Journal of Number Theory **79**, 58–66 (1999)
Article ID jnth.1999.2426, available online at http://www.idealibrary.com on IDEAL®

A Factorization Formula for Class Number Two

Scott T. Chapman

Department of Mathematics, Trinity University,
715 Stadium Drive, San Antonio, Texas 78212-7200
E-mail: schapman@trinity.edu

Jeremy Herr

University of Oklahoma, Norman, Oklahoma 73069
E-mail: jherr@ou.edu

and

University of Texas at Austin, Austin, Texas 78712
E-mail: natalie@arlut.utexas.edu

Communicated by Alan C. Woods

Received November 2, 1998

Let R be an atomic integral domain. R is a half-factorial domain (HFD) if whenever $x_1 \cdots x_n = y_1 \cdots y_m$ for $x_1, ..., x_n, y_1, ..., y_m$ irreducibles of R, then $n = m$. A well known result of L. Carlitz (1960, *Proc. Amer. Math. Soc.* **11**, 391–392) states that the ring of integers in a finite extension of the rationals is a HFD if and only if the class number of R is less than or equal to 2. If R is such a ring of integers with class number 2, then we use some simple Krull monoids to develop a formula for counting the number of different factorizations of any integer x into products of irreducible elements of R. © 1999 Academic Press

Key Words: factorial; half-factorial; Krull monoid.

Let R be an atomic integral domain (i.e., every nonzero nonunit of R can be written as a product of irreducible elements) and $\mathscr{I}(R)$ its set of irreducible elements. The study of conditions which force R to be factorial has been a central focus of commutative algebra for many years. Much recent literature has been devoted to the study of atomic integral domains where the unique factorization property fails (see [1, 3] for a summary). R is called a *half-factorial domain* if whenever $x_1, ..., x_n, y_1, ..., y_m$ are irreducibles of R with $x_1 \cdots x_n = y_1 \cdots y_m$ then $n = m$. This definition can

0022-314X/99 $30.00

be extended in the obvious manner for atomic monoids, which we will denote by HFM. Carlitz [2] characterized rings of algebraic integers which satisfy the HFD property in terms of the class number. We restate his result below using our terminology.

THEOREM 1(Carlitz [2]). *Let R be the ring of integers in a finite extension of the rationals. R is a HFD if and only if R has class number less than or equal to two.*

In this note, we find a formula to compute the total number of factorizations of an integer x in a ring of algebraic integers with class number two. Our main tool will be the connection of the algebra in such a ring of integers to the monoids

$$\mathbb{X}_n = \{(x_1, ..., x_{n+1}) \mid x_i \in \mathbb{N}_0, x_1 + \cdots + x_n = 2x_{n+1}\},$$

where $n \geqslant 2$ is a positive integer and $\mathbb{N}_0$ represents the nonnegative integers.

Because of Carlitz's result, factorization problems in class number two have drawn some attention in the literature (see [5, 9] for instance). Most of these recent works deal with asymptotic studies of factorization properties or obtaining estimates on the total number of integers having at most k different factorizations. Our work does not touch on these subjects, but answers a question concerning a Krull monoid with infinitely many primes in one divisor class by studying a series of Krull monoids whose nontrivial divisor classes contain finitely many elements. If we wish to count the total number of factorizations of an integer α in R, then our formula does assume knowledge of the prime ideal factorization of the principal ideal αR.

We open with some notation and a brief review of Krull monoids. If x is a nonzero nonunit of R, then set

$$\eta(x) = \#\text{of nonassociated irreducible factorizations of } x \in R.$$

This function has been studied in great detail by Halter-Koch in [6], where he shows (using the notation $\mathbf{f}(x)$) for each positive integer n that

$$\eta(x^n) = An^d + O(n^{d-1})$$

for some $A \in \mathbb{Q}_{>0}$ and $d \in \mathbb{N}_0$.

