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Algebras Generated by Reciprocals of Linear Forms

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Let Δ be a finite set of nonzero linear forms in several variables with coefficients in a field $\mathbf{K}$ of characteristic zero. Consider the $\mathbf{K}$-algebra $C(\Delta)$ of rational functions generated by $\{1/\alpha \mid \alpha \in \Delta\}$. Then the ring $\partial(V)$ of differential operators with constant coefficients naturally acts on $C(\Delta)$. We study the graded $\partial(V)$-module structure of $C(\Delta)$. We especially find standard systems of minimal generators and a combinatorial formula for the Poincaré series of $C(\Delta)$. Our proofs are based on a theorem by Brion–Vergne [4] and results by Orlik–Terao [9].

1. INTRODUCTION AND MAIN RESULTS

Let V be a vector space of dimension ℓ over a field $\mathbf{K}$ of characteristic zero. Let Δ be a finite subset of the dual space V^* of V. We assume that Δ does not contain the zero vector and that no two vectors are proportional throughout this paper. Let $S = S(V^*)$ be the symmetric algebra of V^*. It is regarded as the algebra of polynomial functions on V. Let $S_{(0)}$ be the field of quotients of S, which is the field of rational functions on V.

Definition 1.1. Let $C(\Delta)$ be the $\mathbf{K}$-subalgebra of $S_{(0)}$ generated by the set

$$\left\{\frac{1}{\alpha} \mid \alpha \in \Delta\right\}.$$

Regard $C(\Delta)$ as a graded $\mathbf{K}$-algebra with $\deg(1/\alpha) = 1$ for $\alpha \in \Delta$.

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DEFINITION 1.2. Let $\partial(V)$ be the **K**-algebra of differential operators with constant coefficients. Agree that the constant multiplications are in $\partial(V)$: $\mathbf{K} \subset \partial(V)$.

If $x_1, \ldots, x_\ell$ are a basis for V^*, then $\partial(V)$ is isomorphic to the polynomial algebra $\mathbf{K}[\partial/\partial x_1, \ldots, \partial/\partial x_\ell]$. Regard $\partial(V)$ as a graded **K**-algebra with $\deg(\partial/\partial x_i) = 1$ $(1 \leq i \leq \ell)$. It naturally acts on $S_{(0)}$. We regard $C(\Delta)$ as a graded $\partial(V)$-module. In this paper we study the $\partial(V)$-module structure of $C(\Delta)$. In particular, we find systems of minimal generators (Theorem 1.1) and a combinatorial formula for the Poincaré (or Hilbert) series $\mathrm{Poin}(C(\Delta), t)$ of $C(\Delta)$ (Theorem 1.2).

To present our results we need several definitions. Let $\mathbf{E}_p(\Delta)$ be the set of all p-tuples composed of elements of Δ. Let $\mathbf{E}(\Delta) := \bigcup_{p\geq 0} \mathbf{E}_p(\Delta)$. The union is disjoint. Write $\prod \mathscr{E} := \alpha_1 \ldots \alpha_p \in S$ when $\mathscr{E} = (\alpha_1, \ldots, \alpha_p) \in \mathbf{E}_p(\Delta)$. Then one can write

$$C(\Delta) = \sum_{\mathscr{E}\in\mathbf{E}(\Delta)} \mathbf{K}\left(\prod \mathscr{E}\right)^{-1}.$$

Let

$$\mathbf{E}^i(\Delta) = \{\mathscr{E} \in \mathbf{E}(\Delta) \mid \mathscr{E} \text{ is linearly independent}\},$$
$$\mathbf{E}^d(\Delta) = \{\mathscr{E} \in \mathbf{E}(\Delta) \mid \mathscr{E} \text{ is linearly dependent}\}.$$

Note that $\mathscr{E} \in \mathbf{E}^d(\Delta)$ if $\mathscr{E}$ contains a repetition. In a special lecture at the Japan Mathematical Society in 1992, K. Aomoto suggested the study of the finite-dimensional graded **K**-vector space

$$AO(\Delta) := \sum_{\mathscr{E}\in\mathbf{E}^i(\Delta)} \mathbf{K}\left(\prod \mathscr{E}\right)^{-1}.$$

