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Detecting automorphic orbits in free groups

Peter Brinkmann

Department of Mathematics, 160 Convent Ave., NAC 8/133, New York, NY 10031, USA

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ABSTRACT

We present an effective algorithm for detecting automorphic orbits in free groups, as well as a number of algorithmic improvements of train tracks for free group automorphisms.

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Introduction

The following theorem is the main result of this paper.

Theorem 0.1. *Let ϕ be an automorphism of a finitely generated free group F_n.*

- *There exists an explicit algorithm that, given two elements $u, v \in F_n$, decides whether there exists some exponent N such that $u\phi^N = v$.*
- *There exists an explicit algorithm that, given two elements $u, v \in F_n$, decides whether there exists some exponent N such that $u\phi^N$ is conjugate to v.*

If such an exponent N exists, then the algorithms will compute N as well. The words u, v are specified as words in the generators of F_n, and ϕ is specified in terms of the images of generators.

The results in this paper was motivated by work that first appeared in [Bri03]. Theorem 0.1 plays a role in the computation of fixed subgroups of free group automorphisms [Mas03], and it constitutes one part of the recent solution of the conjugacy problem in free-by-cyclic groups due to Bogopolski, Maslakova, Martino, and Ventura [BMMV06].

E-mail address: brinkman@sci.ccny.cuny.edu.

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Our main technical tool is an algorithmic extension of the theory of relative train track maps [BH92,BH95]. Specifically, we present algorithmic (and possibly even practical) ways of finding *efficient* relative train track maps that share many the properties of *improved* relative train track maps as introduced (in a nonconstructive fashion) in [BFH00].

One intriguing aspect of our argument is that it suggests that the detection of orbits in free groups and the computation of efficient maps are closely related problems. Orbit detection and computation of efficient maps leapfrog each other, with orbit detection providing a crucial step in the computation of efficient maps, and efficient maps enabling the detection of orbits.

In Section 1, we review well-known results on homotopy equivalences of finite graphs, with an emphasis on computational aspects of the constants involved. Section 2 contains a brief review of the theory of relative train track maps, including first steps towards improvements. Section 3 contains the first part of our construction of efficient train track maps. Section 4 presents an algorithm that detects orbits of paths, and Section 5 builds upon the results of Section 3 and Section 4 to provide the last, and most difficult, step in our construction of efficient maps, the detection of fixed points of certain lifts of homotopy equivalences of finite graphs. Finally, in Section 6, we translate our results from the realm of homotopy equivalences of graphs to the realm of automorphisms of free groups.

1. Quasi-isometries and bounded cancellation

The results in this section are well known. We list them here, with detailed proofs, because explicit computations of the constants involved do not seem to appear in the literature.

Let $f : G \to H$ be a homotopy equivalence of finite connected graphs, which we equip with the usual path metric (denoted by $|\cdot|$), and let $g : H \to G$ be a homotopy inverse of f.[1] We denote the set of vertices of G by $\mathcal{V}(G)$ and the set of edges by $\mathcal{E}(G)$. Throughout this paper, we only consider homotopy equivalences that map vertices to vertices and edges to edge paths of constant (but not necessarily identical) speed. We may assume that there exists some vertex $\bar{v}_0$ such that $\bar{v}_0 fg = \bar{v}_0$.

Let $\tilde{f} : \tilde{G} \to \tilde{H}$ be a lift of f to the universal covers, with a lift v_0 of $\bar{v}_0$. Given $x, y \in \tilde{G}$, we denote the unique geodesic path connecting x and y by $[x, y]$. For brevity, we write $|x, y|$ for $|[x, y]|$. We define $[x, y]\tilde{f} = [x\tilde{f}, y\tilde{f}]$.[2]

The lift $\tilde{f}$ extends to a homeomorphism of the boundaries $\partial\tilde{G}, \partial\tilde{H}$. Let $\tilde{g} : \tilde{H} \to \tilde{G}$ be a lift of g such that satisfies $v_0\tilde{f}\tilde{g} = v_0$, and note that $\tilde{f}\tilde{g}$ induces the identity on $\partial\tilde{G}$.

Arguments involving universal covers are generally nonconstructive. The universal cover of a finite connected graph, however, is a tree, and we can construct arbitrarily large subtrees as well as partial lifts of maps to these subtrees, which is enough for the computations we will encounter. We describe this construction here, with the tacit understanding that all computations in universal covers will require it as a preliminary step.

Construction 1.1. Fix some vertex $\bar{v}_0 \in G$. Let $v_0 \in \tilde{G}$ be a lift of $\bar{v}_0$ and $w_0 \in \tilde{H}$ a lift of $\bar{w}_0 = \bar{v}_0 f$. We let $T_0 = \{v_0\}$ and $U_0 = \{w_0\}$ and define $\tilde{f}_0 : T_0 \to U_0$ in the only possible way.

Now, suppose we have subtrees $T_0 \subseteq T_1 \subset \tilde{G}$ and $U_0 \subseteq U_1 \subset \tilde{H}$ as well as a partial lift $\tilde{f}_1 : T_1 \to U_1$, i.e., $\tilde{f}|_{T_1} = \tilde{f}_1$. Our goal is to enlarge T_1 and U_1 and extend $\tilde{f}_1$ accordingly.

There is a bijective relationship between vertices of $\tilde{G}$ and edge paths in G originating at $\bar{v}_0$.[3] Let ρ be an edge path in G originating at $\bar{v}_0$. We want to construct T_2 so that it contains a lift of ρ. To this end, starting with v_0 and the first edge of ρ, we keep track of a current vertex v and a current edge E. If T_1 already contains an edge E' originating at v that projects to E, we make the other endpoint of E' our current vertex and move on to the next edge of ρ. If no such edge exists, we

[1] Given f, we can easily compute g (see, for instance, [LS77]).

[2] Note that the composition of the path $[x, y]$ and $\tilde{f}$ is not, in general, an immersion. The path $[x\tilde{f}, y\tilde{f}]$ is the unique immersed path that is homotopic relative endpoints to this composition.

[3] In our computations, we will always be given such paths for those vertices of $\tilde{G}$ that we are interested in.

attach a new edge at v and map it to E. Then we move on to the terminal endpoint of the new edge and the next edge in ρ.[4]

Now, for each vertex v of $T_2 \setminus T_1$, we compute the image ρ_v of the path $[v_0, v]$ in G, and we construct a lift of $\rho_v f$ to the universal cover. Like before, we construct U_2 by extending U_1 such that it includes these lifts, obtaining a larger subtree of $\tilde{H}$ as well as a partial lift $\tilde{f}_2 : T_2 \to U_2$.

Proceeding in this fashion, we can build arbitrarily large subtrees of $\tilde{G}$ and $\tilde{H}$ along with partial lifts of f. If $G = H$, we can and will arrange that $T_2 \subseteq U_2$.

The lift $\tilde{f}$ is a *quasi-isometry*, i.e., there exist constants K_f, D_f such that for all $x, y \in \tilde{G}$, we have

$$\frac{|x, y|}{K_f} - D_f \leqslant |x\tilde{f}, y\tilde{f}| \leqslant K_f|x, y| + D_f. \tag{1}$$

We need to compute suitable constants K_f, D_f. To this end, define the *size* of f to be $S_f = \max_{E\in\mathcal{E}(G)}\{|Ef|\}$.

