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NONASSOCIATIVE LEFT REGULAR AND BIREGULAR RINGS

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Consider the following equivalent characterizations of associative von Neumann regular rings A with unity:

(1) for each $a \in A$ there exists an $x \in A$ such that $a = axa$;
(2) every principal left ideal of A is generated by an idempotent e;
(3) every principal left ideal of A is a direct summand;
(4) every left A-module is flat;
(5) every left A-module is regular;
(6) every finitely presented left A-module is projective.

Trying to define regularity of nonassociative rings we notice that (1) can only be stated for alternative rings; but even in this case it only implies (2) or (3) under additional conditions (examples 2, 3 in Section 1). However, in any ring A with unity properties (2) and (3) are equivalent if we only demand the idempotent e occuring in (2) to be in the right nucleus of the ring. Of course, the left-right symmetry (which is assured by (1) in the associative case) is lost and so we call a ring satisfying (2) *left regular*. An internal characterization of these rings is given in Section 1.

In the category of submodules of *associatively generated A-modules* (=:A-**mod**^{a}) as introduced in [15] we may define *flat, regular* and *finitely presented* objects in a categorical way (Stenström [12], Fieldhouse [6]). Within this context conditions (4), (5), (6) are equivalent (see Proposition 2.6) and are implied by (2) or (3) (see Section 3). If A itself is finitely presented in A-**mod**^{a} conditions (2–6) are equivalent (Theorem 3.1). For example left alternative and Jordan rings with unity, which are finitely generated modules over their centers, are finitely presented in A-**mod**^{a} (Corollary 3.7). A left regular ring A which is finitely presented in A-**mod**^{a} is a projective generator of A-**mod**^{a}; if — additionaly — A is a finitely generated right module over its right nucleus $n(A)$ then it is a projective generator of the category of modules over the left multiplicationring $L(A)$ (Theorems 3.5, 3.6), i.e. $L(A)$-**mod** is equivalent to $n(A)$-**mod**. The categorical background for these concepts is given in Section 2.

The definition of associative *biregular* rings (every principal ideal is generated by a central idempotent, Arens–Kaplansky [3]) can literally be taken for nonassociative rings and many of their properties remain valid in the more general situation (Proposition 1.7). Applying again the results of Section 2 we study in Section 4 the characterization of biregular rings in the category of submodules of *central bimodules* (= A-**bimod**^{c} in [16]). Equivalences analogous to (2)–(6) are obtained in case A is finitely presented in A-**bimod**^{c} (this is also new for associative rings). Biregular rings, which are finitely presented in A-**bimod**^{c} and finitely generated modules over their centers are *Azumaya algebras* over a von Neumann regular ring (Theorem 4.4). For associative rings this was proved by Wehlen [14] using sheaf-theoretic techniques. An alternative ring, which is a finitely generated module over its center is biregular if and only if it is left and right regular (Theorem 4.5).

Notations

R denotes a commutative and associative ring with unity and A denotes a nonassociative R-algebra or ring not necessarily with unity. By an *A-module M* we mean an R-module M together with an R-linear map $\rho : A \to \mathrm{End}_R(M)$ and we shall write $\rho(A)$ as left operators on M. The *nucleus* of an A-module M is defined as $n(M) := \{m \in M \mid (ab)m = a(bm) \text{ for all } a, b \in A\}$ and M is called *associatively generated* if $n(M)$ is a generating set for M (see [15]).

An R-module M is an *A-bimodule* if there are two R-linear maps ρ_1, ρ_2: $A \to \mathrm{End}_R(M)$ and we write $\rho_1(A)$ and $\rho_2(A)$ as left and right operators on M. The *center* of M is $c(M) := \{m \in M \mid (ab)m = a(bm),\ m(ab) = (ma)b$ and $am = ma$ for all $a, b \in A\}$ and M is called a *central* A-bimodule if $c(M)$ is a generating set for M (see [16]).

With this notation $n(A)$ will be the *right nucleus* and $c(A)$ the (usual) *center* of the ring A.