Let

$$\mathscr{A}(\Delta) = \{\ker(\alpha) \mid \alpha \in \Delta\}.$$

Then $\mathscr{A}(\Delta)$ is a (central) arrangement of hyperplanes [8] in V. K. Aomoto conjectured, when $\mathbf{K} = \mathbf{R}$, that the dimension of $AO(\Delta)$ is equal to the number of connected components of

$$M(\mathscr{A}(\Delta)) := V \setminus \bigcup_{H\in\mathscr{A}(\Delta)} H.$$

This conjecture was verified in [9]; where explicit **K**-bases for $AO(\Delta)$ were constructed. This paper can be considered as a sequel to [9]. (It should be remarked that constructions in [9] were generalized for oriented matroids by R. Cordovil [5].) We will prove the following

THEOREM 1.1. *Let $\mathscr{B}$ be a* **K**-*basis for* $AO(\Delta)$. *Let $\partial(V)_+$ denote the maximal ideal of $\partial(V)$ generated by the homogeneous elements of degree* 1.

Then

(1) *the set* $\mathcal{B}$ *is a system of minimal generators for the* $\partial(V)$*-module* $C(\Delta)$,

(2) $C(\Delta) = \partial(V)_+ C(\Delta) \oplus AO(\Delta)$, *and*

(3) $\partial(V)_+ C(\Delta) = \sum_{\mathcal{E} \in \mathbf{E}^d(\Delta)} \mathbf{K}(\prod \mathcal{E})^{-1}$. *In particular,* $\partial(V)_+ C(\Delta)$ *is an ideal of* $C(\Delta)$.

Let $\text{Poin}(\mathcal{A}(\Delta), t)$ be the Poincaré polynomial [8, Definition 2.48] of $\mathcal{A}(\Delta)$. (It is defined combinatorially and is known to be equal to the Poincaré polynomial of $M(\mathcal{A}(\Delta))$ when $\mathbf{K} = \mathbf{C}$ [7, 8, Theorem 5.93].) Then we have

THEOREM 1.2. *The Poincaré series* $\text{Poin}(C(\Delta), t)$ *of the graded module* $C(\Delta)$ *is equal to* $\text{Poin}(\mathcal{A}(\Delta), (1-t)^{-1}t)$.

To prove these theorems we essentially use a theorem by M. Brion and M. Vergne [4, Theorem 1] and results from [9]. By Theorem 1.2 and the factorization theorem (Theorem 2.4) in [12], we may easily show the following two corollaries:

COROLLARY 1.1. *If* $\mathcal{A}(\Delta)$ *is a free arrangement with exponents* $(d_1, \ldots, d_\ell)$ [12, 8, *Definitions* 4.15, 4.25], *then*

$$\text{Poin}(C(\Delta), t) = (1-t)^{-\ell} \prod_{i=1}^{\ell} \{1 + (d_i - 1)t\}.$$

EXAMPLE 1.1. Let $x_1, \ldots, x_\ell$ be a basis for V^*. Let $\Delta = \{x_i - x_j \mid 1 \leq i < j \leq \ell\}$. Then $\mathcal{A}(\Delta)$ is known to be free arrangement with exponents $(0, 1, \ldots, \ell - 1)$ [8, Example 4.32]. So, by Corollary 1.1, we have

$$\text{Poin}(C(\Delta), t) = (1-t)^{-\ell+1}(1+t)(1+2t)\cdots(1+(\ell-2)t).$$

For example, when $\ell = 3$, we have

$$\begin{aligned} \text{Poin}\left(\mathbf{K}\left[\frac{1}{x_1 - x_2}, \frac{1}{x_2 - x_3}, \frac{1}{x_1 - x_3}\right], t\right) &= (1+t)/(1-t)^2 \\ &= 1 + 3t + 5t^2 + 7t^3 + 9t^4 + \cdots, \end{aligned}$$

which can be easily checked by direct computation.

When $\mathcal{A}(\Delta)$ is the set of reflecting hyperplanes of any (real or complex) reflection group, Corollary 1.5 can be applied because $\mathcal{A}(\Delta)$ is known to be a free arrangement [10, 13].

