Algebras Generated by Reciprocals of Linear Forms

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Let $\Delta$ be a finite set of nonzero linear forms in several variables with coefficients in a field $K$ of characteristic zero. Consider the $K$-algebra $C(\Delta)$ of rational functions generated by $\{1/\alpha \mid \alpha \in \Delta\}$. Then the ring $\mathcal{A}(V)$ of differential operators with constant coefficients naturally acts on $C(\Delta)$. We study the graded $\mathcal{A}(V)$-module structure of $C(\Delta)$. We especially find standard systems of minimal generators and a combinatorial formula for the Poincaré series of $C(\Delta)$. Our proofs are based on a theorem by Brion–Vergne [4] and results by Orlik–Terao [9].

1. INTRODUCTION AND MAIN RESULTS

Let $V$ be a vector space of dimension $\ell$ over a field $K$ of characteristic zero. Let $\Delta$ be a finite subset of the dual space $V^*$ of $V$. We assume that $\Delta$ does not contain the zero vector and that no two vectors are proportional throughout this paper. Let $S = S(V^*)$ be the symmetric algebra of $V^*$. It is regarded as the algebra of polynomial functions on $V$. Let $S_{(0)}$ be the field of quotients of $S$, which is the field of rational functions on $V$.

**Definition 1.1.** Let $C(\Delta)$ be the $K$-subalgebra of $S_{(0)}$ generated by the set

$$\left\{ \frac{1}{\alpha} \mid \alpha \in \Delta \right\}.$$

Regard $C(\Delta)$ as a graded $K$-algebra with $\deg(1/\alpha) = 1$ for $\alpha \in \Delta$. 

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Definition 1.2. Let $\partial(V)$ be the $K$-algebra of differential operators with constant coefficients. Agree that the constant multiplications are in $\partial(V) : K \subset \partial(V)$.

If $x_1, \ldots, x_\ell$ are a basis for $V^*$, then $\partial(V)$ is isomorphic to the polynomial algebra $K[\partial/\partial x_1, \ldots, \partial/\partial x_\ell]$. Regard $\partial(V)$ as a graded $K$-algebra with $\deg(\partial/\partial x_i) = 1$ ($1 \leq i \leq \ell$). It naturally acts on $S_{(0)}$. We regard $C(\Delta)$ as a graded $\partial(V)$-module. In this paper we study the $\partial(V)$-module structure of $C(\Delta)$. In particular, we find systems of minimal generators (Theorem 1.1) and a combinatorial formula for the Poincaré (or Hilbert) series $\text{Poin}(C(\Delta), t)$ of $C(\Delta)$ (Theorem 1.2).

To present our results we need several definitions. Let $E_p(\Delta)$ be the set of all $p$-tuples composed of elements of $\Delta$. Let $E(\Delta) := \bigcup_{p \geq 0} E_p(\Delta)$. The union is disjoint. Write $\prod S := \alpha_1 \ldots \alpha_p \in S$ when $S = (\alpha_1, \ldots, \alpha_p) \in E_p(\Delta)$. Then one can write

$$C(\Delta) = \sum_{S \in E(\Delta)} K\left(\prod S\right)^{-1}.$$ 

Let

$$E'(\Delta) = \{ S \in E(\Delta) \mid S \text{ is linearly independent} \},$$

$$E^d(\Delta) = \{ S \in E(\Delta) \mid S \text{ is linearly dependent} \}.$$

Note that $S \in E^d(\Delta)$ if $S$ contains a repetition. In a special lecture at the Japan Mathematical Society in 1992, K. Aomoto suggested the study of the finite-dimensional graded $K$-vector space

$$AO(\Delta) := \sum_{S \in E'(\Delta)} K\left(\prod S\right)^{-1}.$$ 

Let

$$\mathcal{B}(\Delta) = \{ \ker(\alpha) \mid \alpha \in \Delta \}.$$ 

Then $\mathcal{B}(\Delta)$ is a (central) arrangement of hyperplanes [8] in $V$. K. Aomoto conjectured, when $K = \mathbb{R}$, that the dimension of $AO(\Delta)$ is equal to the number of connected components of

$$M(\mathcal{B}(\Delta)) := V \setminus \bigcup_{H \in \mathcal{B}(\Delta)} H.$$ 

This conjecture was verified in [9]; where explicit $K$-bases for $AO(\Delta)$ were constructed. This paper can be considered as a sequel to [9]. (It should be remarked that constructions in [9] were generalized for oriented matroids by R. Cordovil [5].) We will prove the following

