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The variational iteration method: A highly promising method for solving the system of integro-differential equations☆

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Abstract

This paper applies He's variational iteration method for solving two systems of Volterra integro-differential equations. The solution process is illustrated and various physically relevant results are obtained. Comparison of the obtained results with exact solutions shows that the used method is an effective and highly promising method for various classes of both linear and nonlinear integro-differential equations.

Keywords: Variational iteration method; Integro-differential equation; Approximate solution; System; Volterra

1. Introduction

Integro-differential equations arise in many physical processes, such as glass-forming process [1], nano-hydrodynamics [2], drop wise condensation [3], and wind ripple in the desert [4]. There are various numerical and analytical methods to solve such problems, for example, the homotopy perturbation method [5], the Adomian decomposition method [6], but each method limits to a special class of integro-differential equations. Recently the variational iteration method [7–10] has been shown to solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly to accurate solutions [11–18]. In Refs. [19–22] the method was applied successfully to solve some integro-differential equations. Xu [20] first applied the variational iteration method to integral equations, Sweilam [21] applied the method to fourth-order integro-differential equations, and Wang et al. [19] found that the variational iteration method is an efficient algorithm for solving integro-differential equations by using some examples. The variational iteration method may be regarded with considerable justification as a versatile and promising method for solving all kinds of integro-differential equations.

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2. Variational iteration method

To illustrate the basic idea of the method, we consider a general nonlinear system as follows:

$$L(u(t)) + N(u(t)) = g(t), \tag{1}$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known continuous function. The basic essence of this method is to construct a correction functional for (1) in the form [7–10]:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)[Lu_n(t) + N\tilde{u}_n(s) - g(s)]\mathrm{d}s, \tag{2}$$

where λ is a Lagrange multiplier which can be identified optimally via the variational theory [23,24], u_n is the approximate solution and $\tilde{u}_n$ denotes restricted variation [9], i.e.

$$\delta\tilde{u}_n = 0.$$

For integro-differential equations [19–22],

$$u^{(n)}(x) = g(x) + f(x)u(x) + \int_0^x k(x,t)u(t)\mathrm{d}t, \tag{3}$$

its variational iteration formulation can be expressed in the form:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(s)[u_n^{(n)} - F(u_n)]\mathrm{d}s, \tag{4}$$

where $F(u) = g(x) + f(x)u(x) + \int_0^x k(x,t)u(t)\mathrm{d}t$.

Now we consider the following integro-differential equation:

$$\begin{cases} \dfrac{\mathrm{d}u_i(t)}{\mathrm{d}t} = g_i(t) + \displaystyle\sum_{j=0}^{n} p_{ij}(t)u_j + \int_0^t k_{ij}(x,s)u_j(s)\mathrm{d}s, \\ u_i(0) = \alpha_i, \quad i = 1, 2, 3, \ldots, n, \end{cases} \tag{5}$$

where g and p_{ij} are known functions.

According to He's variational iteration method; an iteration scheme for (5) can be constructed as follows:

$$u_{in+1} = u_{in} + \int_0^x \lambda(s)[u'_{in}(s) - F(U_n(s))]\mathrm{d}s, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where,

$$U_n = (u_{1n}, u_{2n}, u_{3n,\ldots})$$

and

$$F_i(U_n) = g_i(t) + \sum_{j=0}^{n} p_{ij}u_j + \int_0^x k_{ij}(x,s)u_j(s)\mathrm{d}s.$$

The details of this procedure are explained in Examples 1 and 2.

3. Applications

In this section, in order to illustrate the method, we solve two examples and then we will compare the obtained results with the exact solutions.

Example 1. Consider the following system of linear Volterra integro-differential equations

$$\begin{cases} u_1'(t) = 1 + t + t^2 - u_2(t) - \int_0^t (u_1(s) + u_2(s))\mathrm{d}s, & u_1(0) = 1, \\ u_2'(t) = -1 - t + u_1(t) - \int_0^t (u_1(s) - u_2(s))\mathrm{d}s, & u_2(0) = -1, \end{cases} \tag{7}$$

with exact solutions $u_1(t) = t + \mathrm{e}^t$, $u_2(t) = t - \mathrm{e}^t$.

To use the variational iteration formulation illustrated above, we define the following:

$$F_1(u_n) = 1 + t + t^2 - u_{2n}(t) - \int_0^t (u_{1n}(s) + u_{2n}(s))\mathrm{d}s, \quad n = 0, 1, 2, \ldots, \tag{8}$$

$$F_2(u_n) = 1 + t + u_{1n}(t) - \int_0^t (u_{1n}(s) - u_{2n}(s))\mathrm{d}s, \quad n = 0, 1, 2, \ldots, \tag{9}$$

$$u_{1\,n+1} = u_{1n} + \int_0^x \lambda(t)[u_{1n}'(t) - F_1(u_n(t))]\mathrm{d}t, \quad n = 0, 1, 2, \ldots, \tag{10}$$

$$u_{2n+1} = u_{2n} + \int_0^x \lambda(t)[u_{2n}'(t) - F_2(u_n(t))]\mathrm{d}t, \quad n = 0, 1, 2, \ldots. \tag{11}$$

We begin with initial guesses with some unknown parameters in the forms:

$$u_{10} = 1 + ax, \quad u_{20} = -1 + bx, \tag{12}$$

where a and b are unknown constants which will be determined later on.