COROLLARY 1.2. *If* $\mathcal{A}(\Delta)$ *is generic* (*i.e.*, $|\Delta| \geq \ell$, *and any* ℓ *vectors in* Δ *are linearly independent*), *then*

$$\text{Poin}(C(\Delta), t) = (1-t)^{-\ell} \sum_{i=0}^{\ell-1} \binom{|\Delta| - \ell + i - 1}{i} t^i.$$

2. PROOFS

In this section we prove Theorems 1.1 and 1.2. For $\varepsilon \in \mathbf{E}(\Delta)$, let $V(\varepsilon)$ denote the set of common zeros of ε: $V(\varepsilon) = \bigcap_{i=1}^{p} \ker(\alpha_i)$ when $\varepsilon = (\alpha_1, \ldots, \alpha_p)$.
Define

$$L = L(\Delta) = \{V(\varepsilon) \mid \varepsilon \in \mathbf{E}(\Delta)\}.$$

Agree that $V(\varepsilon) = V$ if ε is the empty tuple. Introduce a partial order $\leq$ into L by reverse inclusion: $X \leq Y \Leftrightarrow X \supseteq Y$. Then L is equal to the intersection lattice of the arrangement $\mathscr{A}(\Delta)$ [8, Definition 2.1]. For $X \in L$, define

$$\mathbf{E}_X(\Delta) := \{\varepsilon \in \mathbf{E}(\Delta) \mid V(\varepsilon) = X\}.$$

Then

$$\mathbf{E}(\Delta) = \bigcup_{X \in L} \mathbf{E}_X(\Delta) \quad \text{(disjoint)}.$$

Define

$$C_X(\Delta) := \sum_{\varepsilon \in \mathbf{E}_X(\Delta)} \mathbf{K}\left(\prod \varepsilon\right)^{-1}.$$

Then $C_X(\Delta)$ is a $\partial(V)$-submodule of $C(\Delta)$. The following theorem is equivalent to Lemma 3.2 in [9]. Our proof is a rephrasing of the proof there.

PROPOSITION 2.1.

$$C(\Delta) = \bigoplus_{X \in L} C_X(\Delta).$$

Proof. It is obvious that $C(\Delta) = \sum_{X \in L} C_X(\Delta)$. Suppose that $\sum_{X \in L} \phi_X = 0$ with $\phi_X \in C_X(\Delta)$. We will show that $\phi_X = 0$ for all $X \in L$. By taking out the degree p part, we may assume that $\deg \phi_X = p$ for all $X \in L$. Let $\mathscr{S} = \{X \in L \mid \phi_X \neq 0\}$. Suppose $\mathscr{S}$ is not empty. Then there exists a minimal element X_0 in $\mathscr{S}$ (with respect to the partial order by reverse inclusion). Let $X \in \mathscr{S} \setminus \{X_0\}$ and write

$$\phi_X = \sum_{\varepsilon \in \mathbf{E}_X(\Delta)} c_\varepsilon \left(\prod \varepsilon\right)^{-1}$$

with $c_\varepsilon \in \mathbf{K}$. Let $\varepsilon \in \mathbf{E}_X(\Delta)$. Because of the minimality of X_0, one has $X_0 \not\subseteq X$. Thus there exists $\alpha_0 \in \varepsilon$ such that $X_0 \not\subseteq \ker(\alpha_0)$. Let $I(X_0)$ be the prime ideal of S generated by the polynomial functions vanishing on X_0. Then $\alpha_0 \notin I(X_0)$. Thus

$$\left(\prod \Delta\right)^p \left(\prod \varepsilon\right)^{-1} \in I(X_0)^{p|\Delta_{X_0}|-p+1},$$

where $\prod \Delta := \prod_{\alpha \in \Delta} \alpha$ and $\Delta_{X_0} = \Delta \cap I(X_0)$. Multiply $(\prod \Delta)^p$ to both sides of

$$\phi_{X_0} = - \sum_{\substack{X \in S \\ X \neq X_0}} \phi_X$$

to get

$$\begin{aligned}(\textstyle\prod \Delta)^p \phi_{X_0} &= - \sum_{\substack{X \in S \\ X \neq X_0}} (\textstyle\prod \Delta)^p \phi_X \\ &= - \sum_{\substack{X \in S \\ X \neq X_0}} \sum_{\varepsilon \in \mathbf{E}_X(\Delta)} c_\varepsilon (\textstyle\prod \Delta)^p (\textstyle\prod \varepsilon)^{-1} \in I(X_0)^{p|\Delta_{X_0}| - p + 1}.\end{aligned}$$

Since $(\prod \Delta)/(\prod \Delta_{X_0}) \in S \backslash I(X_0)$ and $I(X_0)^{p|\Delta_{X_0}|-p+1}$ is a primary ideal, one has

$$(\textstyle\prod \Delta_{X_0})^p \phi_{X_0} \in I(X_0)^{p|\Delta_{X_0}|-p+1}.$$