Theorem 1.1. Let $\mathcal{B}$ be a $K$-basis for $AO(\Delta)$. Let $\partial(V)_+$ denote the maximal ideal of $\partial(V)$ generated by the homogeneous elements of degree 1.
Then

(1) the set $B$ is a system of minimal generators for the $\partial(V)$-module $C(\Delta)$,

(2) $C(\Delta) = \partial(V)_+C(\Delta) \oplus AO(\Delta)$, and

(3) $\partial(V)_+C(\Delta) = \sum_{x \in E_0(\Delta) \subseteq K(\prod K)^{-1}}$. In particular, $\partial(V)_+C(\Delta)$ is an ideal of $C(\Delta)$.

Let $\text{Poin}(\mathcal{A}(\Delta), t)$ be the Poincaré polynomial [8, Definition 2.48] of $\mathcal{A}(\Delta)$. (It is defined combinatorially and is known to be equal to the Poincaré polynomial of $M(\mathcal{A}(\Delta))$ when $K = C$ [7, 8, Theorem 5.93].) Then we have

**Theorem 1.2.** The Poincaré series $\text{Poin}(C(\Delta), t)$ of the graded module $C(\Delta)$ is equal to $\text{Poin}(\mathcal{A}(\Delta), (1 - t)^{-1}t)$.

To prove these theorems we essentially use a theorem by M. Brion and M. Vergne [4, Theorem 1] and results from [9]. By Theorem 1.2 and the factorization theorem (Theorem 2.4) in [12], we may easily show the following two corollaries:

**Corollary 1.1.** If $\mathcal{A}(\Delta)$ is a free arrangement with exponents $(d_1, \ldots, d_\ell)$ [12, 8, Definitions 4.15, 4.25], then

$$\text{Poin}(C(\Delta), t) = (1 - t)^{-\ell} \prod_{i=1}^{\ell} (1 + (d_i - 1)t).$$

**Example 1.1.** Let $x_1, \ldots, x_\ell$ be a basis for $V^\ast$. Let $\Delta = \{x_i - x_j | 1 \leq i < j \leq \ell\}$. Then $\mathcal{A}(\Delta)$ is known to be free arrangement with exponents $(0, 1, \ldots, \ell - 1)$ [8, Example 4.32]. So, by Corollary 1.1, we have

$$\text{Poin}(C(\Delta), t) = (1 - t)^{-\ell+1}(1 + t)(1 + 2t) \cdots (1 + (\ell - 2)t).$$

For example, when $\ell = 3$, we have

$$\text{Poin}(K \left[ \frac{1}{x_1 - x_2}, \frac{1}{x_2 - x_3}, \frac{1}{x_1 - x_3} \right], t) = (1 + t)/(1 - t)^2$$

$$= 1 + 3t + 5t^2 + 7t^3 + 9t^4 + \cdots,$$

which can be easily checked by direct computation.

When $\mathcal{A}(\Delta)$ is the set of reflecting hyperplanes of any (real or complex) reflection group, Corollary 1.5 can be applied because $\mathcal{A}(\Delta)$ is known to be a free arrangement [10, 13].

**Corollary 1.2.** If $\mathcal{A}(\Delta)$ is generic (i.e., $|\Delta| \geq \ell$, and any $\ell$ vectors in $\Delta$ are linearly independent), then

$$\text{Poin}(C(\Delta), t) = (1 - t)^{-\ell} \sum_{i=0}^{\ell-1} \binom{|\Delta| - \ell + i - 1}{i} t^i.$$
2. PROOFS

In this section we prove Theorems 1.1 and 1.2. For $\varepsilon \in \mathbb{E}(\Delta)$, let $V(\varepsilon)$ denote the set of common zeros of $\varepsilon$: $V(\varepsilon) = \bigcap_{i=1}^{p} \ker(\alpha_i)$ when $\varepsilon = (\alpha_1, \ldots, \alpha_p)$.

