ELSEVIER

Discrete Mathematics 207 (1999) 277–284

DISCRETE MATHEMATICS

Note

Transversals in uniform hypergraphs with property (7,2)

Dmitry G. Fon-Der-Flaass[a, 1], Alexandr V. Kostochka[a, *,2], Douglas R. Woodall[b]

[a] *Institute of Mathematics, Novosibirsk, 630090, Russia*

[b] *School of Mathematical Sciences, University of Nottingham, Nottingham, NG7 2RD, UK*

Received 2 September 1998; accepted 22 December 1998

Abstract

Let $f(r,p,t)$ $(p>t\geqslant 1,\ r\geqslant 2)$ be the maximum of the cardinality of a minimum transversal over all r-uniform hypergraphs $\mathscr{H}$ possessing the property that every subhypergraph of $\mathscr{H}$ with p edges has a transversal of size t. The values of $f(r,p,2)$ for $p=3,4,5,6$ were found in Erdős et al. (Siberian Adv. Math. 2 (1992) 82–88). We give bounds on $f(r,7,2)$, partially answering a question in Erdős et al. (1992). © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Transversal; Uniform hypergraph

1. Introduction

A *transversal* of a family $\mathscr{F}$ of sets is a subset of $\bigcup_{F\in\mathscr{F}} F$ meeting all members of $\mathscr{F}$. The smallest cardinality $\tau(\mathscr{F})$ of a transversal of $\mathscr{F}$ is called the *transversal number* of $\mathscr{F}$. For a hypergraph $\mathscr{H}=(V,\mathscr{E})$, a *transversal* is a transversal of $\mathscr{E}$.

Say that $\mathscr{B}$ *possesses the property* (p,t) if $\tau(\mathscr{F})\leqslant t$ for every $\mathscr{F}\subset\mathscr{B}$ with $|\mathscr{F}|=p$. Erdős, Hajnal and Tuza [2] raised the following problem:

For given integers r, p, and t $(p>t\geqslant 1,\ r\geqslant 2)$, *determine the largest value,* $f(r,p,t)$, *of* $\tau(\mathscr{F})$ *taken over the class of r-uniform families* $\mathscr{F}$ *possessing the property* (p,t).

* Corresponding author.

E-mail address: sasha@math.nsc.ru (A.V. Kostochka)

[1] The work of this author was partly supported by the grant 96-01-01614 of the Russian Foundation for Fundamental Research.

[2] This work was carried out while the second author was visiting Nottingham, funded by Visiting Fellowship Research Grant GR/L54585 from the Engineering and Physical Sciences Research Council. The work of this author was also partly supported by grants 96-01-01614 and 97-01-01075 of the Russian Foundation for Fundamental Research.

PII: S0012-365X(99)00114-4

Erdős, Fon-Der-Flaass, Kostochka, and Tuza [1] found some bounds for $f(r,p,t)$ in general and determined the exact values of $f(r,p,2)$ for $3 \leqslant p \leqslant 6$. In particular, $f(r,6,2)=r$.

In this note we give bounds for $f(r,7,2)$. For $3 \leqslant p \leqslant 6$, the extremal hypergraphs were the maximal (w.r.t. the number of vertices) complete r-uniform hypergraphs possessing the property (p,t). It appears that $p=7$ is the first number such that this is not the case. We will prove that for $k \geqslant 10$, $f(4k,7,2) \geqslant 3k+1$ while the worst complete $4k$-uniform hypergraph possessing property $(7,2)$ has transversal number $3k$. We will also show that $f(r,7,2) \leqslant \lceil \frac{7r}{8} \rceil$. Note that if the lower bound holds for $k=2$, then the upper bound is exact for $r=8$.

2. Lower bound

Clearly, the transversal number of the complete r-uniform hypergraph $K_t(r)$ on t vertices is $t-r+1$. Observe that $K_{7k-1}(4k)$ possesses property $(7,2)$. Indeed, let $A_1,\dots,A_7$ be arbitrary $4k$-element subsets of the ground set $U=\{1,\dots,7k-1\}$. Since $\sum_{i=1}^{7} |A_i| = 28k > 4|U|$, there exists $u_1 \in U$ belonging to at least five sets A_i. Since $7k-1 < 2(4k)$, there exists u_2 meeting the sets A_j not containing u_1. It follows that $K_{7k-1}(4k)$ possesses property $(7,2)$ and $f(4k,7,2) \geqslant (7k-1)-4k+1=3k$.

