uncountable and hence $\text{Lcl } H$ is uncountable.

The subgroup $N$ is constructed as the union of a chain of subgroups $N_k = \langle X_k^G \rangle$ such that, for each $k$, $H \cap N_k$ has $2^k$ distinct conjugates $H g_i(k) \cap N_k$ ($i = 1, ..., 2^k$), where

(a) $g_i(k) \in N_{k+1}$ ($i = 1, ..., 2^k$),
(b) $g_{2i}(k) = g_i(k - 1)$,
(c) $g_{2i}(k) g_i(k - 1)^{-1} \in C_G(N_{k-1})$. 
It follows from (b) and (c) that
\[ H^g(k-1) \cap N_{k-1} \leq H^{g_{2k-1}}(k) \cap N_k \quad \text{and} \quad H^g(k-1) \cap N_{k-1} \leq H^{g_2(k)} \cap N_k. \]

Therefore the subgroups \( H^{g(k)} \cap N_k \) form a tree with \( 2^{k_0} \) branches. If the subgroups \( H^{g(k)} \cap N_k \) form one of these branches then it follows that \( g(k) \in N_k \) for all \( k \), and \( g(k) g(k-1)^{-1} \in C_G(N_{k-1}) \). Therefore we can define a locally inner automorphism \( \phi \) of \( N \) by \( x\phi = g(k)^{-1} x g(k) \), if \( x \in N_k \). It is clear that \( (H \cap N) \varphi = \bigcup \{ (H \cap N_k) \varphi : k \geq 1 \} = \bigcup \{ H^{g(k)} \cap N_k : k \geq 1 \} \) and so the union of each branch is a subgroup locally conjugate in \( N \) to \( H \cap N \).

It only remains to construct the subgroups \( N_k \). Suppose that we have constructed \( N_k \) and the \( 2^{k-1} \) subgroups \( H^{g(k)} \cap N_{k-1} \). Since \( N_G(H) \not\supset C_G(N_{k-1}) \) there is an element \( g \in C_G(N_{k-1}) - N_G(H) \). We choose \( N_k = \langle X^G \rangle \) so that \( g_i(k-1) \in N_k \) for all \( i = 1, \ldots, 2^{k-1} \) and \( H^{g} \cap N_k \neq H \cap N_k \). Then we define \( g_{2^i-1}(k) = g_i(k-1) \) and \( g_{2^i}(k) = g g_i(k-1) \) and all the above conditions are satisfied.

The above result is true without any restriction on the \( CC \)-group \( G \). As we have already seen most of our results require the condition that \( G/Z \) be periodic. In this situation we have a further, equivalent condition, that \( H^G/H_G \) be \( \check{\text{C}} \)ernikov (as usual \( H^G \) denotes the normal closure of \( H \) in \( G \) and \( H_G \) denotes the core of \( H \) in \( G \)). This is similar to conditions used for FC-groups in [12] (see Lemma 4.21, for example) and is also related to some of the questions considered in Section 4 of [4].

First we require a variation on Lemma 4.3.

**Lemma 5.2.** Let \( G \) be a residually \( \check{\text{C}} \)ernikov \( CC \)-group and let \( X \) be a finite subset of \( G \). Then there is a finite subset \( Y \) of \( G \) such that \( G = \langle Y^G \rangle C_G(X^G) \).

**Proof.** There is a normal subgroup \( N \) of \( G \) such that \( T(X^G) \cap N = 1 \) and \( G/N \) is \( \check{\text{C}} \)ernikov, where \( T(X^G) \) is the torsion subgroup of \( \langle X^G \rangle \) and is a normal \( \check{\text{C}} \)ernikov subgroup of \( G \) (Lemma 1 of [1]). By Lemma 4.3, there is a finite subset \( Y \) such that \( G = \langle Y^G \rangle C_G(G/N) \). But \( C_G(G/N) \leq T(X^G) \cap N = 1 \) and so \( C_G(G/N) \leq C_G(X^G) \). Hence \( G = \langle Y^G \rangle C_G(X^G) \).