Define

$$L = L(\Delta) = \{ V(\varepsilon) \mid \varepsilon \in \mathbb{E}(\Delta) \}.$$ 

Agree that $V(\varepsilon) = V$ if $\varepsilon$ is the empty tuple. Introduce a partial order $\leq$ into $L$ by reverse inclusion: $X \leq Y \iff X \supset Y$. Then $L$ is equal to the intersection lattice of the arrangement $\mathcal{A}(\Delta)$ [8, Definition 2.1]. For $X \in L$, define

$$\mathbb{E}_X(\Delta) := \{ \varepsilon \in \mathbb{E}(\Delta) \mid V(\varepsilon) = X \}.$$ 

Then

$$\mathbb{E}(\Delta) = \bigcup_{X \in L} \mathbb{E}_X(\Delta) \quad \text{(disjoint).}$$

Define

$$\mathbb{C}_X(\Delta) := \sum_{\varepsilon \in \mathbb{E}_X(\Delta)} \mathbb{K}(\prod_{\varepsilon})^{-1}.$$ 

Then $\mathbb{C}_X(\Delta)$ is a $\partial(V)$-submodule of $\mathbb{C}(\Delta)$. The following theorem is equivalent to Lemma 3.2 in [9]. Our proof is a rephrasing of the proof there.

**Proposition 2.1.**

$$\mathbb{C}(\Delta) = \bigoplus_{X \in L} \mathbb{C}_X(\Delta).$$

**Proof.** It is obvious that $\mathbb{C}(\Delta) = \sum_{X \in L} \mathbb{C}_X(\Delta)$. Suppose that $\sum_{X \in L} \phi_X = 0$ with $\phi_X \in \mathbb{C}_X(\Delta)$. We will show that $\phi_X = 0$ for all $X \in L$. By taking out the degree $p$ part, we may assume that $\deg \phi_X = p$ for all $X \in L$. Let $\mathcal{P} = \{ X \in L \mid \phi_X \neq 0 \}$. Suppose $\mathcal{P}$ is not empty. Then there exists a minimal element $X_0$ in $\mathcal{P}$ (with respect to the partial order by reverse inclusion). Let $X \in \mathcal{P} \setminus \{ X_0 \}$ and write

$$\phi_X = \sum_{\varepsilon \in \mathbb{E}_X(\Delta)} c_{\varepsilon}(\prod_{\varepsilon})^{-1}$$

with $c_{\varepsilon} \in \mathbb{K}$. Let $\varepsilon \in \mathbb{E}_X(\Delta)$. Because of the minimality of $X_0$, one has $X_0 \not\subseteq X$. Thus there exists $\alpha_0 \in \varepsilon$ such that $X_0 \not\subseteq \ker(\alpha_0)$. Let $I(X_0)$ be the prime ideal of $S$ generated by the polynomial functions vanishing on $X_0$. Then $\alpha_0 \not\in I(X_0)$. Thus

$$(\prod_{\varepsilon})^p(\prod_{\varepsilon})^{-1} \in I(X_0)^{n|\Delta_0|-p+1},$$
where $\Delta := \prod_{\alpha \in \Delta} \alpha$ and $\Delta_{X_0} = \Delta \cap I(X_0)$. Multiply $(\prod \Delta)^p$ to both sides of
\[ \phi_{X_0} = -\sum_{X \in S} \phi_X \]
to get
\[ (\prod \Delta)^p \phi_{X_0} = -\sum_{X \in S} (\prod \Delta)^p \phi_X \]
\[ = -\sum_{X \in S} \sum_{e \in E(X)} c_e (\prod \Delta)^p (\prod e)^{-1} \in I(X_0)^{\mid \Delta_{X_0}\mid - p + 1}. \]

Since $(\prod \Delta)/(\prod \Delta_{X_0}) \in S \setminus I(X_0)$ and $I(X_0)^{\mid \Delta_{X_0}\mid - p + 1}$ is a primary ideal, one has
\[ (\prod \Delta_{X_0})^p \phi_{X_0} \in I(X_0)^{\mid \Delta_{X_0}\mid - p + 1}. \]