$L(A)$, $R(A)$ and $M(A)$ denote the subalgebras of $\mathrm{End}_R(A)$ generated by the left or right multiplications and the identity map of A or by the left and right multiplications and the identity map of A respectively.

Jac(A) is the (left) Jacobson radical of A as defined in Brown [4] for nonassociative rings.

For an element $a \in A$ we denote by $\langle a \mid$ or $\langle a \rangle$ the left ideal or the two-sided ideal of A generated by a (and containing a).

1. Idempotents in a ring

In this Section we list several properties of rings resulting from the existence of idempotents in certain ideals.

Proposition 1.1. *Let every left ideal of the ring A contain an idempotent $e \neq 0$ with $e \in n(A)$. Then*

(i) *the intersection of the maximal modular left ideals of* A *is zero*;
(ii) *the Jacobson radical of* $L(A)$ *is zero.*

Proof. (i) It follows from the proof of Theorem 1 in Brown [4] that the intersection D of all maximal modular left ideals of A is (left) quasi regular: every $d \in D$ is contained in the left ideal of A generated by the set $\{x - xd / x \in A\}$. For any idemptotent $e \in D \cap n(A)$ the set $\{x - xe / x \in A\}$ is a left ideal, i.e. for some $b \in A$ $e = b - be$, which implies $e = 0$.

(ii) For a maximal modular left ideal M in A the factor module A/M is an irreducible $L(A)$-module and the annihilator of A/M in $L(A)$, the set $\{\lambda \in L(A) \mid \lambda(A) \leq M\}$, is a primitive ideal in $L(A)$. Let $\{M_i\}_{i \in I}$ be the familiy of maximal modular left ideals in A and S_i the annihilators of A/M_i in $L(A)$. Then

$$\left(\bigcap_{i \in I} S_i\right) A \leq \bigcap_{i \in I} S_i A \leq \bigcap_{i \in I} M_i = \{0\};$$

since A is a faithful $L(A)$-module $\bigcap_{i \in I} S_i$ must be zero.

In alternative rings the assumption $e \in n(A)$ in (1.1) is not necessary; this follows from the coincidence of the Kleinfeld and Smiley radical (= Jacobson radical) in these rings. The above proof of (ii) actually shows:

Proposition 1.2. *In an alternative ring* A $\operatorname{Jac}(A) = \{0\}$ *implies* $\operatorname{Jac}(L(A)) = \{0\}$.

We call a ring A *left regular* if every principal left ideal of A is generated by an idempotent $e \in n(A)$. Clearly left regular rings do have the properties given in (1.1) and for the associative case they are *von Neumann regular* rings. Accordingly *right regular* rings can be defined in an obvious way.

Proposition 1.3. *For a ring* A *with unity the following statements are equivalent*:

(i) A *is left regular*;

(ii) *every finitely generated left ideal in* A *is generated by an idempotent* $e \in n(A)$;

(iii) *every principal left ideal is a direct summand in* A;

(iv) *every finitely generated left ideal is a direct summand in* A;

(v) *the left principal ideals of* A *constitute* (*with respect to sum and intersection*) *a complemented modular lattice.*

Proof. The equivalence of (i) and (ii) is shown as soon we know that the sum of two left principal ideals is again a left principal ideal. Since $n(A)$ is an associative ring the proof can be taken from the associative case (von Neumann [9, Lemma 15]). The remaining equivalences are obtained in a well-known way.

Example 1. Left semisimple rings (see [15]) are left regular and left regular rings with ascending chain condition on left ideals are left semisimple.

Example 2. Smiley studied in [10] "regular" alternative rings A (for each $a \in A$ there is an $x \in A$ for which $a = axa$) without nilpotent elements; in this case every idempotent $e \in A$ has the property $ea = ae$ for all $a \in A$ and by [10, Lemma 2] every e with this property is in the nucleus $n(A)$ (i.e. $e \in c(A)$). So these rings are left and right regular.