This is a contradiction because

$$\deg(\textstyle\prod \Delta_{X_0})^p \phi_{X_0} = p|\Delta_{X_0}| - p.$$

Therefore $\mathcal{S} = \phi$. ∎

Next we will study the structure of $C_X(\Delta)$ for each $X \in L$. Let $AO_X(\Delta)$ be the $\mathbf{K}$-subspace of $AO(\Delta)$ generated over $\mathbf{K}$ by

$$\{(\textstyle\prod \varepsilon)^{-1} \mid \varepsilon \in \mathbf{E}(\Delta)^i \cap \mathbf{E}_X(\Delta)\}.$$

Then

$$AO(\Delta) = \bigoplus_{X \in L} AO_X(\Delta)$$

by Proposition 2.1. Let $\mathcal{B}_X$ be a $\mathbf{K}$-basis for $AO_X(\Delta)$. Then we have

PROPOSITION 2.2. *The $\partial(V)$-module $C_X(\Delta)$ can also be regarded as a free $\partial(V/X)$-module with a basis $\mathcal{B}_X$. In other words, there exists a natural graded isomorphism*

$$\partial(V/X) \bigotimes_{\mathbf{K}} AO_X(\Delta) \simeq C_X(\Delta).$$

Proof. First assume that Δ spans V^* and $X = \{\mathbf{0}\}$. Then $\mathbf{E}(\Delta)^i \cap \mathbf{E}(\Delta)_X$ is equal to the set of $\mathbf{K}$-bases for V^*, which are contained in Δ. Thus $AO_X(\Delta)$ is generated over $\mathbf{K}$ by

$$\{(\textstyle\prod \varepsilon)^{-1} \mid \varepsilon \in \mathbf{E}_\ell(\Delta) \text{ is a basis for } V\}.$$

Similarly C_X is spanned over $\mathbf{K}$ by

$$\left\{ \left(\prod \varepsilon \right)^{-1} \mid \varepsilon \in \mathbf{E}(\Delta) \text{ spans } V \right\}.$$

Then Theorem 1 of [4] is exactly the desired result. Next let $X \in L$ and $\overline{V} = V/X$. Regard the dual vector space $\overline{V}^*$ as a subspace of V^* and the symmetric algebra $\overline{S} := S(\overline{V}^*)$ of $\overline{V}^*$ as a subring of S. Then $\Delta_X := I(X) \cap \Delta$ is a subset of $\overline{V}^*$ and Δ_X spans $\overline{V}^*$. Consider $AO(\Delta_X)$ and $C(\Delta_X)$, which are both contained in $\overline{S}_{(0)}$. Note that $C_X(\Delta)$ can be regarded as a $\partial(V/X)$-module because $\partial(X)$ annihilates $C_X(\Delta)$. Denote the zero vector of $\overline{V}$ by $\overline{X}$. Then it is not difficult to see that

$$C_{\overline{X}}(\Delta_X) \simeq C_X(\Delta) \quad (\text{as } \partial(\overline{V})\text{-modules}),$$
$$AO_{\overline{X}}(\Delta_X) \simeq AO_X(\Delta) \quad (\text{as } \mathbf{K}\text{-vector spaces}).$$

Since there exists a natural graded isomorphism

$$C_{\overline{X}}(\Delta_X) \simeq \partial(\overline{V}) \bigotimes_{\mathbf{K}} AO_{\overline{X}}(\Delta_X),$$

one has

$$C_X(\Delta) \simeq \partial(V/X) \bigotimes_{\mathbf{K}} AO_X(\Delta).$$

∎

Proof of Theorem 1.3. By Proposition 2.2, $C_X(\Delta)$ is generated over $\partial(V)$ by $AO_X(\Delta)$. Since

$$C(\Delta) = \bigoplus_{X \in L} C_X(\Delta) \quad (\text{Proposition 2.1}),$$

and

$$AO(\Delta) = \bigoplus_{X \in L} AO_X(\Delta),$$

the $\partial(V)$-module $C(\Delta)$ is generated by $AO(\Delta)$. So $\mathscr{B}$ generates $C(\Delta)$ over $\partial(V)$. Define

$$J(\Delta) := \sum_{\varepsilon \in \mathbf{E}^d(\Delta)} \mathbf{K}\left(\prod \varepsilon \right)^{-1},$$

which is an ideal of $C(\Delta)$. Then it is known by [9, Theorem 4.2] that

$$C(\Delta) = J(\Delta) \oplus AO(\Delta) \quad (\text{as } \mathbf{K}\text{-vector spaces}).$$