This is a contradiction because
\[ \deg (\prod \Delta_{X_0})^p \phi_{X_0} = p \mid \Delta_{X_0}\mid - p. \]
Therefore $\mathcal{E} = \phi$. \[ \square \]

Next we will study the structure of $\mathfrak{C}_X(\Delta)$ for each $X \in L$. Let $AO_X(\Delta)$ be the $K$-subspace of $AO(\Delta)$ generated over $K$ by
\[ \{(\prod e)^{-1} \mid e \in E(\Delta)' \cap E_X(\Delta)\}. \]
Then
\[ AO(\Delta) = \bigoplus_{X \in L} AO_X(\Delta) \]
by Proposition 2.1. Let $\mathcal{B}_X$ be a $K$-basis for $AO_X(\Delta)$. Then we have

**Proposition 2.2.** The $\partial(V)$-module $\mathfrak{C}_X(\Delta)$ can also be regarded as a free $\partial(V/X)$-module with a basis $\mathcal{B}_X$. In other words, there exists a natural graded isomorphism
\[ \partial(V/X) \otimes K AO_X(\Delta) \simeq \mathfrak{C}_X(\Delta). \]

**Proof.** First assume that $\Delta$ spans $V^*$ and $X = \{0\}$. Then $E(\Delta)' \cap E_X(\Delta)$ is equal to the set of $K$-bases for $V^*$, which are contained in $\Delta$. Thus $AO_X(\Delta)$ is generated over $K$ by
\[ \{(\prod e)^{-1} \mid e \in E(\Delta) \text{ is a basis for } V\}. \]
Similarly $C_X$ is spanned over $K$ by
\[
\{(\prod e)^{-1} \mid e \in E(\Delta) \text{ spans } V\}.
\]

Then Theorem 1 of [4] is exactly the desired result. Next let $X \in L$ and $\overline{V} = V/X$. Regard the dual vector space $V^*$ as a subspace of $V^*$ and the symmetric algebra $\overline{S} := S(V^*)$ of $V^*$ as a subring of $S$. Then $\Delta_X := I(X) \cap \Delta$ is a subset of $V^*$ and $\Delta_X$ spans $V^*$. Consider $AO(\Delta_X)$ and $C(\Delta_X)$, which are both contained in $\overline{S}(0)$. Note that $C_X(\Delta)$ can be regarded as a $\partial(\overline{V})$-module because $\partial(X)$ annihilates $C_X(\Delta)$. Denote the zero vector of $\overline{V}$ by $\overline{X}$. Then it is not difficult to see that
\[
C_X(\Delta_X) \simeq C_X(\Delta) \quad \text{as } \partial(\overline{V})\text{-modules},
\]
\[
AO_X(\Delta_X) \simeq AO_X(\Delta) \quad \text{as } K\text{-vector spaces}.
\]
Since there exists a natural graded isomorphism
\[
C_X(\Delta_X) \simeq \partial(\overline{V}) \otimes_{K} AO_X(\Delta_X),
\]
on one has
\[
C_X(\Delta) \simeq \partial(V/X) \otimes_{K} AO_X(\Delta).
\]

**Proof of Theorem 1.3.** By Proposition 2.2, $C_X(\Delta)$ is generated over $\partial(V)$ by $AO_X(\Delta)$. Since
\[
C(\Delta) = \bigoplus_{X \in L} C_X(\Delta) \quad \text{(Proposition 2.1),}
\]
and
\[
AO(\Delta) = \bigoplus_{X \in L} AO_X(\Delta),
\]
the $\partial(V)$-module $C(\Delta)$ is generated by $AO(\Delta)$. So $\otimes$ generates $C(\Delta)$ over $\partial(V)$. Define
\[
J(\Delta) := \sum_{e \in E(\Delta)} K(\prod e)^{-1},
\]
which is an ideal of $C(\Delta)$. Then it is known by [9, Theorem 4.2] that
\[
C(\Delta) = J(\Delta) \oplus AO(\Delta) \quad \text{as } K\text{-vector spaces}.
\]
It is obvious to see that
\[
\partial(V)_+ C(\Delta) \subseteq J(\Delta).
\]
On the other hand, we have
\[
C(\Delta) = \partial(V)AO(\Delta) = \partial(V) + AO(\Delta) + AO(\Delta)
= \partial(V), C(\Delta) + AO(\Delta).
\]
Combining these, we have (2) and (3) at the same time. By (2), we know that \(B\) minimally generates \(C(\Delta)\) over \(\partial(V)\), which is (1).