Setting $n = 0$ in (8) and (10), we obtain:

$$F_1(u_0) = 1 + t + t^2 - u_{2\,0}(t) - \int_0^t (u_{1\,0}(s) + u_{2\,0}(s))\mathrm{d}s,$$

$$u_{1\,1} = u_{1\,0} + \int_0^x \lambda(t)[u_{1\,0}'(t) - F_1(u_0(t))]\mathrm{d}t. \tag{13}$$

Identification of the Lagrange multiplier results in

$$\lambda(t) = -1.$$

Substituting (12) into Eq. (13) yields the result:

$$u_{1\,1} = 1 - \frac{1}{3}\left(-1 + \frac{1}{2}a + \frac{1}{2}b\right)x^3 - \frac{1}{2}(-1 + b)x^2 + 2x. \tag{14}$$

Similarly setting $n = 0$ in (9) and (11), we have

$$F_2(u_0) = 1 + t + u_{1\,0}(t) - \int_0^t (u_{1\,0}(s) - u_{2\,0}(s))\mathrm{d}s,$$

$$u_{2\,1} = u_{2\,0} + \int_0^x \lambda(t)[u_{2\,0}'(t) - F_2(u_0(t))]\mathrm{d}t. \tag{15}$$

The Lagrange multiplier can be determined as follows:

$$\lambda(t) = -1.$$

Substituting (12) into (15), we obtain

$$u_{2\,1} = -1 - \frac{1}{3}\left(\frac{1}{2}a - \frac{1}{2}b\right)x^3 - \frac{1}{2}(3 - a)x^2. \tag{16}$$

Putting the initial conditions into Eq. (14) and (16), we have

$$\begin{cases} u_{1\,1}(0) = 1, \\ u_{21}(0) = -1. \end{cases} \tag{17}$$

Table 1
Comparison of the obtained results with exact solutions of Example 1

x_i	u_1 exact	u_{14} HVIM	u_2 exact	u_{24} HVIM
0.0	1	1	−1	−1
0.2	1.421402758	1.421400010	−1.021402758	−1.021400010
0.4	1.891824698	1.891734699	−1.091824698	−1.091734699
0.6	2.422118800	2.421423883	−1.222118800	−1.221423883
0.8	3.025540928	3.022583085	−1.425540928	−1.422583085
1.0	3.718281828	3.709226190	−1.718281828	−1.709226190

From (17) we can see that the initial conditions are satisfied automatically.

In this case the system (17) is not dependent upon parameters $a$and b. Therefore, we can choose $a$and b arbitrarily, taking $a = 1$ and $b = 1$. This selection makes the initial and second approximations in the following form:

$$u_{1\,0} = 1 + x, \quad u_{2\,0} = -1 + x,$$
$$u_{11} = 1 + 2x, \quad u_{21} = -1 - x^2.$$

By repeating this procedure we get the other approximations as follows:

$$u_{12} = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4,$$
$$u_{13} = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{60}x^5,$$
$$u_{14} = 1 + 2x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{1260}x^7 + \frac{1}{10\,080}x^8,$$
$$u_{22} = -1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{12}x^4,$$
$$u_{23} = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{60}x^5 - \frac{1}{180}x^6,$$
$$u_{24} = -1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{1260}x^7 - \frac{1}{10\,080}x^8.$$

Table 1 shows the fourth approximations of the solutions of (7) and its comparison with the exact solutions.

Example 2. As a second example, we consider the following system:

$$\begin{cases} u_1'(t) = -3t^2 u_1(t) + (\pi - 2t^3)u_2(t) + 6\int_0^t ((t-s)u_1(s) + (t-s)^2 u_2(s))\mathrm{d}s, \\ u_2'(t) = -(\pi + 4t^3)^2 u_1(t) - 6t^2 u_2(t) + 12\int_0^t ((t-s)^2 u_1(s) + (t-s)u_2(s))\mathrm{d}s, \\ u_1(0) + u_1(1/4) = \sqrt{2}/2, \quad u_2(0) + u_2(1/4) = \sqrt{2}/2, \end{cases} \tag{18}$$

with the exact solutions, $u_1(t) = \sin \pi t$ and $u_2(t) = \cos \pi t$.