On the other hand, $K_{7k}(4k)$ does not possess property $(7,2)$. Indeed, let $F_1,\dots,F_7$ be the family of complements of lines in a Fano plane on $V=\{1,\dots,7\}$. Then no two points meet every F_i. Blowing up every element of V into k elements, we produce a family $\{F_1',\dots,F_7'\}$ of seven subsets of $V'=\{1,\dots,7k\}$ with transversal number three. Thus although $K_{7k}(4k)$ has transversal number $7k-4k+1=3k+1$, it does not have property $(7,2)$. In this section, we exhibit a family $\mathscr{B}$ of $4k$-element sets possessing property $(7,2)$ with transversal number $3k+1$.

Theorem 1. *Let* $k \geqslant 10$, $|S| = 7k+1$ *and* $X \subset S$, $|X| = 4k$. *Let* $\mathscr{B}$ *be the family of* $4k$*-element subsets of* S *whose intersection with* X *has an odd cardinality. Then* $\mathscr{B}$ *possesses property* $(7,2)$ *and* $\tau(\mathscr{B})=3k+1$.

Remark. By elaborating the arguments of Claims 3 and 4 below, one can prove that the theorem holds already for $k \geqslant 4$. But for $k=2$, $\mathscr{B}$ does not possess property (7,2).

Proof of Theorem 1. Clearly, $S \backslash X$ is a transversal of $\mathscr{B}$ of cardinality $3k+1$. On the other hand, let T be an arbitrary subset of S with $|T|=3k$. Then there is $a \in (S\backslash T) \cap X$ and $b \in (S \setminus T) \backslash X$. Thus, either $(S \setminus T) - a \in \mathscr{B}$ or $(S \setminus T) - b \in \mathscr{B}$. It follows that $\tau(\mathscr{B}) = 3k+1$.

Below in a series of claims we will prove that $\mathscr{B}$ possesses property (7,2).

Let $\mathscr{A} = \{A_1,\dots,A_7\}$ be an arbitrary family of seven members of $\mathscr{B}$. Assume that $\tau(\mathscr{A}) > 2$. Below we will derive the properties of such an $\mathscr{A}$ which finally will produce a contradiction.

Claim 1. *Every element of S belongs to at most four members of $\mathscr{A}$.*

Proof. Assume that, say, a covers $A_1,\dots,A_5$. There exists $b\in A_6\cap A_7$. Then $\{a,b\}$ is a transversal of $\mathscr{A}$, a contradiction.

Since $|A_1|+\cdots+|A_7|=28k$, we conclude from Claim 1 that almost every element of S (with at most four exceptions) has degree four in $\mathscr{A}$. We shall call such elements *standard*; denote the set of standard vertices by St, and $S\backslash\mathrm{St}$ by $\overline{\mathrm{St}}$. The sequence of the degrees of vertices in $\overline{\mathrm{St}}$ must be one of the following: (a) 3,3,3,3; (b) 3,3,2; (c) 3,1; (d) 2,2; (e) 0.

To shorten notation, we set $\overline{Y}=S\backslash Y$, $A_{ij}=A_i\cap A_j$ and $A^{ij}=A_i\cup A_j$.

Claim 2. *If there exists $a\in\overline{A^{ij}}\cap\mathrm{St}$ such that $a\notin A_k$ $(k\neq i,j)$, then $A_k\cap A_{ij}=\emptyset$.*

Proof. If $b\in A_k\cap A_{ij}$, then $\{a,b\}$ is a transversal of $\mathscr{A}$, a contradiction.

Claim 3. *For any $i\neq j$, $|A_{ij}|\leqslant 2k+2$. Furthermore,*
(i) *if $|A_{ij}|=2k+2$, then $A^{ij}\subseteq\mathrm{St}$ and there exists A_k with $A_{ik}=A_i\backslash A_{ij}$ and $|A_{ik}|=2k-2$;*
(ii) *if $|A_{ij}|=2k+1$, then $|\overline{\mathrm{St}}\backslash A^{ij}|\geqslant 2$ and there exists A_k with $2k-3\leqslant|A_{ik}|\leqslant 2k-1$.*

Proof. Assume that $m=\max\{|A_{ij}|\}$, $|A_{1,2}|\geqslant m-1$ and $m\geqslant 2k+2$. Then $|\overline{A^{1,2}}|\geqslant 7k+1-(8k-(m-1))=m-k\geqslant k+2$ and there exists $a\in\overline{A^{1,2}}\cap\mathrm{St}$. W.l.o.g., suppose that $a\in A_3\cap A_4\cap A_5\cap A_6$. Then, by Claim 2, $A_7\cap A_{1,2}=\emptyset$.