Lemma 1.2. *We can compute a number B_{fg} satisfying*

$$B_{fg} \geqslant \max_{x\in\tilde{G}}\bigl\{|x, x\tilde{f}\tilde{g}|\bigr\}.$$

Proof. We first compute $B = \max_{v\in\mathcal{V}(\tilde{G})}\{|v, v\tilde{f}\tilde{g}|\}$. Let γ be a deck transformation of $\tilde{G}$. Since $\tilde{f}\tilde{g}$ extends to the identity on $\partial\tilde{G}$, we have $\gamma\tilde{f}\tilde{g} = \tilde{f}\tilde{g}\gamma$.

For $v \in \mathcal{V}(\tilde{G})$, we have $|v\gamma, v\gamma\tilde{f}\tilde{g}| = |v\gamma, v\tilde{f}\tilde{g}\gamma| = |v, v\tilde{f}\tilde{g}|$, so that we only need to check one representative of each orbit of vertices. The distance $|v, v\tilde{f}\tilde{g}|$ is the length of the path obtained by concatenating $[v, v_0]$ and $[v_0, v\tilde{f}\tilde{g}]$ and tightening. Hence, we can compute B.

Now consider some point $x \in \tilde{G}$. Then there exists some vertex $v \in \mathcal{V}(\tilde{G})$ such that $|x, v| < 1$, so that $|x, x\tilde{f}\tilde{g}| \leqslant 1 + |v, v\tilde{f}\tilde{g}| + S_{fg} \leqslant 1 + B + S_{fg}$. □

Lemma 1.3. *Inequality* (1) *holds with $K_f = \max\{S_f, S_g\}$ and $D_f = \frac{2B_{fg}}{K_f}$.*

Proof. Let $x, y \in \tilde{G}$. By definition of K_f, we have $|x\tilde{f}, y\tilde{f}| \leqslant K_f|x, y|$, so that the upper bound in Inequality (1) holds.

Similarly, we have $|x\tilde{f}\tilde{g}, y\tilde{f}\tilde{g}| \leqslant K_f|x\tilde{f}, y\tilde{f}|$. The triangle inequality implies that $|x, y| \leqslant |x, x\tilde{f}\tilde{g}| + |x\tilde{f}\tilde{g}, y\tilde{f}\tilde{g}| + |y\tilde{f}\tilde{g}, y| \leqslant |x\tilde{f}\tilde{g}, y\tilde{f}\tilde{g}| + 2B_{fg} \leqslant K_f|x\tilde{f}, y\tilde{f}| + 2B_{fg}$. We conclude that $|x, y| - 2B_{fg} \leqslant K_f|x\tilde{f}, y\tilde{f}|$, and the claim follows. □

Thurston's *Bounded Cancellation Lemma* [Coo87] is a fundamental tool in the theory of free group automorphisms. We present a proof here because we require an explicit bound on the constant involved.

Let p, x, y be points in $\tilde{G}$ and let $\alpha = [p, x]$ and $\beta = [p, y]$. We denote the common (possibly trivial) initial segment of α and β by $\alpha \wedge \beta$. If α is a prefix of β, we write $\alpha \leqslant \beta$.

Lemma 1.4 *(Bounded Cancellation Lemma). Let $C_f = (B_{fg} + D_g + S_g)K_g$. If $|\alpha \wedge \beta| = 0$, then*

$$|\alpha\tilde{f} \wedge \beta\tilde{f}| \leqslant C_f.$$

[4] An alternative approach is to attach an entire lift of ρ at v_0 and then fold as necessary [Sta83].

Proof. Let $L = |\alpha\tilde{f} \wedge \beta\tilde{f}|$. Inequality (1) implies that $|(\alpha\tilde{f} \wedge \beta\tilde{f})\tilde{g}| \geqslant \frac{L}{K_g} - D_g$, so that $|\alpha\tilde{f}\tilde{g} \wedge \beta\tilde{f}\tilde{g}| \geqslant \frac{L}{K_g} - D_g - S_g$. Now Lemma 1.2 implies that

$$|\alpha \wedge \beta| \geqslant \frac{L}{K_g} - D_g - S_g - B_{fg}.$$

Hence, if $L > C_f$, then $|\alpha \wedge \beta| > 0$. □

Finally, we record a basic property of homotopy equivalences of graphs.

Lemma 1.5. *Let $f : G \to G$ be a homotopy equivalence of a finite graph. If α is a path in G whose endpoints are fixed by f, then there exists some path β with the same endpoints satisfying $\beta f = \alpha$.*

Proof. Let v be the initial endpoint of α. Then there exists some loop σ based at v so that αf is homotopic (relative endpoints) to the concatenation $\sigma\alpha$. Since f is a homotopy equivalence, there exists a loop σ' satisfying $\sigma' f = \sigma$, and we conclude that $(\bar{\sigma}'\alpha)f = \alpha$. □

2. Relative train track maps

In this section, we review the theory of relative train tracks maps [BH92,DV96] as well as first steps towards our take on improvements of relative train track maps.

Given an automorphism $\phi \in Aut(F)$, we can find a based homotopy equivalence $f : G \to G$ of a finite connected graph G such that $\pi_1(G) \cong F$ and f induces ϕ. This observation allows us to apply topological techniques to automorphisms of free groups. In many cases, it is convenient to work with outer automorphisms. Topologically, this means that we work with homotopy equivalences rather that based homotopy equivalences.

Oftentimes, a homotopy equivalence $f : G \to G$ will respect a *filtration* of G, i.e., there exist subgraphs $G_0 = \emptyset \subset G_1 \subset \cdots \subset G_k = G$ such that for each filtration element G_r, the restriction of f to G_r is a homotopy equivalence of G_r. The subgraph $H_r = \overline{G_r \setminus G_{r-1}}$ is called the *r-th stratum* of the filtration. We say that a path ρ has *nontrivial intersection* with a stratum H_r if ρ crosses at least one edge in H_r.

If $H_r = \{E_1, \ldots, E_m\}$, then the *transition matrix* of H_r is the nonnegative $m \times m$-matrix M_r whose ij-th entry is the number of times the f-image of E_j crosses E_i, regardless of orientation. M_r is said to be *irreducible* if for every tuple $1 \leqslant i, j \leqslant m$, there exists some exponent $n > 0$ such that the ij-th entry of M_r^n is nonzero. If M_r is irreducible, then it has a maximal real eigenvalue $\lambda_r \geqslant 1$ [Gan59]. We call λ_r the *growth rate* of H_r.

Given a homotopy equivalence $f : G \to G$, we can always find a filtration of G such that each transition matrix is either a zero matrix or irreducible. A stratum H_r in such a filtration is called *zero stratum* if M_r is a zero matrix. H_r is called *exponential* if M_r is irreducible with $\lambda_r > 1$, and it is called *nonexponential* if M_r is irreducible with $\lambda_r = 1$.

An unordered pair of edges in G originating from the same vertex is called a *turn*. A turn is called *degenerate* if the two edges are equal. We define a map $Df : \{\text{turns in } G\} \to \{\text{turns in } G\}$ by sending each edge in a turn to the first edge in its image under f. A turn is called *illegal* if its image under some iterate of Df is degenerate; otherwise, it is called *legal*.

An edge path $\rho = E_1 E_2 \cdots E_s$ is said to contain the turns (E_i^{-1}, E_{i+1}) for $1 \leqslant i < s$; ρ is legal if all its turns are legal, and it is r-legal if $\rho \subset G_r$ and no illegal turn in ρ involves an edge in H_r.