**Theorem 5.3.** Let \( G \) be a \( CC \)-group with \( G/Z \) periodic and let \( H \leq G \). Then \( H^G/H_G \) is a \( \check{\text{C}} \)ernikov group if and only if \( N_G(H) \geq C_G(X^G) \), for some finite set \( X \).

**Proof.** If \( H^G/H_G \) is \( \check{\text{C}} \)ernikov then \( G/C_G(H^G/H_G) \) is isomorphic to a periodic group of automorphisms of the \( \check{\text{C}} \)ernikov group \( H^G/H_G \).
Hence $G/C_G(H^G/H_G)$ is Černikov [11, Theorem 3.29]. Clearly $N_G(H) \geq C_G(H^G/H_G)$ and so $[G/N_G(H)]$ satisfies the minimal condition. As noted earlier, this is equivalent to $N_G(H)$ containing some $C_G(X^G)$.

Conversely, assume that $N_G(H) \geq C_G(X^G)$ for some finite set $X$. This condition is preserved when we pass to homomorphic images since the equivalent condition $[G/N_G(H)]$ satisfies the minimal condition is obviously preserved. Therefore we may assume that $H_G = 1$ and, in particular $H \cap Z = 1$. Consider $\bar{H} = HZ/Z \leq G/Z = \bar{G}$. There is a finite set $\bar{X}$ such that $N_G(\bar{H}) \geq C_G(\bar{X}^G)$ and so, by Lemma 5.2, there is a finite subset $\bar{Y}$ of $\bar{G}$ such that $\bar{G} = \langle \bar{Y}^G \rangle N_G(\bar{H})$. Therefore $\bar{H} \cap C_G(\bar{Y}^G) \leq \bar{H}_G$ and hence $HZ/(HZ)_G$ is Černikov.

Let $Q = (HZ)_G$ so that $H/H \cap Q$ is Černikov. Note also that $H \cap Q \leq (H \cap Q)Z = Q$ and so $Q/H \cap Q$ is abelian. Therefore $Q' \leq H \cap Q$. But $Q' \leq G$ and $H_G = 1$ so that $Q' = 1$ and $Q$ is abelian. Hence, by Lemma 4.2, $Q \leq FC(G)$. Also, $N_G(H \cap Q) \geq N_G(H)$ so that $[G/N_G(H \cap Q)]$ satisfies the minimal condition and, since $H \cap Q$ is FC-central in $G$, it follows that $|G:N_G(H \cap Q)|$ is finite. Therefore there is a finite set $W$ of $G$ such that $G = \langle W^G \rangle N_G(H \cap Q)$. Again $H \cap Q \cap C_G(W^G) \leq G$ so that $H \cap Q \cap C_G(W^G) = 1$ and $H \cap Q$ is Černikov.

It follows that $H$ is Černikov and so $H = DF$, where $D$ is the divisible part of $H$ and $F$ is finite. Since $D$ is characteristic in $H$, $N_G(D) \geq N_G(H) \geq C_G(X^G)$. But $D \leq FC(G)$ since $D$ is contained in the torsion subgroup of the radicable part of $G$ (see Lemma 2.1 of [6]). Hence $|G:N_G(D)|$ is finite and $D^G$ is Černikov. Also, $F^G$ is Černikov and so $H^G = D^G F^G$ is Černikov.

It should be noted that both directions of this result fail if we omit the condition that $G/Z$ is periodic. In Example 4.1 the subgroup $A$ satisfies $A^G/A_G$ is Černikov, since $A^G = A \oplus B$, but $N_G(A) = A \oplus B$ so that $G/N_G(A)$ is not even periodic.

On the other hand $N_G(R) = BR = C_G(i) = C_G(iG)$ but $R_G = 1$ so that $R/R_G$ is not Černikov.

**Theorem 5.4.** Let $G$ be a CC-group with $G/Z$ periodic and let $H$ be a subgroup of $G$ such that $N_G(H) \not\geq C_G(X^G)$ for any finite subset of $G$. Then $|Lcl H| = 2^{[H/N_G]}$.