If \(M = \bigoplus_{p \geq 0} M_p\) is a graded vector space with \(\dim M_p < +\infty \ (p \geq 0)\), we let
\[
Poin(M, t) = \sum_{p=0}^{\infty} (\dim M_p)t^p
\]
be its Poincaré (or Hilbert) series. Recall [8, Sect. 2.42] the (one variable) Möbius function \(\mu: L(\Delta) \to \mathbb{Z}\) defined by \(\mu(V) = 1\) and for \(X > V\) by \(\sum_{Y \subseteq X} \mu(Y) = 0\). Then the Poincaré polynomial \(Poin(\partial(\Delta), t)\) of the arrangement \(\partial(\Delta)\) is defined by
\[
Poin(\partial(\Delta), t) = \sum_{X \in L} \mu(X)(-t)^{\codim X}.
\]

PROPOSITION 2.3 [9, Theorem 4.3]. For \(X \in L\) we have
\[
\dim AO_X(\Delta) = (-1)^{\codim X} \mu(X) \text{ and } Poin(AO(\Delta), t) = Poin(\partial(\Delta), t).
\]

Recall that \(C(\Delta)\) is a graded \(\partial(V)\)-module. Since \(C(\Delta)\) is infinite-dimensional, \(Poin(C(\Delta), t)\) is a formal power series. We now prove Theorem 1.2, which gives a combinatorial formula for \(Poin(C(\Delta), t)\).

Proof of Theorem 1.2. We have
\[
Poin(C(\Delta), t) = \sum_{X \in L} Poin(C_X(\Delta), t) = \sum_{X \in L} Poin(\partial(V/X), t)Poin(AO_X(\Delta), t)
\]
by Propositions 2.1 and 2.2. Since the \(K\)-algebra \(\partial(V/X)\) is isomorphic to the polynomial algebra with \(\codim X\) variables, we have
\[
Poin(C(\Delta), t) = \sum_{X \in L} (1 - t)^{-\codim X} Poin(AO_X(\Delta), t).
\]
By Proposition 2.3, we have
\[
Poin(AO_X(\Delta), t) = (-1)^{\codim X} \mu(X) t^{\codim X}.
\]
Thus
\[
Poin(C(\Delta), t) = \sum_{X \in L} (-1)^{\codim X} \mu(X) t^{\codim X (1 - t)^{-1} t}.
\]
Let \( \text{Der} \) be the \( S \)-module of derivations:
\[
\text{Der} = \{ \theta \mid \theta: S \to S \text{ is a } K\text{-linear derivations} \}.
\]

Then \( \text{Der} \) is naturally isomorphic to \( S \otimes_K V \). Define
\[
D(\Delta) = \{ \theta \in \text{Der} \mid \theta(\alpha) \in \alpha S \text{ for any } \alpha \in \Delta \},
\]
which is naturally an \( S \)-submodule of \( \text{Der} \). We say that the arrangement \( \mathcal{A}(\Delta) \) is free if \( D(\Delta) \) is a free \( S \)-module [8, Definition 4.15]. An element \( \theta \in D(\Delta) \) is said to be homogeneous of degree \( p \) if
\[
\theta(x) \in S_p \text{ for all } x \in V^*.
\]

When \( \mathcal{A}(V) \) is a free arrangement, let \( \theta_1, \ldots, \theta_\ell \) be a homogeneous basis for \( D(\Delta) \). The \( \ell \) nonnegative integers \( \deg \theta_1, \ldots, \deg \theta_\ell \) are called the exponents of \( \mathcal{A}(\Delta) \). Then one has

**Proposition 2.4 (Factorization Theorem [12], [8, Theorem 4.137]).** If \( \mathcal{A}(\Delta) \) is a free arrangement with exponents \( d_1, \ldots, d_\ell \), then
\[
\text{Poin}(\mathcal{A}(\Delta), t) = \prod_{i=1}^\ell (1 + d_it).
\]

By Theorem 1.2 and Proposition 2.4, we immediately have Corollary 1.1.