Let ρ be a path in G. In general, the composition $\rho \circ f^k$ is not an immersion, but there is a unique immersion that is homotopic to $\rho \circ f^k$ relative endpoints. We denote this immersion by ρf^k, and we say that we obtain ρf^k from $\rho \circ f^k$ by *tightening*. If σ is a circuit in G, then σf^k is the immersed circuit homotopic to $\sigma \circ f^k$.

Theorem 2.1. *(See [BH92, Theorem 5.12].) Every outer automorphism of F is represented by a homotopy equivalence $f : G \to G$ such that each exponential stratum H_r has the following properties*:

1. *If E is an edge in H_r, then the first and last edges in Ef are contained in H_r.*
2. *If β is a nontrivial path in G_{r-1} with endpoints in $G_{r-1} \cap H_r$, then βf is nontrivial.*
3. *If ρ is an r-legal path, then ρf is an r-legal path.*

We call f a *relative train track map*. A detailed, explicit algorithm for computing relative train track maps appeared in [DV96].

We conclude this section with the introduction of some terminology that will be needed later.

A path ρ is a *(periodic) Nielsen path* if $\rho f^k = \rho$ for some $k > 0$. In this case, the smallest such k is the *period* of ρ. A Nielsen path ρ is called *indivisible* if it cannot be expressed as a concatenation of shorter Nielsen paths.

A decomposition of a path $\rho = \rho_1 \cdot \rho_2 \cdot \cdots \cdot \rho_s$ into subpaths is called a *k-splitting* if $\rho f^k = \rho_1 f^k \cdots \rho_s f^k$, i.e., there is no cancellation between $\rho_i f^k$ and $\rho_{i+1} f^k$ for $1 \leqslant i < s$. Such a decomposition is a *splitting* if it is a k-splitting for all $k > 0$. We will also use the notion of k-splittings of circuits $\sigma = \rho_1 \cdot \rho_2 \cdot \cdots \cdot \rho_s$, which requires, in addition, that there be no cancellation between $\rho_s f^k$ and $\rho_1 f^k$.

The *r-length* of a path ρ in G, denoted by $|\rho|_r$, is the number of edges in H_r that ρ crosses. A path ρ in G is said to be of *height* r if ρ is contained in G_r but not in G_{r-1}. If $H_r = \{E_r\}$ is a nonexponential stratum, then *basic paths* of height r are of the form $E_r\gamma$ or $E_r\gamma E_r^{-1}$, where γ is a path in G_{r-1}.

Definition 2.2. We say that a relative train track map $f : G \to G$ is *normalized* if the following properties hold:

1. For every vertex $v \in \mathcal{V}(G)$, vf is a fixed vertex of f.
2. Every nonexponential stratum H_r contains only one edge E_r and $E_r f = E_r u_r$ for some path u_r in G_{r-1}.
3. If $H_r = \{E_r\}$ is a nonexponential stratum, u_r is of height s, and $s < t < r$, then H_t is nonexponential and u_t is also of height s.
4. If E is an edge in an exponential stratum H_r, then $|Ef|_r \geqslant 2$.
5. Every isolated fixed point of f is a vertex.
6. If C is a noncontractible component of some filtration element G_r, then $C = Cf$.

Lemma 2.3. *Every outer automorphism $\mathcal{O}$ has a positive power $\mathcal{O}^k$ that is represented by a normalized relative train track map $f : G \to G$. Both k and f can be computed.*

Proof. First, we compute a relative train track map $f' : G' \to G'$ representing $\mathcal{O}$ [BH92,DV96]. We easily read off an exponent k such that f'^k satisfies the first, fourth, and sixth properties of normalized maps, and we have $Ef'^k = vEw$ for every edge E in a nonexponential stratum H_r.

After replacing f by a power f^k, we may need to refine the filtration of G because an irreducible matrix may have reducible powers. We may also need to permute some filtration elements in order to achieve the desired alignment of nonexponential strata.

If v is nontrivial and w is trivial, we reverse the orientation of E. If both v and w are nontrivial, we split E into two edges E', E'' such that $E = \bar{E}'E''$ and $E'f'^k = E'\bar{v}$ and $E''f'^k = E''w$.

By refining the filtration of G' so that each nonexponential stratum contains exactly one edge and subdividing at isolated fixed points if necessary, we obtain a normalized representative $f : G \to G$ of $\mathcal{O}^k$. $\square$

Lemma 2.4. *Let $f : G \to G$ be a normalized relative train track map with an exponential stratum H_r. If C is a noncontractible component of G_{r-1} and v is a vertex in $H_r \cap C$, then $v = vf$.*

Proof. This argument is contained in the proof of [BFH00, Theorem 5.1.5]. We repeat it here because it is short.

Let v be a vertex in $H_r \cap C$. Since f is normalized, we have $C = Cf$, so that there exists a path α in C that starts at v and ends at vf. The vertex vf is fixed, and there exists some path β in C that starts and ends at vf such that $\alpha f = \beta f$. Then $(\alpha\bar{\beta})f$ is trivial, so that $\alpha\bar{\beta}$ is trivial because of the second property of relative train track maps. □

Lemma 2.5. *Let $f : G \to G$ be a normalized train track map with a nonexponential stratum H_r. If ρ is a path in G_r, then it splits as a concatenation of basic paths of height r and paths in G_{r-1}.*

Proof. This is essentially [BFH00, Lemma 4.1.4]. The lemma follows immediately from the second property of normalized train track maps. □

Lemma 2.6. *Let $f : G \to G$ be a normalized train track map with an exponential stratum H_r. If ρ is a circuit or edge path of height r containing an r-legal subpath of r-length $L > 2C_f$ (where C_f is the bounded cancellation constant of f), then ρf contains an r-legal subpath of r-length greater than L.*

Proof. This is an immediate consequence of Lemma 1.4 and the fourth property of normalized maps, which implies $\lambda_r \geqslant 2$. □

We will need the following consequence of [Bri00, Proposition 6.2].

Lemma 2.7. *Let $f : G \to G$ be a relative train track map with an exponential stratum H_r. If ρ is an edge path of height r and $L_0 > 0$, then at least one of the following three possibilities occurs*:

- *ρf^M contains an r-legal segment of r-length greater than L_0.*
- *ρf^M contains fewer r-illegal turns than ρ.*
- *ρf^M is a concatenation of indivisible Nielsen paths of height r and paths in G_{r-1}.*

3. Improving nonexponential strata

In [BFH00], the authors improve the behavior of nonexponential strata in a nonconstructive fashion. We retrace some of their steps here, replacing the nonconstructive parts by constructive arguments.

Let $H_r = \{E_r\}$ be a nonexponential stratum of a normalized train track map $f : G \to G$, and let ρ be a path in G_{r-1} originating at the terminal vertex of E_r. We define a new map $f' : G' \to G'$ by removing E_r and adding an edge E'_r whose initial vertex is the initial endpoint of E_r and whose terminal vertex is the terminal vertex of ρ. We obtain u'_r by tightening $\bar{\rho}u_r(\rho f)$, so that $E'_r f' = E'_r u'_r$. There is an obvious homotopy equivalence $g : G \to G'$ that sends E_r to $E'_r\bar{\rho}$. With this marking, f' induces the same outer automorphism as f. We say the E'_r is obtained from E_r by *sliding along* ρ.