It is obvious to see that

$$\partial(V)_+ C(\Delta) \subseteq J(\Delta).$$

On the other hand, we have

$$C(\Delta) = \partial(V)AO(\Delta) = \partial(V)_+ AO(\Delta) + AO(\Delta)$$
$$= \partial(V)_+ C(\Delta) + AO(\Delta).$$

Combining these, we have (2) and (3) at the same time. By (2), we know that $\mathscr{B}$ minimally generates $C(\Delta)$ over $\partial(V)$, which is (1). ∎

If $M = \bigoplus_{p\geq 0} M_p$ is a graded vector space with $\dim M_p < +\infty$ $(p \geq 0)$, we let

$$\mathrm{Poin}(M, t) = \sum_{p=0}^{\infty} (\dim M_p) t^p$$

be its *Poincaré (or Hilbert) series*. Recall [8, Sect. 2.42] the (one variable) Möbius function μ: $L(\Delta) \to \mathbf{Z}$ defined by $\mu(V) = 1$ and for $X > V$ by $\sum_{Y \leq X} \mu(Y) = 0$. Then the *Poincaré polynomial* $\mathrm{Poin}(\mathscr{A}(\Delta), t)$ of the arrangement $\mathscr{A}(\Delta)$ is defined by

$$\mathrm{Poin}(\mathscr{A}(\Delta), t) = \sum_{X \in L} \mu(X)(-t)^{\operatorname{codim} X}.$$

PROPOSITION 2.3 [9, Theorem 4.3]. *For $X \in L$ we have*

$$\dim AO_X(\Delta) = (-1)^{\operatorname{codim} X} \mu(X) \text{ and } \mathrm{Poin}(AO(\Delta), t) = \mathrm{Poin}(\mathscr{A}(\Delta), t).$$

Recall that $C(\Delta)$ is a graded $\partial(V)$-module. Since $C(\Delta)$ is infinite-dimensional, $\mathrm{Poin}(C(\Delta), t)$ is a formal power series. We now prove Theorem 1.2, which gives a combinatorial formula for $\mathrm{Poin}(C(\Delta), t)$.

Proof of Theorem 1.2. We have

$$\mathrm{Poin}(C(\Delta), t) = \sum_{X \in L} \mathrm{Poin}(C_X(\Delta), t)$$
$$= \sum_{X \in L} \mathrm{Poin}(\partial(V/X), t) \mathrm{Poin}(AO_X(\Delta), t)$$

by Propositions 2.1 and 2.2. Since the $\mathbf{K}$-algebra $\partial(V/X)$ is isomorphic to the polynomial algebra with codim X variables, we have

$$\mathrm{Poin}(C(\Delta), t) = \sum_{X \in L} (1-t)^{-\operatorname{codim} X} \mathrm{Poin}(AO_X(\Delta), t).$$

By Proposition 2.3, we have

$$\mathrm{Poin}(AO_X(\Delta), t) = (-1)^{\operatorname{codim} X} \mu(X) t^{\operatorname{codim} X}.$$

Thus

$$\mathrm{Poin}(C(\Delta), t) = \sum_{X \in L} (-1)^{\operatorname{codim} X} \mu(X) \left(\frac{t}{1-t}\right)^{\operatorname{codim} X}$$
$$= \mathrm{Poin}(\mathscr{A}(\Delta), (1-t)^{-1} t).$$

∎

Let Der be the S-module of derivations:

$$\mathrm{Der} = \{\theta \mid \theta\colon S \to S \text{ is a } \mathbf{K}\text{-linear derivations}\}.$$

Then Der is naturally isomorphic to $S \bigotimes_{\mathbf{K}} V$. Define

$$D(\Delta) = \{\theta \in \mathrm{Der} \mid \theta(\alpha) \in \alpha S \text{ for any } \alpha \in \Delta\},$$

which is naturally an S-submodule of Der. We say that the arrangement $\mathscr{A}(\Delta)$ is *free* if $D(\Delta)$ is a free S-module [8, Definition 4.15]. An element $\theta \in D(\Delta)$ is said to be *homogeneous of degree* p if

$$\theta(x) \in S_p \text{ for all } x \in V^*.$$

When $\mathscr{A}(V)$ is a free arrangement, let $\theta_1, \dots, \theta_\ell$ be a homogeneous basis for $D(\Delta)$. The ℓ nonnegative integers $\deg \theta_1, \dots, \deg \theta_\ell$ are called the *exponents* of $\mathscr{A}(\Delta)$. Then one has

PROPOSITION 2.4 (Factorization Theorem [12], [8, Theorem 4.137]). *If $\mathscr{A}(\Delta)$ is a free arrangement with exponents $d_1, \dots, d_\ell$, then*

$$\mathrm{Poin}(A(\Delta), t) = \prod_{i=1}^{\ell}(1 + d_i t).$$

By Theorem 1.2 and Proposition 2.4, we immediately have Corollary 1.1.