If there exists $b\in\overline{A^{1,2}}\cap\mathrm{St}\cap A_7$, then b misses A_i for some $i\in\{3,4,5,6\}$, say A_6. Again by Claim 2, $A_6\cap A_{1,2}=\emptyset$, and $|A_{6,7}|=|A_6|+|A_7|-|A^{6,7}|\geqslant 8k-(7k+1-(m-1))=k-2+m$, a contradiction to the choice of m. It follows that

$$A_7\subseteq(A^{1,2}\backslash A_{1,2})\cup\overline{\mathrm{St}}. \tag{1}$$

Observe that

$$|(A^{1,2}\backslash A_{1,2})\cup\overline{\mathrm{St}}|=8k-2|A_{1,2}|+|\overline{\mathrm{St}}\backslash A^{1,2}|.$$

If $|A_{1,2}|\geqslant 2k+3$ then the last expression is at most $4k-2$. This contradicts the fact that $|A_7|=4k$. If $|A_{1,2}|=2k+2$ then to satisfy (1) we need $|\overline{\mathrm{St}}\backslash A^{1,2}|=4$ and $A_7=(A^{1,2}\backslash A_{1,2})\cup\overline{\mathrm{St}}=(A_1\backslash A_{1,2})\cup(A_2\backslash A_{1,2})\cup\overline{\mathrm{St}}$, so that $A^{1,2}\subseteq\mathrm{St}$, $|A_1\backslash A_{1,2}|=|A_2\backslash A_{1,2}|=2k-2$ and $A_{1,7}=A_1\backslash A_{1,2}$. This proves (i).

Finally, if $|A_{1,2}|=2k+1$ then to satisfy (1) we need $|\overline{\mathrm{St}}\backslash A^{1,2}|\geqslant 2$ and $|(A^{1,2}\backslash A_{1,2})\backslash A_7|\leqslant 2$. Since $|A_{1,7}|\leqslant|A_1|-|A_{1,2}|=2k-1$, this proves (ii).

Claim 4. *For any $i\neq j$, $|A_{ij}|\geqslant 2k-6$.*

Proof. Let $i=1$, $j=2$.

Case 1. There exists A_3 with $|A_{1,3}|\geqslant 2k+2$. Then by Claim 3, $|A_{1,3}|=2k+2$, $A_1\subseteq \mathrm{St}$, and for some A_4, $|A_{1,4}|=2k-2$. Then $\sum_{l=2}^{7}|A_l\cap A_1|=3|A_1|=12k$, and again by Claim 3,

$$\sum_{l=3}^{7}|A_l\cap A_1| = |A_3\cap A_1|+|A_4\cap A_1|+\sum_{l=5}^{7}|A_l\cap A_1|\leqslant 4k+3(2k+2)=10k+6.$$

Thus $|A_{1,2}|=\sum_{l=2}^{7}|A_l\cap A_1|-\sum_{l=3}^{7}|A_l\cap A_1|\geqslant 12k-(10k+6)=2k-6$.

Case 2. $\max\{|A_l\cap A_1|: 3\leqslant l\leqslant 7\}=2k+1$. Let $|A_{1,3}|=2k+1$. Then by part (ii) of Claim 3, $\sum_{l=2}^{7}|A_l\cap A_1|\geqslant 3|A_1|-2=12k-2$ and for some A_s, $2k-3\leqslant |A_{1,s}|\leqslant 2k-1$. If $s=2$, then we are done. Thus we may assume $s=4$. It follows that

$$\sum_{l=3}^{7}|A_l\cap A_1| = |A_3\cap A_1|+|A_4\cap A_1|+\sum_{l=5}^{7}|A_l\cap A_1|\leqslant 4k+3(2k+1)=10k+3,$$

and $|A_{1,2}|\geqslant(12k-2)-(10k+3)=2k-5$.

Case 3. $\max\{|A_l\cap A_1|: 3\leqslant l\leqslant 7\}\leqslant 2k$. Then $\sum_{l=2}^{7}|A_l\cap A_1|\geqslant 12k-4$ and $\sum_{l=3}^{7}|A_l\cap A_1|\leqslant 10k$. Consequently, $|A_{1,2}|\geqslant 2k-4$.