Let $\tilde{f} : \tilde{G} \to \tilde{G}$ be a lift of f that fixes the initial endpoint of a lift $\tilde{E}_r$ of E_r. Then $\tilde{f}$ leaves invariant a copy H of the universal cover of the connected component of G_{r-1} that contains u_r. Let $h = \tilde{f}|_H$, and let v_0 be the terminal endpoint of $\tilde{E}_r$. Note that $v_0 \in H$, and that $[v_0, v_0 h]$ projects to u_r.

Lemma 3.1. *There exists a slide of E_r to E'_r with $E'_r f' = E'_r$ if and only if h fixes a point in H.*

Proof. If h fixes $v \in H$, then sliding E_r along $[v_0, v]$ yields a fixed edge E'_r. Conversely, if there exists a path ρ such that sliding E_r along ρ yields a fixed edge, then the terminal endpoint of the lift of ρ is fixed by h. □

In Section 5, we present an algorithm for detecting fixed points of h.

Lemma 3.2. *Assume that h has no fixed points. Let $U_k = [v_0, v_0 h^k]$ and $V_k = U_k \wedge U_{k+1}$ for $k \geqslant 0$. Then V_k is a proper prefix of V_{k+1}.*

Proof. This follows from the discussion of preferred edges in the proof of [BFH00, Proposition 5.4.3]. □

As an immediate consequence of Lemma 3.2, we obtain the following lemma.

Lemma 3.3. *If h has a periodic point, then h has a fixed point.*

The following proposition is the main result of this section; it replaces a nonconstructive argument in [BFH00].

Proposition 3.4. *Assume that h has no fixed points. We can compute a vertex in $v \in H$ and an exponent $m \geqslant 1$ such that sliding E_r along $[v_0, v]$ yields $E_r'(f^m)' = E_r' \cdot u_r'$ and u_r' is a closed path starting and ending at a fixed vertex.*

Proof. Let v_k equal the terminal vertex of the path V_k (Lemma 3.2),[5] and let $w_k = [v_k, v_{k+1}]$. The path w_{k+m} is a subpath of $w_k h^m$ for all $k, m \geqslant 0$.

The idea of the proof is to compute $w_0, w_1, w_2, \ldots, w_k$ until we identify a suitable vertex v in w_k. Since w_{k+1} is a subpath of $w_k h$, we have *height*$(w_{k+1}) \leqslant$ *height*(w_k), so that the height of the paths w_k has to stabilize eventually. The following procedure assumes that the height remains constant; should the height drop while the procedure is in progress, we simply start over.

Assume the height stabilizes at r. This means that H_r cannot be a zero stratum. Now, if H_r is nonexponential, we have $|w_{k+1}|_r \leqslant |w_k|_r$. We keep iterating until we find w_k such that $|w_k|_r = |w_{k+1}|_r \geqslant 1$. Let v be the initial endpoint of an occurrence of E_r in w_k. Then v has the desired properties (and we do not need to replace f by a higher power in this case).

Now, assume that H_r is exponential. If we encounter a path w_k that contains an r-legal subpath of r-length at least $2(C_f + 1)$, then w_{k+1} contains a vertex v that projects to a fixed vertex of f and whose r-distance from the closest r-illegal turn is at least C_f. Now Lemma 1.4 yields that v has the desired properties.

Assume that the length of r-legal subpaths remains bounded below $2(C_f + 1)$. The number of illegal turns cannot go up and must stabilize eventually, so that eventually we will end up in the third case of Lemma 2.7 and see a composition of Nielsen paths of height r and paths in G_{r-1}. We can detect this case in a brute-force fashion, by checking all subpaths of w_k in order to see whether they are Nielsen.

Let v be the initial point of one of the Nielsen paths. Then v is periodic of period m, so that sliding E_r along $[v_0, v]$ yields the desired improvement of f^m. □

Definition 3.5. Let $f : G \to G$ be a normalized relative train track map with a nonexponential stratum $H_r = \{E_r\}$. We say that H_r is *efficient*:

1. if $E_r f$ splits as $E_r \cdot u_r$ and u_r is a closed path in G_{r-1},
2. if u_r is a periodic Nielsen path, then its period is one (in this case, we say that E_r is *linear*), and
3. if u_r is nontrivial, then there exists no slide of E_r to E_r' such that $E_r' f' = E_r'$.

We say that a relative train track map is *efficient* if it is normalized, all its nonexponential strata are efficient, and the nonexponential strata are sorted in such a way that if u_r and u_s are of the same height but u_r is Nielsen and u_s is not, then $s > r$.

Lemma 3.6. *There exists a slide of E_r to E_r' with $E_r' f' = E_r' u_r'$ and u_r' a periodic Nielsen path if and only if h commutes with a nontrivial deck transformation.*

[5] This agrees with our original definition of v_0.

Proof. This lemma follows from [BFH00, Proposition 5.4.3]. □

Remark 3.7. Lemma 3.6 implies that if H_r is efficient and u_r is nontrivial and non-Nielsen, then there exists no slide that takes u_r to a periodic Nielsen path.

An infinite ray ρ starting at a fixed vertex v_0 is a *fixed ray* if $\rho f = \rho$. It is *attracting* if there exists some N such that if η is a ray starting at v_0 and $|\rho \wedge \eta| > N$, then ηf^n converges to ρ, i.e., $|\rho \wedge \eta f^n|$ goes to infinity. A *repelling* fixed ray is an attracting fixed ray for a homotopy inverse of f. See [LL04] for a detailed discussion attracting and repelling fixed points for free group automorphisms.

Lemma 3.8. *Let $f : G \to G$ be an efficient relative train track map with a nonexponential stratum $H_r = \{E_r\}$ that is neither linear nor constant. Let*

$$R_r = E_r u_r (u_r f)\big(u_r f^2\big) \cdots .$$

Then R_r is the unique attracting fixed ray of the form $E_r\gamma$, for $\gamma \subset G_r$, and there are no Nielsen paths of the form $E_r\gamma$. In particular, we have $\lim_{k\to\infty} \rho f^k = R_r$ for all basic paths ρ of height r.

Proof. This lemma follows from the proof of [BFH00, Lemma 5.5.1]. The assumptions of [BFH00] are stronger that our assumptions, but a close inspection of the proof shows that only our assumptions are needed for the results that we use here. □

If ρ is a path starting and ending at fixed points, then we can find at most one path ρ' with the same endpoints such that $\rho' f = \rho$. In this case, we write $\rho' = \rho f^{-1}$. We define ρf^{-k} in the obvious fashion. If ρ is closed, then ρf^{-k} exists for all k.

Lemma 3.9. *Let $f : G \to G$ be an efficient relative train track map with a nonexponential stratum $H_r = \{E_r\}$ that is neither linear nor constant. Let*

$$S_r = E_r\big(\bar{u}_r f^{-1}\big)\big(\bar{u}_r f^{-2}\big) \cdots .$$

Then S_r is the unique repelling fixed ray of the form $E_r\gamma$, for $\gamma \subset G_r$. In particular, we have $\lim_{k\to\infty} \rho f^{-k} = S_r$ for all basic paths ρ of height r.

Proof. Lemma 3.8 implies that h only has one repelling fixed ray. Since S_r is clearly fixed, it is the unique repelling fixed ray. □

4. Detecting orbits of paths

If H_r is an exponential stratum and ρ is a path of height r, we let $\iota_r(\rho)$ equal the number of r-illegal turns in ρ.