The arrangement $\mathscr{A}(\Delta)$ is *generic* if $|\Delta| \geq \ell$ and any ℓ vectors in Δ are linearly independent. In this case, it is easy to see that [8, Lemma 5.122]

$$\mathrm{Poin}(\mathscr{A}(\Delta), t) = (1 + t) \sum_{i=0}^{\ell-1} \binom{|\Delta| - 1}{i} t^i.$$

Proof of Corollary 1.2. By Theorem 1.2, one has

$$\begin{aligned}
\mathrm{Poin}(C(\Delta), t) &= \left(1 + \frac{t}{1-t}\right) \sum_{i=0}^{\ell-1} \binom{|\Delta| - 1}{i} \left(\frac{t}{1-t}\right)^i \\
&= (1-t)^{-\ell} \sum_{i=0}^{\ell-1} (1-t)^{\ell-i-1} \binom{|\Delta| - 1}{i} t^i \\
&= (1-t)^{-\ell} \sum_{i=0}^{\ell-1} \binom{|\Delta| - 1}{i} t^i \sum_{j=0}^{\ell-i-1} \binom{\ell - i - 1}{j} (-1)^j t^j \\
&= (1-t)^{-\ell} \sum_{k=0}^{\ell-1} t^k \sum_{j=0}^{k} (-1)^j \binom{|\Delta| - 1}{k-j} \binom{\ell - k + j - 1}{j}.
\end{aligned}$$

On the other hand, we have

$$\sum_{j=0}^{k} (-1)^j \binom{|\Delta| - 1}{k-j} \binom{\ell - k + j - 1}{j} = \binom{|\Delta| - \ell + k - 1}{k}$$

by equating the coefficients of x^k in $(1+x)^{|\Delta|-\ell+k-1}$ and $(1+x)^{|\Delta|-1}(1+x)^{-(\ell-k)}$. This proves the assertion. ∎

We now consider the **nbc** (no broken circuit) bases [1–3, 6, 9, p. 72]. Suppose that Δ is linearly ordered: $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Let $X \in L$ with codim $X = p$. Define

$$\mathbf{nbc}_X(\Delta) := \{\varepsilon \in \mathbf{E}_X(\Delta) \mid \varepsilon = (\alpha_{i_1}, \ldots, \alpha_{i_p}), i_1 < \cdots < i_p, \text{ contains no broken circuits}\}.$$

Let $\mathscr{B}_X = \{(\prod \varepsilon)^{-1} \mid \varepsilon \in \mathbf{nbc}_X(\Delta)\}$ for $X \in L$. Then we have

PROPOSITION 2.5 [9, Theorem 5.2]. *Let $X \in L$. The set $\mathscr{B}_X$ is a* **K**-*basis for $AO_X(\Delta)$.*

Thanks to Propositions 2.1, 2.2, and 2.5 we easily have

PROPOSITION 2.6. *Let $\mathscr{B} = \bigcup_{X \in L} \mathscr{B}_X = \{\phi_1, \ldots, \phi_m\}$. Write* supp$(\phi_i) = X$ *if $\phi_i \in \mathscr{B}_X$. Then, for any $\phi \in C(\Delta)$ and $j \in \{1, \ldots, m\}$, there uniquely exists $\theta_j \in \partial(V/\text{supp}(\phi_j))$ such that*

$$\phi = \sum_{j=1}^{m} \theta_j(\phi_j).$$

Remark 2.1. Suppose that Δ spans V^* and that $AO_{\{0\}}(\Delta) = \sum_{j=1}^{q} \mathbf{K}\phi_j$, where $q = |\mu(\{0\})|$. Then the mapping

$$\phi \mapsto \sum_{j=1}^{q} \theta_j^{(0)}(\phi_j) \in AO_{\{0\}}(\Delta)$$

is the restriction to $C(\Delta)$ of the Jeffrey–Kirwan residue [4, Definition 6, 11]. Here $\theta_j^{(0)}$ is the degree zero part of θ_j $(j = 1, \ldots, q)$.

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