For $a\in S$ let *the spectrum* $s(a)$ be the set of indices i such that $a\in A_i$.

Claim 5. *For any $i\neq j$ and any $a,b\in\overline{A^{ij}}\cap \mathrm{St}$, $s(a)=s(b)$.*

Proof. If $s(a)\neq s(b)$, let $k\in s(b)\backslash s(a)$ and $l\in s(a)\backslash s(b)$. By Claim 2, $A^{kl}\cap A_{1,2}=\emptyset$. Then, by Claim 4,

$$|A_{kl}|=|A_k|+|A_l|-|A^{kl}|\geqslant 8k-(7k+1-(2k-6))=3k-7>2k+2,$$

a contradiction to Claim 3.

For any $i\neq j$, let $c(i,j)$ denote the number k such that $k\neq i,j$ and $A_k\cap\overline{A^{ij}}\cap\mathrm{St}=\emptyset$. By Claim 5, this number is unique. In particular, $A_{c(i,j)}\cap\mathrm{St}\subseteq A^{ij}\backslash A_{ij}$. It follows that $c(i,c(i,j))=j$ and $c(j,c(i,j))=i$. In other words, St is the disjoint union of seven sets $D_1,\dots,D_7$ with equal spectra inside each set. And the A_i-s form the complements of lines of the Fano plane on these D_j-s.

Claim 6. *No element of $\overline{\mathrm{St}}$ has a spectrum $\{i,j,c(i,j)\}$ for some i,j.*

Proof. Suppose that $s(a)=\{i,j,c(i,j)\}$. By the above, each $b\in\overline{A^{ij}}$ has the spectrum $\{1,\dots,7\}\backslash\{i,j,c(i,j)\}$. Then a and b cover $\mathscr{A}$.

Let A_i, A_j and A_k be such that $k=c(i,j)$. Since each of them has an odd intersection with X, the number of elements in X belonging to an odd number of members of $\{A_i,A_j,A_k\}$ is odd. But by the above, each $a\in\mathrm{St}$ belongs to an even number of members of $\{A_i,A_j,A_k\}$. Thus,

$$|A_i\cap\overline{\mathrm{St}}\cap X|+|A_j\cap\overline{\mathrm{St}}\cap X|+|A_k\cap\overline{\mathrm{St}}\cap X|$$

is odd. In particular, $\overline{\mathrm{St}} \cap X \neq \emptyset$. Moreover, if $\overline{\mathrm{St}} \cap X = \{a\}$, then by Claim 6, there are some i, j, $c(i,j)$ such that none of these contains a. Thus, $|\overline{\mathrm{St}} \cap X| \geqslant 2$.

On the other hand, since the cardinality of each A_i is $4k$,

$$|A_i \cap \overline{\mathrm{St}}| + |A_j \cap \overline{\mathrm{St}}| + |A_k \cap \overline{\mathrm{St}}|$$

is even for every $i \neq j$. By the reasons similar to above, $|\overline{\mathrm{St}} \backslash X| \geqslant 2$.

The only possibility we are left with is that $|\overline{\mathrm{St}}| = 4$ and the degree sequence is 3,3,3,3. Furthermore, $|\overline{\mathrm{St}} \cap X| = 2$.

However, since $\sum_{i=1}^{7} |A_i \cap X|$ is odd, the sum of degrees of the nonstandard vertices in X must be odd. This is a final contradiction.

3. Upper bound

It will be easier to prove the upper bound in the following form.

Theorem 2. *Let $\mathscr{B}$ be an r-uniform family. If $\tau(\mathscr{B}) > \lceil 7r/8 \rceil$, then there exists $\mathscr{F} \subset \mathscr{B}$ with $|\mathscr{F}| \leqslant 7$ such that $\tau(\mathscr{F}) > 2$.*

Proof. Let $r = 8k + s$, where $k \geqslant 1$ and $0 \leqslant s \leqslant 7$.

Suppose that there exists an r-uniform family $\mathscr{B}$ possessing the property $(7,2)$ with $\tau(\mathscr{B}) > \lceil 7r/8 \rceil = 7k + s$. For each set A, we set

$$\mathscr{B}_A = \{B \in \mathscr{B} \mid B \cap A = \emptyset\}.$$

Below, any triple (A_1, A_2, A_3) of members of $\mathscr{B}$ with $A_1 \cap A_2 \cap A_3 = \emptyset$ will be called a *good triple*. To shorten notation, below we set $A_{ij} = A_i \cap A_j$ and $a_{ij} = |A_{ij}|$. Our main tool will be the following fact.