Lemma 4.1. *Let $f : G \to G$ be an efficient relative train track map. If ρ is a circuit or edge path in G, then we can determine algorithmically whether ρ is a periodic Nielsen path; if ρ is Nielsen, then we can compute its period as well.*

Proof. Assume inductively that we can detect periodic Nielsen paths and circuits in G_{r-1}. We want to show that if ρ is of height r, then we can determine whether ρ is Nielsen.

We first assume that $H_r = \{E_r\}$ is nonexponential. Then ρ splits as a concatenation of basic paths of height r and paths in G_{r-1} (Lemma 2.5), and it is Nielsen if and only if each of these constituent paths is Nielsen. Hence, we may assume that ρ is a basic path of height r, i.e., $\rho = E_r\gamma$ or $\rho = E_r\gamma\bar{E}_r$ for some $\gamma \in G_{r-1}$. If $E_r f = E_r$, then ρ is Nielsen if and only if γ is Nielsen so that we are done by induction. If E_r is neither constant nor linear, then Lemma 3.8 yields that ρ cannot be Nielsen.

This leaves the case that E_r is linear. If $\rho = E_r\gamma$, then it cannot be Nielsen (if $E_r\gamma$ were Nielsen, then Lemma 3.3 would imply that we can slide E_r to a constant edge, in violation of efficiency of f). Clearly, a path of the form $E_r\gamma\bar{E}_r$ can only be Nielsen if γ is a (possibly negative) power of u_r, which completes the proof for nonexponential H_r.

Now, assume that H_r is exponential. If an endpoint of ρ is not fixed, then ρ cannot be Nielsen. If both endpoints of ρ are fixed, we compute $\rho, \rho f, \rho f^2, \ldots$ until one of the following three cases occurs:

- We encounter some image ρf^k that contains an r-legal path whose length exceeds $2C_f$. Then Lemma 2.6 implies that ρ is not Nielsen.
- We encounter some image ρf^k that contains fewer r-illegal turns than ρ. Since f does not increase the number of r-illegal turns, ρ is not Nielsen.
- We can express ρ as $\rho = \alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_m\beta_m$, where the α_i are Nielsen paths of height r, and the β_i are subpaths in G_{r-1}, such that we encounter some $\rho f^k = \alpha_1(\beta_1 f^k)\cdots\alpha_m(\beta_m f^k)$. In this case, ρ is Nielsen if and only if the β_i are Nielsen.

One of these three cases must occur eventually, and we can detect the third case in a brute-force way by checking all possible decompositions of ρ.

Finally, if H_r is a zero stratum, then ρ cannot possibly be Nielsen, so that the proof is complete. □

If u is a closed path and ρ is an arbitrary edge path, we let $p_u(\rho)$ equal the largest exponent m so that u^m is a prefix of ρ.

Lemma 4.2. *Let $f : G \to G$ be a relative train track map with an exponential stratum H_r and a closed Nielsen path u of height r. If ρ is an edge path of height r and $k \geqslant 0$ an exponent such that $p_u(\rho) = m$ and $p_u(\rho f^k) = l$, then $\iota_r(\rho) \geqslant (2m - l - 1)\iota_r(u)$.*

Proof. We express ρ as $\rho = u^m\gamma$. Since we have $p_u(\rho f^k) = l$, we conclude that $p_{\bar{u}}(\gamma f^k) \geqslant m - l - 1$, so that $\iota_r(\gamma f^k) \geqslant (m - l - 1)\iota_r(u)$. Since f does not introduce new illegal turns, we have $\iota_r(\gamma) \geqslant (m - l - 1)\iota_r(u)$, so that $\iota_r(\rho) \geqslant (2m - l - 1)\iota_r(u)$. □

Lemma 4.3. *Let $f : G \to G$ be an efficient train track map and let ρ be a non-Nielsen path whose endpoints are fixed. Then for any $L > 0$, we can compute an exponent $k_0 > 0$ such that $|\rho f^k| > L$ and $|\rho f^{-k}| > L$ (if ρf^{-k} exists) for all $k \geqslant k_0$.*

Proof. We assume inductively that the lemma holds for the restriction of f to G_{r-1}. We first assume that $H_r = \{E_r\}$ is a nonexponential stratum. Then ρ splits as a concatenation of basic paths of height r and paths in G_{r-1}, so that we may assume that ρ is a non-Nielsen basic path of height r, i.e., $\rho = E_r\gamma\bar{E}_r$ or $\rho = E_r\gamma$.

Assume that E_r is neither constant nor linear. Then we can find a prefix R of R_r as well as a prefix S of S_r (see Lemmas 3.8 and 3.9) of length greater than L for which $|Rf| > |R| + C_f$ and $|Sf^{-1}| > |S| + C_f$. Now Lemmas 3.8, 3.9, and 1.4 imply that we can find some exponent k_0 such that R is a prefix of ρf^k and S is a prefix of ρf^{-k} for all $k \geqslant k_0$. We conclude that $|\rho f^{\pm k}| > L$ for all $k \geqslant k_0$.

If E_r is constant, then the inductive hypothesis applied to γ completes the proof. This leaves the case that E_r is linear. Let s be the height of γ. If s is smaller than the height of u_r, we conclude that no copy of u_r will cancel completely in ρf^k for any $k > 0$, so that we have $|\rho f^{\pm k}| > L$ for all $k > L$.

If s equals the height of u_r and H_s is nonexponential, then no more than $|\gamma|$ copies of u_r cancel in $|\rho f^k|$, so that we have $|\rho f^{\pm k}| > L$ for all $k > L + |\gamma|$. If H_s is exponential, then for all

$k \geqslant 0$, the number of copies of u_r that cancel in ρf^k is bounded by $\iota_s(\gamma)$, so that $|\rho f^k| > L$ if $k > L + \iota_s(\gamma)$.

We still need to study the length of ρf^{-k} for $k \geqslant 0$. Let $m = p_{u_r}(\gamma f^{-k})$ and $l = p_{u_r}(\gamma)$. Then Lemma 4.2 implies that $\iota_r(\gamma f^{-k}) \geqslant (2m - l - 1)\iota_r(u_r)$. This implies that $\iota_r(\rho f^{-k}) \geqslant k\iota_r(\bar{u}_r) + (2m - l - 1)\iota_r(u_r) - 2m\iota_r(u_r) = (k - l - 1)\iota_r(u_r)$, so that $|\rho f^{-k}| > L$ if $k > L + l + 1$.

If s exceeds the height of u_r, then, by definition of efficiency, H_s is also linear, and ρ splits into subpaths of the form $E_r\eta$, $E_s\eta$, and $E_r\eta\bar{E}_s$, where $\eta \subset G_{s-1}$. The first two cases are done by induction on s, so that we only need to consider the case $E_r\eta\bar{E}_s$. This case is essentially the same as the previous one (we need to apply Lemma 4.2 to both η and $\bar{\eta}$), except we need to consider the possibility that there is a closed Nielsen path τ such that $u_r = \tau^a$, $u_s = \tau^b$, and $\eta = \tau^c$. In this case, we have $a \neq b$ (or else $E_r\bar{E}_s$ would be Nielsen, in violation of efficiency), so that $|(E_r\eta\bar{E}_s)f^k| \geqslant k - c$, so that $|(E_r\eta\bar{E}_s)f^k| > L$ if $k > L + c$.