Example 3. More generally, in Amemiya–Halperin [1] "regular" alternative rings A (as in Example 2) are considered, for which every idempotent is in the nucleus ("idempotent-associative"). These rings are also left and right regular.

Further examples are given in Section 3.

An associatively generated A-module M is *almost faithful* if the annihilator of M in $A = \{a \in A \mid am = 0$ for all $m \in M\}$ does not contain a two-sided ideal of A. According to the associative case, a ring A is called *left primitive* if there exists an almost faithful, associatively generated, irreducible A-module. It follows from the definition that A is left primitive if and only if A contains a modular maximal left ideal which does not contain a two-sided ideal of A (see Brown[4]).

Proposition 1.6. *Let every ideal of the ring A contain an idempotent $e \neq 0$ with $e \in c(A)$. Then*

(i) *the intersection of the maximal modular ideals in A (= Brown–McCoy radical) is zero*;

(ii) *the Jacobson radical of $M(A)$ is zero*;

(iii) *A is left primitive iff A is a simple ring with unity.*

Proof. Using the characterization of the Brown–McCoy radical of nonassociative rings (Smiley [11]) we can apply the proofs of (1.1) to show (i) and (ii).

(iii) Let M be an almost faithful, associatively generated, irreducible A-module with nucleus $n(M)$. For $a \in A$ and any central idempotent e in $\langle a \rangle$ the set $e \cdot M = e \cdot (A \cdot n(M)) = A(e \cdot n(M))$ is an A-submodule of M and therefore $e \cdot M = M$. The ideal $D := \{x - xe \mid x \in A\}$ has the property $D \cdot e = \{0\}$ and so $D \cdot M = D(e \cdot M) = \{0\}$; since M is almost faithful this means $D = \{0\}$ and e is the unity of A, i.e. A is a simple ring with unity.

Extending a definition given in Arens–Kaplansky [3] for associative rings we call a nonassociative ring A *biregular* if every principal (two-sided!) ideal in A is generated by a central idempotent. This is equivalent to the condition that every finitely generated ideal in A is generated by a central idempotent: if the principal ideals $\langle a \rangle$ and $\langle b \rangle$ in A are generated by the central idempotents e, f then $\langle a \rangle + \langle b \rangle$ is generated by the central idempotent $e + f - ef$. A ring A with unity is biregular if and only if every principal ideal is a direct summand in A.

Proposition 1.7. *A biregular ring A has the following properties:*

(i) *every homomorphic image of A is biregular;*

(ii) *primitive ideals are modular maximal ideals in A;*

(iii) *every ideal in A is the intersection of the modular maximal ideals containing it;*

(iv) *the Brown–McCoy radical of A and* $\mathrm{Jac}(M(A))$ *are zero.*

Proof. (i) is easy to see and the other properties are immediate consequences of (i) and Propostion 1.6.

Observe that the Brown–McCoy radical of any ring A is equal to the intersection of all ideals B in A for which A/B is a biregular ring.

c-semisimple rings, i.e. rings which are finite direct sums of simple rings with unity [16, Satz (2.3)], are biregular and biregular rings with ascending chain condition on (two-sided) ideals are c-semisimple.

Proposition 1.8. *Let A be a ring with unity for which $M(A)$ is a finitely generated $c(A)$-module. If A is left and right regular then A is biregular.*

Proof. Let J be a principal ideal in the left and right regular ring A. J is a finitely generated $c(A)$-module and hence finitely generated as left and right ideal; there are idempotents e, f in the right or left nucleus of A respectively with $Ae = J = fA$ and from $e = fe = f$ and $ea = (ea)e = ae$ for all $a \in A$ it follows $e \in c(A)$.

Suppose A to be an alternative ring with unity of the type described in example 2 or 3, which is a finitely generated $c(A)$-module. Then $M(A)$ is a finitely generated $c(A)$-module and by (1.8) A is biregular. Further examples of biregular rings will occur in Section 4.