Lemma 1. *Let $\mathscr{B}$ be an r-uniform family containing a good triple (A_1, A_2, A_3) satisfying the following inequalities*:

$$a_{12} \leqslant 4k + \lceil s/2 \rceil; \tag{2}$$

$$\max\{a_{13}, a_{23}\} \leqslant 3k + s/2; \tag{3}$$

$$a_{13} + a_{23} \leqslant 5k + s. \tag{4}$$

Then the theorem holds for $\mathscr{B}$.

Proof. Suppose that $\tau(\mathscr{B}) > 7k + s$. For the proof, we may clearly assume

$$a_{12} \geqslant a_{13} \geqslant a_{23}, \tag{5}$$

since reordering to assure (5) will not violate (2), (3) or (4).

Case 1. $a_{13} \leqslant (k + a_{12})/2$. Then, by (5), $|A_3 - A_1 - A_2| \geqslant r - 2a_{13} \geqslant 7k + s - a_{12}$, and so there exists $B_0 \subseteq A_3 - A_1 - A_2$ with $|B_0| = 7k + s - a_{12}$ such that $A_3 - B_0$ can be partitioned into two parts B_1 and B_2 so that $|B_1| = \lfloor (k+a_{12})/2 \rfloor$, $|B_2| = \lceil (k+a_{12})/2 \rceil$ and $A_{i3} \subseteq B_i$ for $i = 1, 2$. Furthermore, for $i = 1, 2$, there exists $B_i' \subseteq A_i - A_{i3} - A_{12}$ with $|B_i'| = b_i = 7k + s - a_{12} - |B_{3-i}|$, provided that $b_i \geqslant 0$. But this is always the case since by (2),

$$a_{12} + |B_{3-i}| \leqslant \lfloor \tfrac{3}{2} a_{12} + \tfrac{1}{2}k + \tfrac{1}{2} \rfloor \leqslant \lfloor 6k + \tfrac{3}{2}\lceil s/2 \rceil + \tfrac{1}{2}k + \tfrac{1}{2} \rfloor \leqslant 7k + s.$$

Since $\tau(\mathscr{B}) > 7k+s$, there exist $A_4 \in \mathscr{B}_{A_{12} \cup B_0}$, $A_5 \in \mathscr{B}_{A_{12} \cup B_1 \cup B_2'}$ and $A_6 \in \mathscr{B}_{A_{12} \cup B_2 \cup B_1'}$. For $B_4 = (A_1 \cup A_2) - A_{12} - B_1' - B_2'$, we have

$$|B_4| = 2(8k+s) - 2a_{12} - (7k + s - a_{12} - |B_2|) - (7k + s - a_{12} - |B_1|)$$

$$= 2k + |B_1| + |B_2| = 3k + a_{12} \leqslant 7k + s.$$

Hence there exists $A_7 \in \mathscr{B}_{B_4}$.

Suppose that there are two elements x and y covering $\mathscr{F} = \{A_1, \ldots, A_7\}$. In order to cover A_1, A_2 and A_3, at least one of them, say x, belongs to some A_{ij}, where $1 \leqslant i < j \leqslant 3$. Assume first that $x \in A_{12}$. Then $y \in A_3 = B_0 \cup B_1 \cup B_2$. It follows that one of the edges A_4, A_5 and A_6 is not covered. If $x \in A_{i3}$ $(i = 1, 2)$ and $y \notin A_{12}$, then to meet both A_{3-i} and A_7, $y \in A_{3-i} - A_1 - B_4 = B_{3-i}'$. But in this case, y misses A_{i+4}. This completes Case 1.