Finally, assume that H_r is exponential. In this case, we compute $\rho, \rho f, \ldots$ until we either find some k_0 such that ρf^{k_0} has an r-legal subpath of r-length greater than $L + 2C_f$ (in which case Lemma 2.6 yields that $|\rho f^k| > L$ for all $k \geqslant k_0$), or, by Lemma 2.7, we encounter some k such that ρf^k is a composition of indivisible Nielsen paths of height r and paths in G_{r-1}. Since ρ is non-Nielsen, one of the subpaths in G_{r-1} must be non-Nielsen, so that we are done by induction.

In order to understand lengths under backward iteration, we need to consider two cases: If ρ is not a composition of indivisible Nielsen paths of height r and paths in G_{r-1}, then Lemma 2.7 implies that the number of r-illegal turns has to go up under backward iteration. In this case, we simply compute $\rho f^{-1}, \rho f^{-2}, \ldots$ until we find some k_0 for which ρf^{-k_0} contains L r-illegal turns, and we conclude that $|\rho f^{-k}| > L$ for all $k \geqslant k_0$.

If ρ is a concatenation of indivisible Nielsen paths of height r and paths in G_{r-1}, then one of the subpaths γ in G_{r-1} is not Nielsen, so that the inductive hypothesis applies to γ. Lemma 1.5 guarantees that γf^{-k} exists for all $k \geqslant 0$, so that we are done. □

Proposition 4.4. *Let $f : G \to G$ be an efficient train track map, and let ρ_1 and ρ_2 be paths whose endpoints are fixed. Then we can determine algorithmically whether ρ_2 is the image of ρ_1 under some power of f^k, and we can compute the exponent k if it exists.*

Proof. Using Lemma 4.1, we determine whether ρ_1 is a periodic Nielsen path. If it is, we simply enumerate all distinct images of ρ_1 and check whether ρ_2 is among them. If ρ_1 is not Nielsen, we apply Lemma 4.3 with $L = |\rho_2|$ to obtain an exponent k_0. Now we compute $\rho, \rho f, \ldots, \rho_1 f^{k_0}$ and check whether ρ_2 is contained in this list.

If ρ_2 is contained in this list, we obtain a positive answer as well as the desired exponent k. If not, we switch ρ_1 and ρ_2 and repeat the argument. □

Theorem 4.5. *Let $f : G \to G$ be an efficient train track map with an exponential stratum H_r. Then we can compute all indivisible periodic Nielsen paths of height r as well as their periods.*

Proof. Let α be an indivisible Nielsen path of height r. Then α contains exactly one r-illegal turn, and the r-length of its two r-legal subpaths is bounded by C_f (Lemma 1.4). Moreover, the first and last (possibly partial) edges of α are contained in H_r.

For an edge E in H_r, let P_E be the set of maximal subpaths in G_{r-1} of Ef, and let $P = \bigcup_{E \in H_r} P_E$. If β is a maximal subpath in G_{r-1} of α, then there exists some $\gamma \in P$ and $k \geqslant 0$ such that $\beta = \gamma f^k$.

Let γ be a path in P. If γ is Nielsen, we let $L_\gamma = \max_k\{|\gamma f^k|\}$. If γ is not Nielsen, Lemma 4.3 with $L = C_f$ yields an exponent k_0 such that $|\rho f^k| > L$ for all $k \geqslant k_0$. We let $L_\gamma = \max_{0 \leqslant k < k_0}\{|\rho f^k|\}$.

Let $M = \max_{\gamma \in P}\{L_\gamma\}$ and observe that α has no subpaths in G_{r-1} whose length exceeds M. Let Q be the set of all edge paths ρ such that ρ contains exactly one r-illegal turn, the r-length of r-legal subpaths is bounded by C_f, the length of subpaths in G_{r-1} is bounded by M, and the first and last edges are contained in H_r. Clearly, if α is an indivisible Nielsen path of height r, then α is a subpath of some $\rho \in Q$.

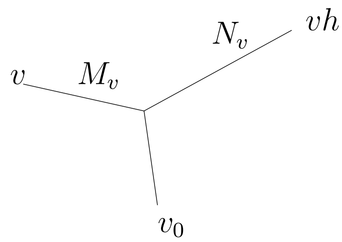

Fig. 1. Looking for fixed points.

We define a map $g : Q \to G \cup \{*\}$[6] by letting ρg equal the unique maximal subpath of ρf contained in Q if ρf contains an r-illegal turn, and we let $\rho g = *$ if ρf contains no r-illegal turn.

For each $\rho \in G$, we compute $\rho, \rho g, \rho g^2, \ldots$ until we either encounter $*$ (in which case ρ has no Nielsen subpath) or we find that $\rho g^k = \rho g^m$ for some $0 \leqslant k < m$. Then ρg^k contains an indivisible Nielsen subpath α, and we can easily compute the endpoints of α. Moreover, if k and m are as small as possible, then $m - k$ is the period of α. Since all indivisible Nielsen paths of height r show up in this fashion, the proof is complete. □

Corollary 4.6. *Given an efficient relative train track map $f : G \to G$, we can compute an exponent $k \geqslant 1$ such that all periodic Nielsen paths of f^k have period one.*

5. Detecting fixed points

Let $f : G \to G$ be a normalized relative train track map with a nonexponential stratum $H_r = \{E_r\}$. Assume that the restriction of f to G_{r-1} is efficient. The purpose of this section is to present an algorithm for determining whether E_r has a slide to a constant edge (Proposition 5.6). This is the last missing piece in our computation of efficient maps (Theorem 5.7).

We have $E_r f = E_r u_r$, and we want to express u_r as the path obtained by tightening $\bar{\rho}(\rho f)$ for some path ρ in G_{r-1}, if possible. To this end, choose a fixed vertex $\bar{v}_0 \in G_{r-1}$. The main idea is to perform a breadth-first search of edge paths ρ originating at $\bar{v}_0$, keeping track of the paths obtained by tightening $\bar{\rho}(\rho f)$ until we either encounter u_r or we determine that further searching will not yield u_r. If we encounter u_r along the way, then sliding E_r along $\bar{\rho}$ will turn it into a constant edge.

It will be convenient to work in the universal cover H of G_{r-1}, constructing partial lifts h of f as we go along (Construction 1.1), beginning with $T_0 = U_0 = \{v_0\}$. For a vertex v in H, we define ρ_v to be the path $[v_0, v]$ and w_v to be the projection of $[v, vh]$. Note that w_v is the projection of the path obtained by tightening $\bar{\rho}_v(\rho_v h)$.

We let $M_v = |v_0, v| - |[v_0, v] \wedge [v_0, vh]|$ and $N_v = |v_0, vh| - |[v_0, v] \wedge [v_0, vh]|$ (Fig. 1). Note that $|w_v| = M_v + N_v$.

The following is a partial list of conditions under which we need not extend our search beyond a vertex v:

- The path w_v was encountered before in our search. In this case, searching beyond v will not yield any new results.
- If $|w_v| > |u_r| + C_f$, $M_{v'} > 0$ and $N_{v'} > C_f$ for some vertex $v' \in [v_0, v]$, then Lemma 1.4 implies that $|w_{v'}| > |u_r|$ for all vertices v' beyond v, so that we will not encounter u_r if we search beyond v.