Proceeding the same way as illustrated in Example 1, we obtain:

$$F_1(u_n) = -3t^2 u_{1n}(t) + (\pi - 2t^3)u_{2n}(t) + 6\int_0^t ((t-s)u_{1n}(s) + (t-s)^2 u_{2n}(s))\mathrm{d}s, \quad n = 0, 1, \ldots, \tag{19}$$

$$F_2(u_n) = -(\pi + 4t^3)u_{1n}(t) - 6t^2 u_{2n}(t) + 12\int_0^t ((t-s)^2 u_{1n}(s) + (t-s)u_{2n}(s))\mathrm{d}s, \quad n = 0, 1, \ldots, \tag{20}$$

$$u_{1\,n+1} = u_{1\,n} + \int_0^x \lambda(t)[u'_{1\,n}(t) - F_1(u_n(t))]\mathrm{d}t, \quad n = 0, 1, \ldots, \tag{21}$$

$$u_{2\,n+1} = u_{2\,n} + \int_0^x \lambda(t)[u'_{2\,n}(t) - F_2(u_n(t))]\mathrm{d}t, \quad n = 0, 1, \ldots. \tag{22}$$

Table 2
Comparison of the obtained results with exact solutions of Example 2

x_i	u_1 exact	u_{14} HVIM	u_2 exact	u_{24} HVIM
0.00	0	0	1	1
0.05	0.1564344651	0.1563260778	0.9876883406	0.9877021974
0.10	0.3090169944	0.3080962749	0.9510565163	0.951614638
0.15	0.4539904998	0.4486385966	0.8910065242	0.8916537964
0.20	0.5877852524	0.5684781508	0.8090169943	0.8119976597
0.25	0.7071067813	0.6570929992	0.7071067811	0.7171726094

Further, we set $n = 0$ in (19) and (20) to get

$$F_1(u_0) = -3t^2 u_{10}(t) + (\pi - 2t^3)u_{20}(t) + 6\int_0^t ((t-s)u_{10}(s) + (t-s)^2 u_{20}(s))\mathrm{d}s, \tag{23}$$

$$F_2(u_0) = -(\pi + 4t^3)u_{10}(t) - 6t^2 u_{20}(t) + 12\int_0^t ((t-s)^2 u_{10}(s) + (t-s)u_{20}(s))\mathrm{d}s. \tag{24}$$

The initial approximations are assumed to have the forms

$$u_{10}(t) = a\pi t^3 + b\pi t, \qquad u_{20}(t) = c\pi t^2 + d,$$

where a, b, c and d are unknown constants which will be determined afterwards.

The Lagrange multiplier in this example can be determined easily, which reads $\lambda(t) = -1$. After computing $F_1(u_0)$, $F_2(u_0)$ and putting $\lambda(t) = -1$ into (21) and (22), the first approximation is obtained as follows:

$$u_{11} = a\pi\, x^3 - \frac{1}{6}\left(\frac{27}{10}a\pi + \frac{9}{5}c\pi\right)x^6 - \frac{1}{2}b\pi\, x^4 - \frac{1}{3}(3a\pi - c\pi^2)x^3 + \pi \mathrm{d}x, \tag{25}$$

$$u_{21} = c\pi x^2 + d - \frac{19}{35}a\pi x^7 - \frac{1}{5}(5c\pi - 3b\pi)x^5 - \frac{1}{4}a\pi^2 x^2 - \frac{1}{2}(2c\pi + b\pi^2)x^2. \tag{26}$$

Using the initial conditions in (25) and (26), we have

$$\begin{cases} u_{11}(0) + u_{11}(1/4) = -\dfrac{9}{81920}a\pi - \dfrac{3}{40960}c\pi - \dfrac{1}{512}b\pi + \dfrac{1}{192}c\pi^2 + \dfrac{1}{4}d\pi = \dfrac{\sqrt{2}}{2}, \\ u_{21}(0) + u_{21}(1/4) = 2d - \dfrac{1}{1024}c\pi - \dfrac{19}{573440}a\pi - \dfrac{3}{5120}b\pi - \dfrac{1}{1024}a\pi^2 - \dfrac{1}{32}b\pi^2 = 1 + \dfrac{\sqrt{2}}{2}. \end{cases} \tag{27}$$

Solving the system (27) by Maple 9, we find that the parameters c and b are free parameters, for simplicity, we choose $c = d = 1$, as a result we obtain

$$a = 808.931869, \qquad b = -24.4655243.$$

Thus the first approximation is determined and the other approximations have also been calculated by Maple 9 and the results have been illustrated in Table 2.

Remark. The selection of initial approximation is arbitrary but a suitable selection is effective for fast convergence and fit accuracy. We suggest the initial approximations to be selected well-set with $g(x)$.

4. Conclusion

In this article, first, we have outlined He's variational iteration method and next we have applied this method to two systems of Volterra integro-differential equations. In order to illustrate the method, we solve two examples. The results compared with the corresponding values of exact solutions show that as long as the values of the variable x become large the accuracy gets weak. To overcome this difficulty, it seems that one needs to increase the number of iterations in this case. Although the examples given in this paper are linear, it can be applicable to nonlinear integro-differential equations.

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