Case 2. $a_{13} > (k + a_{12})/2$. Let $B_1 \subseteq A_2 - A_1 - A_3$ with $|B_1| = 7k + s - a_{12} - a_{13}$. Let $B_2 \subseteq A_3 - A_1 - A_2$ with $|B_2| = \min\{7k + s - a_{12}, 8k + s - a_{13} - a_{23}\}$. Let $B_3 = A_3 - A_1 - B_2$. Observe that $B_3 \supset A_{23}$ and

$$|B_3| = 8k + s - a_{13} - \min\{7k + s - a_{12}, 8k + s - a_{13} - a_{23}\}$$

$$= \max\{a_{12} - a_{13} + k, a_{23}\}.$$

Let $B_4 \subseteq A_1 - A_3 - A_2$ with $|B_4| = \min\{7k + s - a_{12} - |B_3|, 8k + s - a_{12} - a_{13}\}$. By the cardinality constraints on B_1, B_2, B_3 and B_4, there exist $A_4 \in \mathscr{B}_{A_{12} \cup A_{13} \cup B_1}$, $A_5 \in \mathscr{B}_{A_{12} \cup B_2}$, and $A_6 \in \mathscr{B}_{A_{12} \cup B_3 \cup B_4}$.

Let $B_5 = (A_1 \cup A_2) - A_{12} - B_1 - B_4$. If we prove that

$$|B_5| \leqslant 7k + s, \tag{6}$$

then the lemma would follow. Indeed, in this case there exists $A_7 \in \mathscr{B}_{B_5}$. Thus, if two elements x and y cover $\mathscr{F} = \{A_1, \ldots, A_7\}$ and $x \in A_{12}$, then $y \in A_3 \subset A_{13} \cup B_2 \cup B_3$ and hence at least one of the edges A_4, A_5 and A_6 is not covered. If $x \in A_{13}$ and $y \notin A_{12}$, then $y \in A_2 - A_1 \subseteq B_1 \cup B_5$ and hence A_4 or A_7 is not covered. Finally, if $x \in A_{23}$ and $y \notin A_{12} \cup A_{13}$, then $y \in A_1 - A_2 \subseteq B_4 \cup B_5$ and hence A_6 or A_7 is not covered.

By the definition,

$$|B_5| = 2(8k+s) - 2a_{12} - (7k + s - a_{12} - a_{13})$$

$$- \min\{7k + s - a_{12} - |B_3|, 8k + s - a_{12} - a_{13}\}$$

$$= 9k + s - a_{12} + a_{13} - (7k + s - a_{12}) - \min\{-|B_3|, k - a_{13}\}$$

$$= 2k + a_{13} + \max\{|B_3|, -k + a_{13}\}.$$

Hence if $|B_3| \leqslant -k + a_{13}$, then by (2),

$$|B_5| = 2k + a_{13} - k + a_{13} \leqslant k + 2(3k + s/2) = 7k + s.$$

Otherwise,

$$|B_5| = 2k + a_{13} + |B_3| = 2k + a_{13} + \max\{a_{12} - a_{13} + k, a_{23}\}$$

$$= \max\{3k + a_{12}, 2k + a_{13} + a_{23}\}.$$

By (2) and (4), in both cases $|B_5| \leqslant 7k + s$. This proves the lemma.

Proof of the theorem. *Case* 1. There are $A_1, A_3 \in \mathscr{B}$ with $2k \leqslant |A_1 \cap A_3| \leqslant 3k + s/2$. Let $A_{13} \subset B_1 \subset A_1$ with $|B_1| = 4k + \lfloor s/2 \rfloor$ and $A_{13} \subset B_3 \subset A_3$ with $|B_3| = 3k + a_{13} + \lceil s/2 \rceil$. Since $|B_1 \cup B_3| = 7k + s$, there exists $A_2 \in \mathscr{B}_{B_1 \cup B_3}$. By the definition of B_1 and B_3, (A_1, A_2, A_3) is a good triple, $|A_2 \cap A_1| \leqslant 4k + \lceil s/2 \rceil$ and $|A_2 \cap A_3| \leqslant 5k + \lfloor s/2 \rfloor - a_{13}$. Hence, all conditions (2)–(4) of Lemma 1 are satisfied and it can be applied.

Case 2. There are $A_1, A_2 \in \mathscr{B}$ with $3k + s/2 < |A_1 \cap A_2| \leqslant 4k + s/2$. For $i = 1, 2$, let $A_{12} \subset B_i \subset A_i$ with $|B_i| = 5k + \lceil s/2 \rceil$. Since

$$|B_1 \cup B_2| = 10k + 2\lceil s/2 \rceil - a_{12} \leqslant 7k + \lceil s/2 \rceil,$$

there exists $A_3 \in \mathscr{B}_{B_1 \cup B_2}$. By the definition of B_1 and B_2, (A_1, A_2, A_3) is a good triple and $|A_3 \cap A_i| \leqslant 3k$ for $i = 1, 2$. Moreover, since Case 1 does not hold, $|A_3 \cap A_i| < 2k$ for $i = 1, 2$. Again, Lemma 1 can be applied.