Assume that there exists an infinite sequence $v_0, v_1, v_2, \ldots$ such that $v_k \neq v_{k+2}$, $|v_k, v_{k+1}| = 1$ for all k, and none of the two cases above occurs. Then $|w_{v_k}|$ goes to infinity (or else there would be some repetition along the way), and we have $M_{v_k} = 0$ or $N_{v_k} \leqslant C_f$ for all k. In fact, we have $M_{v_k} = 0$ for

[6] $*$ is merely some termination symbol.

all k or $N_{v_k} \leqslant C_f$ for all k (otherwise we would encounter a fixed interior vertex, i.e., a vertex $v_k \neq v_0$ for which w_{v_k} is trivial, so that we would have reached our first termination criterion because w_{v_0} is trivial). In the first case, the v_k define an attracting fixed ray of h. In the second case, they define a repelling fixed ray of h.

5.1. Attracting fixed rays

If $v_0, v_1, v_2, \ldots$ is an attracting fixed ray with no interior fixed vertices, then this sequence is determined by v_0 and v_1 alone because the first edge of $[v_k, v_k h]$ is the same as the edge $[v_k, v_{k+1}]$; otherwise we would encounter a trivial w_k along the way. For the same reason, the edge $[v_0, v_1]$ cannot project to a constant edge. In other words, we need to consider at most one attracting fixed ray for each nonconstant edge originating at v_0, and we can easily compute arbitrarily long prefixes of each ray.

In order to determine when to stop following an attracting ray, we will identify some k_0 such that $|v_k, v_k h| > |u_r| + C_f$ for all $k \geqslant k_0$. This implies that $|w_{v_k}| > |u_r| + C_f$ for all $k \geqslant k_0$. Moreover, if v is a vertex such that $v_{k_0} \in [v_0, v]$, then Lemma 1.4 implies that $|w_v| > |u_r|$, so that we can terminate our search at v_{k_0}.

First, assume that the edge bounded by v_0 and v_1 is contained in an exponential stratum H_s. Then $[v_0, v_k]$ projects to an r-legal path for all k, and we have $|v_k, v_k h|_r \geqslant |v_0, v_k|_r$ because f is normalized. Hence, we only need to compute $v_0, \ldots, v_k$ until the r-length of $[v_0, v_k]$ exceeds $|u_r| + C_f$.

Now, assume that $[v_0, v_1]$ projects to a nonexponential edge E_s. Since $v_0, v_1, \ldots$ is a fixed ray, $[v_0, v_1]$ cannot project to $\bar{E}_s$, and so $\lim_{k\to\infty}[v_0, v_k]$ equals R_s. If E_s is linear, then we reach our first termination criterion after at most $|u_s|$ steps, so that we may assume that E_s is neither constant nor linear.

Lemma 5.1. *Let $L > 0$ and assume that v is a vertex in H such that $|v, vh| \geqslant L$, $|vh, vh^2| \geqslant L$, and $vh \in [v, vh^2]$. Then, for all $x \in [v, vh]$, we have*

$$|x, xh| \geqslant \frac{2L}{K_f + 1} - D_f.$$

Proof. Let $t = |x, v|$. Then Inequality (1) implies that $|xh, vh| \geqslant \frac{t}{K_f} - D_f$ and $|xh, vh^2| \leqslant K_f(L - t) + D_f$. We conclude that $|x, xh| \geqslant L - t + \max\{\frac{t}{K_f} - D_f, L - K_f(L - t) - D_f\}$. The minimum of the right-hand side of this inequality is attained for $t = \frac{LK_f}{K_f+1}$, and substituting this value yields a lower bound of $\frac{2L}{K_f+1} - D_f$. □

We choose L such that $\frac{2L}{K_f+1} - D_f > |u_r| + C_f$. Now Lemma 4.3 yields an exponent k_0 such that $|u_s f^k| > L$ for all $k \geqslant k_0$. We only need to compute $v_0, \ldots, v_k$ until $[v_0, v_k]$ projects to $E_s u_s \cdots (u_s f^{k_0})$, and Lemma 5.1 guarantees that $|w_v| > |u_r| + C_f$ for all v beyond v_k. This completes our algorithm in the case of attracting fixed rays.

5.2. Repelling fixed rays

In the attracting case, we construct fixed rays edge by edge, and an attracting fixed ray that contains no interior fixed points is determined by its first edge. In the repelling case, the situation is more complicated, but the following lemma still give us a way of computing successive edges in potential fixed rays given a sufficiently long prefix.

Lemma 5.2. *Let $v_0, v_1, \ldots, v_k$ be a sequence such that $N_{v_j} \leqslant C_f$ for all $0 \leqslant j \leqslant k$ and $M_{v_k} > C_f$. Then at most one vertex v adjacent to v_k, other than v_{k-1}, can be contained in a repelling ray originating at v_0, and we can find v algorithmically or determine that there is no such v. Moreover, if v' is a vertex satisfying $v_k \in [v_0, v']$ and $v \notin [v_0, v']$, then $M_{v'} \geqslant M_{v_k} + |v_k, v'| - C_f$.*

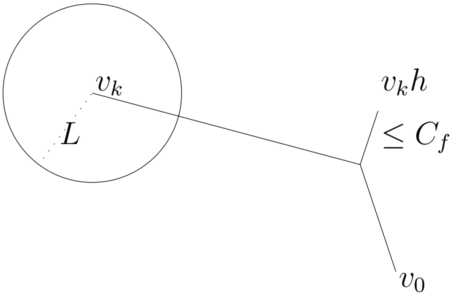

Fig. 2. Finding repelling fixed rays.

Proof. Using Inequality (1), we find some $L > 0$ such that if ρ is a path of length at least L, then $|\rho f| \geqslant 2C_f + 1$. Now we enumerate all vertices $p_1, \dots, p_m$ such that $|v_k, p_i| = L$ and $v_k \in [v_0, p_i]$ for all i (Fig. 2). Lemma 1.4 yields that $|[v_k, p_i]h \wedge [v_k, p_j]h| \leqslant C_f$ if $|[v_k, p_i] \wedge [v_k, p_j]| = 0$.

If p_i and p_j are contained in fixed rays, then $N_{p_i} < C_f$ and $N_{p_j} < C_f$. This implies that $|[v_k, p_i]h \wedge [v_k, p_j]h| > C_f$, so that $|[v_k, p_i] \wedge [v_k, p_j]| > 0$. Hence, if there exists some p_i such that $|[v_k, p_i]h \wedge [v_k, v_k h]| > C_f$, then the second vertex v in $[v_k, p_i]$ is uniquely determined by this property.

The last claim is an immediate consequence of Lemma 1.4. □

Another complication in the repelling case is that the height may go up as we apply Lemma 5.2 to compute subsequent vertices. The following lemmas provide a means of handling this possibility.

Lemma 5.3. *Assume that H_s is an exponential stratum and let $C = S_f(1 + \#\mathcal{E}(G))$. If η is a repelling fixed ray of height s with a maximal prefix α in G_{s-1}, then $|\alpha f| + C \geqslant |\alpha|$.*

Proof. If the initial vertex v_0 is contained in a contractible component of G_{s-1}, then the claim is trivial, so that we may assume that v_0 is contained in a noncontractible component of G_{s-1}. By Lemma 2.4, the terminal endpoint of α is fixed.

Choose β so that $\eta = \alpha\beta$. By definition, the first edge in β is contained in H_s. Let γ be the maximal subpath in G_{s-1} of βf. It suffices to show that $|\gamma| \leqslant C$.