2. The categorical framework

Let S-**mod** be the category of unitary left modules over an associative ring S with unity. We say an object in S-**mod** is *subgenerated* by an S-module D, if it is contained in an S-module which is generated by D. $\sigma[D]$ denotes the greatest subcategory of S-**mod** for which every object is subgenerated by D. $\sigma[D]$ is a Grothendieck category with generator: let Ω be the set of all S-homomorphisms $\psi : S \to \sigma[D]$; then $\bigoplus_{\psi\in\Omega} \psi(S)$ is a generator of $\sigma[D]$.

If D is a faithful S-module which is a fintely generated right module over $\mathrm{End}_S(D)$ (i.e. M is finendo, Faith [5]) with generating elements $d_1, \ldots, d_k \in D$, the map $S \to D^k$, $s \to s \cdot (d_1, \ldots, d_k)$, is a monomorphism and we have $S \in \sigma[D]$ which implies that S-**mod** is subgenerated by D.

An S-module is called *$\sigma[D]$-projective* if it is projective relative to all short exact sequences in $\sigma[D]$. It follows from well-known theorems that a finitely generated S-module is $\sigma[D]$-projective if and only if it is D-projective (Anderson–Fuller [2, §16]). Accordingly we define $\sigma[D]$-*injective* modules and we get that any S-module is $\sigma[D]$-injective if and only if it is D-injective.

An object in the category $\sigma[D]$ is *finitely generated* (= *of finite type*) in $\sigma[D]$ if and only if it is a finitely generated S-module. A finitely generated F in $\sigma[D]$ is *finitely presented* in $\sigma[D]$, if for every exact sequence $0 \to K \to L \to F \to 0$ in $\sigma[D]$ with L finitely generated it follows that K is finitely generated. Any F in $\sigma[D]$ which is finitely presented in S-**mod** is finitely presented in $\sigma[D]$. A finitely generated D-projective object in $\sigma[D]$ is also finitely presented in $\sigma[D]$.

2.1. Let $0 \to K \to E \to F \to 0$ be an exact sequence in $\sigma[D]$ with E finitely presented in $\sigma[D]$. If K is finitely generated then F is finitely presented in $\sigma[D]$.

Since $\sigma[D]$ is a locally finitely generated Grothendieck category we have (Stenström [13, Chap. V]):

2.2. An object F in $\sigma[D]$ is finitely presented in $\sigma[D]$ if and only if the functor $\mathrm{Hom}_S(F, -)$ commutes with inductive direct limits in $\sigma[D]$.

From Stenström [12] and Fieldhouse [6] we are led to the following definitions:

2.3. A short exact sequence is *pure in* $\sigma[D]$ if every finitely presented object in $\sigma[D]$ is projective relative to it. An object Q is *regular in* $\sigma[D]$ if every short exact sequence with center Q is pure in $\sigma[D]$. An object V is *flat in* $\sigma[D]$ if every short exact sequence $0 \to K \to L \to V \to 0$ in $\sigma[D]$ is pure in $\sigma[D]$.

Proposition 2.4. *For an S-module Q in $\sigma[D]$ there are equivalent:*

(i) *Q is regular in $\sigma[D]$;*

(ii) *every finitely generated submodule of Q is pure in $\sigma[D]$;*

(iii) *every finitely presented module in $\sigma[D]$ is Q-projective.*

If Q is finitely presented in $\sigma[D]$ these properties are equivalent to:

(iv) *every finitely generated submodule of Q is a direct summand.*

Proof. (i) $\Leftrightarrow$ (iii) follows immediately from the definition, (i) $\Rightarrow$ (ii) is clear and (ii) $\Rightarrow$ (i) is a consequence of (2.2). (iii) $\Rightarrow$ (iv) is derived from (2.1) and (iv) $\Rightarrow$ (ii) is trivial.