Case 3. There are $A_1, A_2 \in \mathscr{B}$ with $k \leqslant |A_1 \cap A_2| < 2k$. For $i = 1, 2$, let $A_{12} \subset B_i \subset A_i$ with $|B_i| = 4k + \lfloor s/2 \rfloor$. Since $|B_1 \cup B_2| = 8k + 2\lfloor s/2 \rfloor - a_{12} \leqslant 7k + s$, there exists $A_3 \in \mathscr{B}_{B_1 \cup B_2}$. By the definition of B_1 and B_2, (A_1, A_2, A_3) is a good triple and $|A_3 \cap A_i| \leqslant 4k + \lceil s/2 \rceil$ for $i = 1, 2$. Moreover, since Cases 1 and 2 do not hold, $|A_3 \cap A_i| < 2k$ for $i = 1, 2$. Once more, Lemma 1 can be applied.

Case 4. For any $A, B \in \mathscr{B}$, either $|A \cap B| > 4k + \lceil s/2 \rceil$ or $|A \cap B| < k$. Let $x = \max\{|A \cap B| : A, B \in \mathscr{B},\ |A \cap B| \leqslant 4k + s/2\}$ and $A_1, A_3 \in \mathscr{B}$ with $|A_1 \cap A_3| = x$. For $i = 1, 3$, let $A_{13} \subset B_i \subset A_i$ with $|B_1| = \lceil (7k + s + x)/2 \rceil$ and $|B_3| = \lfloor (7k + s + x)/2 \rfloor$. Since $|B_1 \cup B_3| = 7k + s$, there exists $A_2 \in \mathscr{B}_{B_1 \cup B_3}$. By the construction,

$$\max\{|A_1 \cap A_2|, |A_3 \cap A_2|\} \leqslant 8k + s - \lfloor (7k + s + x)/2 \rfloor.$$

Since (A_1, A_2, A_3) is a good triple, either $|A_1 \cap A_2| \leqslant 4k + s/2$ or $|A_3 \cap A_2| \leqslant 4k + s/2$. We will assume that $|A_3 \cap A_2| \leqslant 4k + s/2$. Then, under conditions of the case, $|A_3 \cap A_2| \leqslant x$.

Since $a_{13} + a_{23} \leqslant 2x \leqslant 2k - 2 < \lfloor (5k + s + x)/2 \rfloor \leqslant 7k + s - a_{12}$, we can partition A_3 into four parts $B_1, \ldots, B_4$ so that

$$|B_1| = 7k + s - a_{12}, \quad |B_2| = 7k - a_{12},$$

$$|B_3| = |B_4| = a_{12} - 3k \quad \text{and} \quad B_1 \supseteq A_{13} \cup A_{23}.$$

For $i = 1, 2$, there exists $A_{i+3} \in \mathscr{B}_{A_{12} \cup B_i}$.

Let $i \in \{1,2\}$ and let $C_i = A_4 \cap A_i$. Since $A_4 \cap A_{12} = \emptyset$, we have $|C_i| \leqslant x$. Let $D_i = A_{12} \cup B_{i+2} \cup A_{i3} \cup C_{3-i}$. Since

$$|D_i| \leqslant a_{12} + (a_{12} - 3k) + x + x \leqslant 5k + 2\lceil s/2 \rceil + 2(k-1) < 7k + s,$$

there exists $A_{i+5} \in \mathscr{B}_{D_i}$.

Assume that some two elements x and y cover $\mathscr{F} = \{A_1, \ldots, A_7\}$. If $x \in A_{12}$ then $y \in B_j$ for some $j \in \{1,2,3,4\}$. But in this case A_{j+3} is not covered. So, let $x \in A_{i3}$ for some $i \in \{1,2\}$. In order to cover A_4 and A_{3-i}, we need $y \in C_{3-i}$. Then A_{i+5} is missed. This is the final contradiction.

References

[1] P. Erdős, D.G. Fon-Der-Flaass, A.V. Kostochka, Zs. Tuza, Small transversals in uniform hypergraphs, Siberian Adv. Math. 2 (1992) 82–88.

[2] P. Erdős, A. Hajnal, Zs. Tuza, Local constraints ensuring small representing sets, J. Combin. Theory, Ser. A 58 (1991) 78–84.