If γ is a subpath of Ef for some edge $E \subset H_s$, then $|\gamma| \leqslant S_f$. If γ is the image of some subpath $\gamma' \subset G_{s-1}$ of β, then Lemma 2.4 implies that γ' is contained in a contractible component of G_{s-1},[7] so that $|\gamma'| \leqslant \#\mathcal{E}(G)$. This implies that $|\gamma| \leqslant S_f \#\mathcal{E}(G)$. □

Lemma 5.4. *If H_s is an exponential stratum and the sequence $v_0, v_1, \dots$ defines a repelling fixed ray η of height s without interior fixed points, then $\iota_s(w_{v_k})$ is an unbounded nondecreasing function of k.*

Proof. Since η has no interior fixed points, it cannot be a concatenation of Nielsen paths of height r and subpaths in G_{r-1}. This implies that η contains infinitely many r-illegal turns. Now Lemma 2.6 implies that the distance between two r-illegal turns is bounded by some constant L. Since η is repelling, $|w_{v_k}|$ is unbounded, which proves the claim. □

Lemma 5.5. *Assume that H_s is a nonexponential stratum and that η is a repelling fixed ray of height s with no fixed interior vertices. Then $\eta = S_s$.*

Proof. This is an immediate consequence of Lemmas 2.5 and 3.9. □

[7] Otherwise β would have an initial subpath η of height s, starting and ending at fixed vertices, so that ηf is trivial. This is impossible because f is a homotopy equivalence.

Lemma 5.5 implies that if the height goes up as we follow a potential repelling fixed ray, then the height must eventually stabilize at an exponential stratum.

We now continue our breadth-first traversal of vertices in H. If we encounter a vertex v_k such that $[v_0, v_k]$ satisfies the hypotheses of Lemma 5.2, then we need to consider the possibility that $[v_0, v_k]$ is a prefix of a repelling fixed ray. In this case, we use Lemma 5.2 to compute subsequent vertices v. (In this process, M_v may drop below C_f, so that Lemma 5.2 no longer applies; in this case, we simply continue our breadth-first search. This is not a problem, however, because it can only happen finitely many times before we encounter our first termination criterion.)

Let s be the height of the potential repelling ray computed so far. If H_s is nonexponential, then our ray must converge to S_s. Using arguments similar to those in the attracting case, we follow S_s until we recognize a vertex k_0 such that for all vertices v beyond v_{k_0}, we have $M_v > \max\{C, |u_r|\}$ (where C is the constant from Lemma 5.3). $M_v > C$ guarantees that we are not following a prefix of a ray of greater height, and $M_v > |u_r|$ implies that we will not encounter u_r as we follow the ray.

If H_s is exponential, then we follow our ray until we encounter a vertex v for which $\iota_s(w_v) > \max\{C, |u_r|\}$. Once again, Lemma 5.3 guarantees that the height will not go up if we continue following our ray, and we will not encounter u_r if we continue our search. Hence, our algorithm terminates in all possible cases.

5.3. *Picking up the pieces*

Proposition 5.6. *If $H_r = \{E_r\}$, then we can determine algorithmically whether there exists a path $\rho \subset G_{r-1}$ such that u_r is obtained by tightening $\bar{\rho}(\rho f)$, and we can compute ρ if it exists.*

Proof. If ρ exists, then its initial vertex is a fixed vertex in G_{r-1}. Repeating the procedure above for each fixed vertex in G_{r-1} yields the desired algorithm. □

Theorem 5.7. *Given an outer automorphism $\mathcal{O}$ of F_n, we can compute an efficient relative train track map $f: G \to G$ as well as an exponent $k \geqslant 1$ such that f represents $\mathcal{O}^k$.*

Proof. We can compute an exponent $k \geqslant 1$ and a normalized relative train track map $f: G \to G$ representing $\mathcal{O}^k$. Now we assume inductively that the restriction of f to G_{r-1} is efficient. If H_r is zero or exponential, then there is nothing to do. If $H_r = \{E_r\}$ is nonexponential, then we first use Proposition 5.6 to determine whether there exists a slide of E_r to a constant edge. If no such edge exists, we use Proposition 3.4 to achieve efficiency of H_r. □

6. Proof of the main result

Lemma 6.1. *Let $f: G \to G$ be an efficient relative train track map. There exists an algorithm that, given a circuit σ in G and a constant $L > 0$, determines whether σ is Nielsen. If σ is not Nielsen, then the algorithm finds an exponent k_0 such that $|\sigma f^k| > L$ for all $k \geqslant k_0$.*

Proof. Lemma 4.1 takes care of the detection of Nielsen circuits. If σ is not Nielsen, then we consider the height r of σ. If H_r is nonexponential, then it splits as a concatenation of basic paths of height r (Lemma 2.5), so that Lemma 4.3 completes the proof in this case.

If H_r is exponential, then we compute $\sigma, \sigma f, \sigma f^2, \ldots$ until we encounter an image $\sigma' = \sigma f^k$ for some $k > 0$ such that σ' contains an r-legal path of length greater than $2(C_f + 1)$ or σ' is a concatenation of Nielsen paths of height r and paths in G_{r-1}.

We can recognize both possibilities algorithmically. In the first case, $\sigma' f$ splits at a fixed vertex in a long r-legal subpath. In the second case, σ' splits at the terminal endpoint of a subpath in G_{r-1}. In either case, Lemma 4.3 completes the proof. □

Theorem 6.2. *Let ϕ be an automorphism of F_n. The exists an algorithm that, given two elements $u, v \in F_n$, determines whether there exists some exponent N such that $u\phi^N$ is conjugate to v. If such an N exists, then the algorithm will compute N as well.*

Proof. Theorem 5.7 yields an exponent k and an efficient relative train track map $f:G\to G$ that represents the outer automorphism defined by ϕ^k. We can find some constant $Q\geqslant 1$ such that if σ is a circuit in G representing a conjugacy class ω in F_n, then $\frac{1}{Q}|\omega|\leqslant|\sigma|\leqslant Q\,|\omega|$.[8]

Represent the conjugacy class of u by a circuit σ. If σ is a Nielsen circuit of period p, then we conclude that $u\phi^{kp}$ is conjugate to u. Now we compute $u, u\phi,\dots,u\phi^{kp-1}$ and check whether any conjugate of v is in this list.

If σ is not Nielsen, we let $L=Q\cdot S_\phi^k\cdot|v|$, and we find some exponent k_0 such that $|\sigma f^j|>L$ for all $j\geqslant k_0$. We conclude that the length of the conjugacy class of $u\phi^j$ exceeds $|v|$ for all $j\geqslant kk_0$. Now we list $u, u\phi, u\phi^2,\dots,u\phi^{kk_0-1}$ and check whether any conjugate of v is in this list. If no conjugate is contained in this list, then we exchange u and v and repeat the argument. This completes the proof. □

Theorem 6.3. *Let ϕ be an automorphism of F_n. The exists an algorithm that, given two elements $u, v\in F_n$, determines whether there exists some exponent N such that $u\phi^N=v$. If such an N exists, then the algorithm will compute N as well.*

Proof. We use a trick from [BFH97]. Let $F'=F_n*\langle a\rangle$ and define $\psi\in Aut(F')$ by letting $x\psi=x\phi$ if $x\in F_n$ and $a\psi=a$. If $w\in F_n$, then wa is cyclically reduced in F', so that $u\phi^N=v$ if and only if $(ua)\psi^N$ is conjugate to va. Now Theorem 6.2 completes the proof. □

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[8] The length of a conjugacy class ω is defined to be the length of the shortest element in ω.