If $\mathscr{S}$ is a commutative, cancellative monoid, then denote the quotient group of $\mathscr{S}$ by $\mathscr{Q}(\mathscr{S})$. Let $\leqq$ be the divisibility relation (or quasi-ordering) on $\mathscr{S}$ induced by the following: $x \leqq y$ in $\mathscr{Q}(\mathscr{S})$ if and only if $xz = y$ for some $z \in \mathscr{S}$. Let $\mathscr{F}$ be a nonempty family of homomorphisms $f \not\equiv 0$ of $\mathscr{Q}(\mathscr{S})$ into $\mathbb{Z}$ such that for each $x \in \mathscr{Q}(\mathscr{S})$ the set $\mathscr{F}^* = \{f \in \mathscr{F} \mid f(x) \neq 0\}$ is finite. An element of $\mathscr{F}^*$ is called a *state* on $\mathscr{S}$. If the monoid $\mathscr{S}$ is defined by $\mathscr{F}$ (i.e., $\mathscr{S} = \{x \in \mathscr{Q}(\mathscr{S}) \mid f(x) \geqslant 0 \ \forall f \in \mathscr{F}\}$) then $\mathscr{S}$ is a *Krull*

Monoid. A state on $\mathcal{S}$ is *essential* if for any $x, y \in \mathcal{Q}(\mathcal{S})$, there exists $z \in \mathcal{Q}(\mathcal{S})$ such that $x \leqq z$, $y \leqq z$ and $f(z) = \max\{f(x), f(y)\}$. Let $\mathcal{J}$ denote the set of essential states f which are *normalized* (i.e., $f(\mathcal{Q}(\mathcal{S})) = \mathbb{Z}$). Consider the map $\varphi\colon \mathcal{S} \to \mathbb{Z}_+^{(\mathcal{J})}$ defined by $\varphi(x)_{(f)} = f(x)$. The essential states of a Krull monoid yield a *divisor theory* for $\mathcal{S}$. The factor monoid

$$\mathbb{Z}_+^{(\mathcal{J})}/\varphi(\mathcal{S}) \cong \mathbb{Z}^{(\mathcal{J})}/\varphi(\mathcal{Q}(\mathcal{S}))$$

is a group known as the *divisor class group of* $\mathcal{S}$ and denoted $\mathrm{Cl}(\mathcal{S})$. The unit basis vectors e_i in $\mathbb{Z}_+^{(\mathcal{J})}$ are the *prime divisors* of $\mathcal{S}$. The interested reader may consult [3] for further information on Krull monoids.

The monoids $\mathbb{X}_n$ are examples of the monoids

$$\mathcal{M}(a_1, ..., a_n; b) = \{(x_1, ..., x_{n+1}) \mid x_i \in \mathbb{N}_0, a_1x_1 + \cdots + a_nx_n = bx_{n+1}\},$$

where $a_1, ..., a_n, b$ are positive integers, studied by the first author, Krause and Oeljeklaus in [4]. For our purposes we need the following facts.

LEMMA 2. *Let $n > 1$ be a positive integer.*

(1) *Each $\mathbb{X}_n$ is a half-factorial Krull monoid with divisor class group $\mathbb{Z}_2$.*

(2) *The essential states of $\mathbb{X}_n$ are the projections $\pi_i((x_1, ..., x_{n+1})) = x_i$ for $i = 1, ..., n$.*

(3) *The nontrivial divisor class of $\mathbb{X}_n$ contains exactly n prime divisors (namely $e_1, ..., e_n$).*

(4) *$\mathbb{X}_n$ contains no prime elements.*

Proof. Proofs of these facts can be found in [4, Sect. 2]. That each $\mathbb{X}_n$ is an HFM follows from [7, Proposition 2]. ∎

It is of interest to note that the monoid $\mathbb{X}_n$ cannot appear as the multiplicative monoid of a Krull domain [4, Proposition 2.2]. The irreducible elements of $\mathbb{X}_n$ can be retrieved from the divisor class group by considering the sum of two (not necessary distinct) prime divisors e_i with $1 \leqslant i \leqslant n$. To demonstrate, the irreducible element of $\mathbb{X}_3$ are

$$\begin{aligned} \mathbf{i}_1 &= (2, 0, 0, 1), & \mathbf{i}_4 &= (1, 1, 0, 1) \\ \mathbf{i}_2 &= (0, 2, 0, 1), & \mathbf{i}_5 &= (1, 0, 1, 1) \\ \mathbf{i}_3 &= (0, 0, 2, 1), & \mathbf{i}_6 &= (0, 1, 1, 1). \end{aligned} \tag{1}$$

When referring to the function $\eta(x)$ for a particular $\mathbb{X}_n$, we will use the notation $\eta_{\mathbb{X}_n}(x)$. Since our argument will be recursive, we begin with $\mathbb{X}_2$.