**Proof.** Using Corollary 2.6, we may assume that $H_G = 1$. We show that $|L : N_L(H)| = 2^{[H/N_G]}$ using Theorem 3.2.

Let $X$, $\lambda \in A$, be the finite subsets of $H$ and let $U_\lambda = N_L(H \cap X^G)$; then $N_L(H) = \bigcap \{U_\lambda : \lambda \in A\}$ and we must show that the subgroups $U_\lambda$ satisfy the conditions of Theorem 3.2. We first note that $U_\lambda \not\geq C_L(X^G)$ so that condition (a) is satisfied and the hypothesis that $N_L(H) \not\geq C_L(X^G)$ for any $X$ implies that $\bigcap \{U_\lambda : \lambda \in A\} \not\geq K_i$, for any $i$, so that condition (c) is also satisfied.
Suppose that \( U_\mu \supseteq U_{\lambda_1} \cap \cdots \cap U_{\lambda_k} \). Then \( U_\mu \supseteq C_L(X^G) \) where \( X = X_{\lambda_1} \cup \cdots \cup X_{\lambda_k} \) is a finite subset of \( H \). Let \( V = C_L(X^G) \) and consider \( H_V = \bigcap \{ H \varphi : \varphi \in V \} \). It is clear that \( N_L(H_V) \supseteq V = C_L(X^G) \) and hence \( N_G(H_V) \supseteq C_G(X^G) \). It follows from Theorem 5.3 that \( H_V \) is a Černikov group. Since we have \( N_L(H \cap X^G) = U_\mu \supseteq C_L(X^G) = V \), it follows that \( X_{\mu} \subseteq X_{\mu} \cap H \subseteq H_V \) and so there are only countably many distinct groups \( U_\mu \) containing \( V \). Thus condition (b) of Theorem 3.2 is satisfied.

Now let \( F_\lambda^* = \bigcap \{ H \varphi : \varphi \in U_\lambda \} \subseteq \bigcap \{ H \varphi : \varphi \in C_L(X^G) \} \). Then \( F_\lambda^* \) is a Černikov group, as above, and \( X_\lambda^* \cap H \subseteq F_\lambda^* \). Therefore \( H = \langle F_\lambda^* : \lambda \in \Lambda \rangle \) and if \( H \) is not Černikov then it is clear that there are \( |H| \) distinct subgroups \( F_\lambda^* \) and hence \( |H| \) distinct subgroups \( U_\lambda \). If \( H \) is a Černikov group we must note that \( H \neq \langle F_{\lambda_1}^*, \ldots, F_{\lambda_k}^* \rangle \) for if this were the case we would have \( N_L(H) \supseteq N_L(F_{\lambda_1}^*) \cap \cdots \cap N_L(F_{\lambda_k}^*) \supseteq C_L(X^G) \), where \( X = X_{\lambda_1} \cup \cdots \cup X_{\lambda_k} \), and this is contrary to our hypothesis. It follows that in this case there are infinitely many distinct subgroups \( F_\lambda^* \) and hence infinitely many distinct subgroups \( U_\lambda \).

In both cases we can apply Theorem 3.2 to obtain \( |L\text{cl } H| = |L : N_L(H)| = 2^{|H|} \).

### 6. Conjunctivity of Subgroups

For FC-groups, it is frequently the case that the subgroups in a local conjugacy class are all conjugate if and only if they are only finitely many of them. Similar questions have been asked for certain subgroups of CC-groups (for example, Sylow subgroups and Carter subgroups [6, 8] and locally nilpotent injectors [7]). For CC-groups it is more typical that the local conjugacy class is countable, although in the special cases mentioned above results about the structure of the group \( G \) can also be obtained.

**Lemma 6.1.** Let \( G \) be a CC-group with \( G/Z \) periodic and let \( H \leq G \). If \( H^G/H_G \) is Černikov, then \( \text{Lcl } H = \text{Cl } H \).