Proposition 2.5. *Let $0 \to L \to Q \to N \to 0$ be an exact sequence in $\sigma[D]$.*

(i) *If Q is regular in $\sigma[D]$ then L and N are regular in $\sigma[D]$.*

(ii) *If L and N are regular in $\sigma[D]$ and the above sequence is pure in $\sigma[D]$ then Q is regular in $\sigma[D]$.*

Proof. Using property (3) in (2.4) (i) follows from Proposition (16.12) in Anderson–Fuller [2]. (ii) can be obtained by applying the Five Lemma.

As a consequence of (2.5) a direct sum of modules in $\sigma[D]$ is regular if and only if every summand is regular in $\sigma[D]$ and we get:

Proposition 2.6. *The following conditions for an S-module D are equivalent:*
(i) *D is regular in $\sigma[D]$;*
(ii) *every module in $\sigma[D]$ is regular in $\sigma[D]$;*
(iii) *every module in $\sigma[D]$ is flat in $\sigma[D]$;*
(iv) *every short exact sequence in $\sigma[D]$ is pure in $\sigma[D]$.*

Theorem 2.7. *If D is a finitely generated, faithful S-module which is a finitely generated right module over* $\mathrm{End}_S(D)$ *then the following statements are equivalent:*
(i) *D is regular and finitely presented in $\sigma[D]$;*
(ii) *D is finitely presented in S-**mod** and S is von Neumann regular;*
(iii) *D is a projective, regular S-module;*
(iv) *D is a projective generator for S-**mod** and* $\mathrm{End}_S(D)$ *is von Neumann regular.*

Proof. (i) $\Longrightarrow$ (ii) By (2.4) D is D-projective; since D is finitely generated and S is subgenerated by D this implies that D is projective in S-**mod**. By (2.6) S is regular in $\sigma[D]$; S is finitely presented and hence a von Neumann regular ring by (2.4).

(ii) $\Longrightarrow$ (iii) Every module over a von Neumann regular ring is regular (Fieldhouse [6]). (iii) $\Longrightarrow$ (i) is obvious.

(iii) $\Longrightarrow$ (iv) S is a finitely generated submodule of a projective regular S-module D^k (for appropriate $k \in N$) and hence direct summand of D^k, i.e. S is generated by D. $\mathrm{Hom}_S(D, -)$ defines an equivalence between S-**mod** and $\mathrm{End}_S(D)$-**mod** and S is von Neumann regular; therefore $\mathrm{End}_S(D)$ is von Neumann regular, too.

(iv) $\Longrightarrow$ (ii) is shown by a similar argument.

3. One-sided A-modules

To apply the theory just described let A be a nonassociative ring with unity and $L(A)$ the left multiplicationring of A. Considering A as an $L(A)$-module, $\sigma[A]$ is the subcategory of $L(A)$-**mod** whose objects are submodules of associatively generated A-modules (=: A-**mod**^{a}). The $\sigma[A]$-projective and $\sigma[A]$-injective modules are the *a-projective* and *a-injective* $L(A)$-modules introduced in [15].

According to (2.3) we can define *pure* short exact sequences, *regular* and *flat* modules in A-**mod**^{a} and Proposition 2.6 gives a relationship between these concepts. As a consequence of (2.4) we know that a left regular ring A is regular in A-**mod**^{a}.

From Propositions 1.3, 2.4 and 2.6 we obtain:

Theorem 3.1. *For a ring A with unity, which is finitely presented in A-**mod**^{a}, the following conditions are equivalent*:

(i) *A is a left regular ring*;
(ii) *A is regular in A-**mod**^{a}*;
(iii) *every finitely presented module in A-**mod**^{a} is a-projective*;
(iv) *every module in A-**mod**^{a} is regular in A-**mod**^{a}*;
(v) *every module in A-**mod**^{a} is flat in A-**mod**^{a}*;
(vi) *every short exact sequence in A-**mod**^{a} is pure in A-**mod**^{a}*.