PROPOSITION 3. *If* $x=(x_1, x_2, x_3)\in\mathbb{X}_2$, *then*

$$\eta_{\mathbb{X}_2}(x)=\lfloor \min\{x_1, x_2\}/2\rfloor+1.$$

Proof. First note that the irreducibles in $\mathbb{X}_2$ are

$$\mathbf{i}_1=(2, 0, 1), \qquad \mathbf{i}_2=(0, 2, 1), \qquad \text{and} \qquad \mathbf{i}_3=(1, 1, 1).$$

Assume without loss that $\min\{x_1, x_2\}=x_1$. If $a_1\mathbf{i}_1+a_2\mathbf{i}_2+a_3\mathbf{i}_3$ is a factorization of x into irreducibles, then

$$\{\underbrace{2, ..., 2}_{a_1 \text{ times}}, \underbrace{1, ..., 1}_{a_3 \text{ times}}\}$$

forms a partition of x_1 consisting of 1's and 2's. Conversely, if

$$\{\underbrace{2, ..., 2}_{\alpha \text{ times}}, \underbrace{1, ..., 1}_{\beta \text{ times}}\}$$

is such a partition of x_1, then x can be written as $\alpha\mathbf{i}_1+\beta\mathbf{i}_3+\gamma\mathbf{i}_2$ where $\gamma\geqslant 0$ (since $x_1=\min\{x_1, x_2\}$). Hence $\eta_{\mathbb{X}_2}(x)=\lfloor x_1/2\rfloor+1$ and in general

$$\eta_{\mathbb{X}_2}(x)=\left\lfloor\frac{\min\{x_1, x_2\}}{2}\right\rfloor+1. \quad \blacksquare$$

We next derive $\eta_{\mathbb{X}_3}(x)$. Notice that for any $n\geqslant 2$, if $x=(x_1, ..., x_{n+1})\in\mathbb{X}_n$, then x can be mapped via a monoid automorphism to an element $x'=(x'_1, ..., x'_{n+1})$ with $x'_1\leqslant x'_2\leqslant\cdots\leqslant x'_n$. Our formulas for $n\geqslant 3$ will assume that the first n coordinates of an element x are monotonically increasing.

PROPOSITION 4. *If* $x=(x_1, x_2, x_3, x_4)\in\mathbb{X}_3$, *with* $x_1\leqslant x_2\leqslant x_3$, *then*

$$\eta_{\mathbb{X}_3}(x)=\sum_{j=0}^{\lfloor x_1/2\rfloor}\sum_{k=0}^{x_1-2j}\left(\left\lfloor\frac{\min\{x_2-k, x_3-x_1+2j+k\}}{2}\right\rfloor+1\right),$$

or, without the minimum function,

$$\eta_{\mathbb{X}_3}(x)=\sum_{j=0}^{\lfloor x_1/2\rfloor}\left(\sum_{k=0}^{(1/2)(x_1+x_2-x_3-2j)}\left(\left\lfloor\frac{x_3-x_1+2j+k}{2}\right\rfloor+1\right)\right)$$
$$+\sum_{j=0}^{(1/2)(x_1-x_2+x_3-2)}\left(\sum_{k=(1/2)(x_1+x_2-x_3-2j)+1}^{x_1-2j}\left(\left\lfloor\frac{x_2-k}{2}\right\rfloor+1\right)\right).$$

Proof. If $x=(x_1, x_2, x_3, x_4) \in \mathbb{X}_3$, then the first entry x_1 is determined by the coefficients a_1, a_4, a_5 of $\mathbf{i}_1$, $\mathbf{i}_4$, and $\mathbf{i}_5$ in an irreducible factorization of x of the form $a_1\mathbf{i}_1 + \cdots + a_6\mathbf{i}_6$ (the irreducibles of $\mathbb{X}_3$ are listed in (1)). The coefficient a_1 may range from 0 to $\lfloor x_1/2 \rfloor$. Choose an allowable value for a_1, and then subtract these $\mathbf{i}_1$'s from x to obtain

$$\begin{aligned} x' = (x'_1, x'_2, x'_3, x'_4) &= x - a_1\mathbf{i}_1 \\ &= (x_1 - 2a_1, x_2, x_3, x_4 - a_1). \end{aligned}$$