**Proof.** Let \( \varphi \in \text{Linn } G \); then \( \varphi \) induces a locally inner automorphism \( \theta \) on \( G/H_G \). By Corollary 2.2, \( \theta \) coincides with an inner automorphism on \( H^G/H_G \) and so \( (H/H_G) \theta = H^x/H_G \), for some \( x \in G \). Hence \( H \varphi = H^x \) and so \( \text{Lcl } H = \text{Cl } H \).

The example following Corollary 2.2 shows that this result is false if \( G/Z \) is not periodic. In general, the converse of this result is false even for periodic FC-groups. Example 3.8 of [12] gives in uncountable extraspecial \( p \)-group containing a countably infinite elementary abelian subgroup \( H \) such that \( H_G = 1 \) and \( \text{Lcl } H = \text{Cl } H \).
However, the converse is true for residually Černikov CC-groups. We observe first some of the properties of residually Černikov CC-groups.

**Lemma 6.2.** If $G$ is a residually Černikov CC-group with centre $Z$, then $G/Z$ is periodic.

**Proof.** Let $x \in G$. Then $[G, x]$ is Černikov and so there is a normal subgroup $N$ of $G$ with $G/N$ Černikov and $[G, x] \cap N = 1$. There is an integer $n$ such that $x^n \in N$. Clearly $[G, x^n] \leq [G, x] \cap N = 1$ and so $x^n \in Z$. It follows that Lemma 6.1 can be applied to residually Černikov CC-groups. Also, if $H$ is a subgroup of a residually Černikov CC-group such that $H^G/H_G$ is not Černikov then Theorems 5.3 and 5.4 show that $|LcH| = 2|H/H_G|$. To prove the converse of Lemma 6.1 it is therefore sufficient to show that $|Cl H| \leq |H/H_G|$. This is obtained in a way similar to the arguments in [12], by showing that a residually Černikov CC-group satisfies a condition similar to the $Z$-groups defined there.

**Lemma 6.3.** If a CC-group is a subgroup of the cartesian product $\prod \{F_i : i \in I\}$ of Černikov groups $F_i$ and $I$ is infinite, then $|G/Z| \leq |I|$.

**Proof.** Let $A$ be the set of finite subsets of $I$ so that $|A| = |I|$ and, for each $\lambda \in A$, let $N_{\lambda} = \bigcap \{F_i : i \notin \lambda\}$ so that $G/N_\lambda$ is Černikov and $\bigcap \{N_\lambda : \lambda \in A\} = 1$. Let $C_\lambda/N_\lambda = Z(G/N_\lambda)$; then by Lemma 4.3 there is a finite subset $X$ of $G$ such that $G = \langle X^G \rangle C_\lambda$. Clearly $Z = Z(G) = C_\lambda(C_\lambda) \cap C_\lambda(X^G)$ and since $G/C_\lambda(X^G)$ is Černikov, we have $C_\lambda(C_\lambda)/Z$ is Černikov, for each $\lambda \in A$.

Let $g \in G$ and let $T$ be the torsion subgroup of $\langle g^G \rangle$. There is a $\lambda \in A$ such that $N_\lambda \cap T = 1$ and so $[C_\lambda, g] \leq N_\lambda \cap T = 1$ so that $g \in C_\lambda(C_\lambda)$. Thus $G/Z$ is generated by the Černikov subgroups $C_\lambda(C_\lambda)/Z$ and the result follows.

**Lemma 6.4.** Let $H$ be a subgroup of the residually Černikov CC-group $G$ with $|H| = \omega$, infinite. Then there is a normal subgroup $N$ of $G$ such that $H \cap N = 1$ and $|G/N| = \omega$.

**Proof.** For each $x \in H$ we choose a normal subgroup $M_x$ of $G$ such that $G/M_x$ is Černikov and $x \notin M_x$. Let $M = \cap \{M_x : x \in H\}$ so that $G/M$ is isomorphic to a subgroup of $\prod \{G/M_x : x \in H\}$. If $C/M = Z(G/M)$ then, by Lemma 6.3, $|G/C| \leq \omega$. Clearly $H \cap M = 1$ and $H \cong HM/M$. By considering the embedding of $C/M$ in its divisible hull it is clear that there is a subgroup $N/M$ of $C/M$ such that $|C/N| \leq \omega$ and $HM \cap N = M$. Then $N \leq G$, $|G/N| \leq \omega$, and $H \cap N = H \cap M = 1$, as required.
Corollary 6.5. Let $S$ be an infinite subset of the residually Černikov CC-group $G$. Then $|G:C_G(S)| \leq |S|$.