The arrangement \( \mathcal{A}(\Delta) \) is generic if \( |\Delta| \geq \ell \) and any \( \ell \) vectors in \( \Delta \) are linearly independent. In this case, it is easy to see that [8, Lemma 5.122]
\[
\text{Poin}(\mathcal{A}(\Delta), t) = (1 + t) \sum_{i=0}^{\ell-1} \binom{|\Delta| - 1}{i} t^i.
\]

**Proof of Corollary 1.2.** By Theorem 1.2, one has
\[
\text{Poin}(\mathcal{A}(\Delta), t) = \left( 1 + \frac{t}{1-t} \right)^{\ell-1} \sum_{i=0}^{\ell-1} \binom{|\Delta| - 1}{i} \left( \frac{t}{1-t} \right)^i
\]
\[
= (1-t)^{-\ell} \sum_{i=0}^{\ell-1} (1-t)^{\ell-i-1} \binom{|\Delta| - 1}{i} i^i
\]
\[
= (1-t)^{-\ell} \sum_{i=0}^{\ell-1} \binom{|\Delta| - 1}{i} t^i \sum_{j=0}^{\ell-i-1} \binom{\ell - i - 1}{j} (-1)^j t^j
\]
\[
= (1-t)^{-\ell} \sum_{k=0}^\ell t^k \sum_{j=0}^k (-1)^j \binom{|\Delta| - 1}{k-j} \binom{\ell - k + j - 1}{j}.
\]

On the other hand, we have
\[
\sum_{j=0}^k (-1)^j \binom{|\Delta| - 1}{k-j} \binom{\ell - k + j - 1}{j} = \binom{|\Delta| - \ell + k - 1}{k}.
\]
by equating the coefficients of \(x^k\) in \((1 + x)^{|\Delta| - \ell + k - 1}\) and \((1 + x)^{|\Delta| - 1}(1 + x)^{(\ell - k)}\). This proves the assertion.

We now consider the \(\text{nbc}\) (no broken circuit) bases \([1–3, 6, 9, \text{p. 72}].\) Suppose that \(\Delta\) is linearly ordered: \(\Delta = \{\alpha_1, \ldots, \alpha_n\}\). Let \(X \in L\) with \(\operatorname{codim} X = p\). Define

\[
\text{nbc}_X(\Delta) := \{e \in E_X(\Delta) \mid e = (\alpha_{i_1}, \ldots, \alpha_{i_p}), i_1 < \cdots < i_p, \text{contains no broken circuits}\}.
\]

Let \(\mathcal{B}_X = \{(\prod e)^{-1} \mid e \in \text{nbc}_X(\Delta)\}\) for \(X \in L\). Then we have

**Proposition 2.5** \([9, \text{Theorem 5.2}].\) Let \(X \in L\). The set \(\mathcal{B}_X\) is a \(K\)-basis for \(AO_X(\Delta)\).

Thanks to Propositions 2.1, 2.2, and 2.5 we easily have

**Proposition 2.6.** Let \(\mathcal{B} = \bigcup_{X \in L} \mathcal{B}_X = \{\phi_1, \ldots, \phi_m\}\). Write \(\operatorname{supp}(\phi_j) = X\) if \(\phi_j \in \mathcal{B}_X\). Then, for any \(\phi \in C(\Delta)\) and \(j \in \{1, \ldots, m\}\), there uniquely exists \(\theta_j \in \partial(V/\operatorname{supp}(\phi_j))\) such that

\[
\phi = \sum_{j=1}^{m} \theta_j(\phi_j).
\]

**Remark 2.1.** Suppose that \(\Delta\) spans \(V^*\) and that \(AO_{\{0\}}(\Delta) = \sum_{j=1}^{q} K\phi_j\), where \(q = |\mu(\{0\})|\). Then the mapping

\[
\phi \mapsto \sum_{j=1}^{q} \theta_j^{(0)}(\phi_j) \in AO_{\{0\}}(\Delta)
\]

is the restriction to \(C(\Delta)\) of the Jeffrey–Kirwan residue \([4, \text{Definition 6, 11}].\) Here \(\theta_j^{(0)}\) is the degree zero part of \(\theta_j\) \((j = 1, \ldots, q)\).

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