Any associative ring with identity is finitely presented and by (3.1) von Neumann regular rings are characterized (Fieldhouse [6]). More general self-projective rings with unity [15, Satz (3.4)] are finitely presented in A-**mod**^{a} and another interesting case is given in

Lemma 3.2. *Let A be a left alternative or a Jordan ring with unity which is a finitely generated module over its center $c(A)$. Then A is finitely presented in $L(A)$-**mod** (and hence in A-**mod**^{a}).*

Proof. In the cases under consideration $L(A)$ is a finitely generated $c(A)$-module. The $L(A)$-epimorphism $\mu : L(A) \to A, \mu(\lambda) := \lambda(1)$, splits as $c(A)$-epimorphism and so the kernel of μ is a finitely generated $c(A)$-module and hence finitely generated as $L(A)$-module. Now by (2.1) A is finitely presented in $L(A)$-**mod**.

Lemma 3.3. *Suppose A is an R-algebra with unity, A is finitely presented as R-module and $L(A)$ is finitely generated as R-module. If R is von Neumann regular and $L(A)$ is a left semisimple R-algebra then A is a left regular ring and A is finitely presented in $L(A)$-**mod**.*

Proof. A principal left ideal J of A is finitely generated as R-module and therefore an R-direct summand of A; since $L(A)$ is left semisimple over R this implies that J is an $L(A)$-direct summand of A. A is a projective R-module and therefore a projective $L(A)$-module.

Corollary 3.4. *Let A be an alternative or Jordan algebra over a von Neumann regular ring R and A finitely presented as R-module. If A is separable over R then A is left regular and finitely presented in A-**mod**^{a}.*

Proof. The Jordan case follows immediately from (3.3). If A is alternative and separable we get under the given circumstances via a separability-criterion (Müller [8, Satz 4]) that $L(A)$ is separable and hence semisimple over R.

In a left regular ring every left ideal is associatively generated. So we obtain from Lemmas 4.1, 4.2 in [15] and Theorem 3.1:

Theorem 3.5. *A left regular ring A which is finitely presented in A-**mod**^{a} is a projective generator of A-**mod**^{a}.*

This gives an example for the situation described in [15, Satz (4.3)]. Since $\mathrm{End}_{L(A)}(A)$ is isomorphic to the right nucleus $n(A)$ Theorem 2.7 implies:

Theorem 3.6. *For a ring A which is finitely generated as a right $n(A)$-module the following statements are equivalent:*

(i) *A is left regular and finitely presented in A-**mod**^{a};*

(ii) *A is finitely presented in $L(A)$-**mod** and $L(A)$ is von Neumann regular;*

(iii) *A is a projective, regular $L(A)$-module;*

(iv) *A is a projective generator of $L(A)$-**mod** and $n(A)$ is von Neumann regular.*

Together with Lemma 3.2 the theorem yields:

Corollary 3.7. *For a left alternative or Jordan ring A which is a finitely generated $c(A)$-module there are equivalent:*

(i) *A is left regular;*

(ii) *$L(A)$ is von Neumann regular;*

(iii) *A is regular in A-**mod**^{a}.*

4. Two-sided A-modules

In analogy to the preceding Section we consider the ring A with unity as module over its multiplicationring $M(A)$. Then $\sigma[A]$ is the subcategory of $M(A)$-**mod** whose objects are submodules of central modules (=: A-**bimod**^{c}) and the $\sigma[A]$-projective $M(A)$-modules are the c-projective $M(A)$-modules in [16].

Again we obtain definitions for *pure* short exact sequences, *regular* and *flat* modules in A-**bimod**^{c} by (2.3) and we know from (2.4) that biregular rings are regular in A-**bimod**^{c}.

According to (3.1) we have

Theorem 4.1. *For a ring A with unity which is finitely presented in A-**bimod**^{c} there are equivalent:*

(i) *A is a biregular ring;*

(ii) *A is regular in A-**bimod**^{c};*

(iii) *every finitely presented module in A-**bimod**^{c} is c-projective.*

Of course the other equivalent properties in (3.1) might as well be formulated in the present case.