Proof. Let $H = S^G$. Then $|H| = |S|$ and, by Lemma 6.4, there is a normal subgroup $N$ of $G$ such that $H \cap N = 1$ and $|G/N| = |S|$. Clearly $N \leq C_G(H) \leq C_G(S)$ and so the result follows.

Corollary 6.6. Let $H$ be a subgroup of the residually Černikov CC-group $G$ such that $H^G/H_G$ is not Černikov. Then $|	ext{Cl } H| = |H/H_G|$.

Proof. We may write $H = SH_G$, where $|S| = |H/H_G|$ has to be infinite. It is clear that $C_G(S) \leq N_G(H)$ and so $|G:N_G(H)| \leq |S| = |H/H_G|$.

Conversely, let $T$ be a transversal to $N_G(H)$ in $G$. If $T$ were finite then $[G/N_G(H)]$ would satisfy the minimal condition and so $H^G/H_G$ would be Černikov. Therefore $T$ is infinite and $|G:C_G(T^G)| \leq |T|$. But $H \cap C_G(T^G) \leq H_G$ and so $|H/H_G| \leq |T| = |	ext{Cl } H|$.

Corollary 6.6 can be combined with Lemma 6.1 and the results of Section 5 to give the following equivalent conditions.

Theorem 6.7. Let $H$ be a subgroup of the residually Černikov CC-group $G$. Then the following are equivalent:

1. $\text{Lcl } H = \text{Cl } H$,
2. $H^G/H_G$ is Černikov,
3. $N_G(H) \geq C_G(X^G)$, for some finite set $X$,
4. $[G/N_G(H)]$ satisfies the minimal condition,
5. $\text{Lcl } H$ is countable.

Finally we give an example to show that condition (2) cannot be weakened to $H/H_G$ being Černikov and condition (5) cannot be weakened to $\text{Cl } H$ being countable.

Example 6.8. For each integer $m \geq 1$ let $C_m = \langle x_{m,i}^i \rangle i \geq 1 \rangle$ be a copy of the quasicyclic $p$-group, where $p$ is an odd prime, and put $C = \text{Dr } \{ C_m \mid m \geq 1 \}$. For each integer $n \geq 1$, there is an automorphism $\alpha_n$ of $C$ of order 2 defined by

$$(x_{m,i})^{\alpha_n} = \begin{cases} (x_{m,i})^{-1}, & m \leq n \\ (x_{m,i}), & m > n. \end{cases}$$

Let $A = \text{Dr } \{ \langle \alpha_n \rangle \mid n \geq 1 \}$ and form the split extension $G$ of $C$ by $A$; clearly $G$ is periodic and for each $x \in G$, $[G, x] \leq C_1 \cdots C_n$, for some $n$, and therefore $G$ is a CC-group. Moreover $Z(G) = 1$ and so $G$ is residually Černikov.
For each $m \geq 1$, we define $y_m = \prod \{ x_{i,m-1+i} \mid 1 \leq i \leq m \}$. Let $H$ be the subgroup generated by all the $y_m$: clearly $H$ is a quasicyclic $p$-subgroup of $G$. It is easy to show that $[H, x_1] = C_1$, $[H, \langle x_1, x_2 \rangle] = C_1 C_2$, etc. Therefore $[H, A] = C$ and so $H^G = C$ so that $H^G$ is not Černikov. Since $G$ is countable we must have $\text{Cl} H$ countable but by Theorem 6.7, $\text{Lcl} H \neq \text{Cl} H$.

In this example the automorphism of $H$ which inverts each one of the elements of $H$ is induced by a locally inner automorphism of $G$ which is not inner. Thus the conclusion of Corollary 2.2(b) is, of course, false for an arbitrary Černikov subgroup of a periodic CC-group.

REFERENCES