Contrary to the one-sided situation associative rings A are not projective in A-**bimod**^{c} and they need not even be finitely presented in A-**bimod**^{c}. c-projective rings as characterized in [16, Satz (2.4)] are finitely presented in A-**bimod**^{c} and also:

Lemma 4.2. *Let A be an alternative or Jordan ring which is a finitely generated module over its center $c(A)$. Then A is finitely presented in $M(A)$-**mod** (and hence in A-**bimod**^{c}).*

Proof. The Jordan case is actually contained in (3.2) and the alternative case is proved in complete analogy to (3.2) since $M(A)$ is a finitely generated A-module.

In a biregular ring every ideal is a central A-bimodule and from Lemma 3.6, 3.7 in [16] and Theorem 4.1 it follows:

Theorem 4.3. *A biregular ring with unity which is finitely presented in A-**bimod**^{c} is a projective generator of A-**bimod**^{c}.*

Equivalent formulations for the latter condition are given in [16, Satz (3.8)].

Theorem 4.4. *If the ring A with unity is a finitely generated $c(A)$-module the following statements are equivalent:*

(i) *A is biregular and finitely presented in A-**bimod**^{c};*

(ii) *A is finitely presented in $M(A)$-**mod** and $M(A)$ is von Neumann regular;*

(iii) *A is a projective, regular $M(A)$-module;*

(iv) *A is a projective generator of $M(A)$-**mod** and $c(A)$ is von Neumann regular;*

(v) *A is a projective $c(A)$-module, A is separable over $c(A)$ and $c(A)$ is von Neumann regular.*

Proof. The first four equivalences follow from Theorem 2.7. (v) characterizes A as an Azumaya algebra over a von Neumann regular ring and it is equivalent to (iv) as a consequence of Satz (3.10) in [16].

For associative rings the equivalence of (i) and (v) was proved in Wehlen [14, Theorem (2.3)] by using sheaf-theoretic techniques.

Theorem 4.5. *For an alternative or Jordan ring A with unity which is a finitely generated $c(A)$-module the following are equivalent:*

(i) *A is biregular;*

(ii) *$M(A)$ is von Neumann regular;*

(iii) *A is regular in A-**bimod**^{c};*

(iv) *A is a $c(A)$-Azumaya algebra and $c(A)$ is von Neumann regular;*

(v) *A is a projective $c(A)$-module, $M(A)$ is a separable $c(A)$-algebra and $c(A)$ is von Neumann regular;*

(vi) *A is a projective $c(A)$-module, $L(A)$ (or $R(A)$) is a separable $c(A)$-algebra and $c(A)$ is von Neumann regular;*

(vii) *$L(A)$ and $R(A)$ are von Neumann regular;*

(viii) *A is left and right regular.*

Proof. Under the given conditions A is finitely presented in $M(A)$-**mod** (Lemma 4.2). So the equivalence of (i), (ii) and (iv) follows from Theorem 4.4 and (iv) $\Leftrightarrow$ (v) follows from Satz (3.10) in [16]. (i) $\Leftrightarrow$ (iii) was shown in Theorem 4.1.

For Jordan rings (i), (vii) and (viii) are identical and the same is true for (v) and (vi). So we assume A to be alternative: (vi) $\Rightarrow$ (viii) is a consequence of Lemma 3.3 and (viii) $\Leftrightarrow$ (vii) follows from Corollary 3.7. (viii) $\Rightarrow$ (i) was proved in Proposition 1.8.

(v) $\Leftrightarrow$ (vi) Using a criterion for separability (Müller [8, Satz 4]) we get that under the given conditions $M(A)$ is separable over $c(A)$ iff $L(A)$ or $R(A)$ is separable over $c(A)$.

Notice that Theorem 4.5 is valid for any class of rings, closed under homomorphisms, and having the property: If A is in the class and is a finitely generated $c(A)$-module then

(1) $L(A), R(A)$ and $M(A)$ are finitely generated $c(A)$-modules;

(2) if A is a simple ring then A is left and right semisimple.

For associative rings the equivalence of (i) and (viii) is a consequence of Theorem 6.3 in Michler–Villamayor [7].

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