Next, consider the coefficient a_4, which may now range from 0 to $x'_1 = x_1 - 2a_1$. We choose an allowable value for a_4, obtaining

$$\begin{aligned} x'' = (x''_1, x''_2, x''_3, x''_4) &= x' - a_4\mathbf{i}_4 \\ &= (x_1 - 2a_1 - a_4, x_2 - a_4, x_3, x_4 - a_1 - a_4). \end{aligned}$$

Now the only irreducible that will affect the first entry is $\mathbf{i}_5$, and we are forced to take $a_5 = x''_1 - x_1 - 2a_1 - a_4$, leaving us with

$$\begin{aligned} x''' = (x'''_1, x'''_2, x'''_3, x'''_4) &= x'' - a_5\mathbf{i}_5 \\ &= (x_1 - 2a_1 - a_4 - a_5, x_2 - a_4, x_3 - a_5, x_4 - a_1 - a_4 - a_5) \\ &= (0, x_2 - a_4, x_3 - (x_1 - 2a_1 - 4a), x_4 - a_1 - a_4 - a_5). \end{aligned}$$

Since $x'''_1 = 0$, we may ignore the first entry (i.e., factoring x''' in $\mathbb{X}_3$ is analogous to factoring its counterpart $y = (x'''_2, x'''_3, x'''_4) = (x_2 - a_4, x_3 - x_1 + 2a_1 + a_4, x_4 - a_1 - a_4 - a_5)$ in $\mathbb{X}_2$) and we have

$$\begin{aligned} \eta_{\mathbb{X}_3}(x''') = \eta_{\mathbb{X}_2}(y) &= \left\lfloor \frac{\min\{y_1, y_2\}}{2} \right\rfloor + 1 \\ &= \left\lfloor \frac{\min\{x_2 - a_4, x_3 - x_1 + 2a_1 + a_4\}}{2} \right\rfloor + 1. \end{aligned}$$

For convenience, we will now substitute j for a_1 and k for a_4, and obtain

$$\begin{aligned} \eta(x) &= \sum_{a_1=0}^{\lfloor x_1/2 \rfloor} \sum_{a_4=0}^{x_1 - 2a_1} \eta(x''') \\ &= \sum_{j=0}^{\lfloor x_1/2 \rfloor} \sum_{k=0}^{x_1 - 2j} \left(\left\lfloor \frac{\min\{x_1 - k, x_3 - x_1 + 2j + k\}}{2} \right\rfloor + 1 \right). \end{aligned}$$

We can eliminate the minimum function by evaluating when $x_3 - x_1 + 2j + k \leqslant x_2 - k$, and vice versa:

$$x_3 - x_1 + 2j + k \leqslant x_2 - k$$

$$\Updownarrow$$

$$2j + 2k \leqslant x_1 + x_2 - x_3$$

$$\Updownarrow$$

$$k \leqslant \tfrac{1}{2}(x_1 + x_2 - x_3 - 2j).$$

So we can split the inner summation into two summations, one in which $0 \leqslant k \leqslant \frac{1}{2}(x_1 + x_2 - x_3 - 2j)$, and the other in which k takes on the remaining values, $\frac{1}{2}(x_1 + x_2 - x_3 - 2j) + 1 < k \leqslant \lfloor x_1/2 \rfloor$. Since there are not always necessarily "remaining values", we must also add on another summation, to account for the case when $\frac{1}{2}(x_1 + x_2 - x_3 - 2j) + 1 \nleqslant x_1 - 2j$. This is equivalent to $j > \frac{1}{2}(x_1 - x_2 + x_3 - 2)$. So we have

$$\begin{aligned}\eta_{\mathbb{X}_3}(x) = \sum_{j=0}^{(1/2)(x_1 - x_2 + x_3 - 2)} \Bigg(\sum_{k=0}^{(1/2)(x_1 + x_2 - x_3 - 2j)} \left(\left\lfloor \frac{x_3 - x_1 + 2j + k}{2} \right\rfloor + 1 \right) \\ + \sum_{k=(1/2)(x_1 + x_2 - x_3 - 2j) + 1}^{x_2 - 2j} \left(\left\lfloor \frac{x_2 - k}{2} \right\rfloor + 1 \right) \Bigg) \\ + \sum_{j=(1/2)(x_1 - x_2 + x_3)}^{\lfloor x_1/2 \rfloor} \left(\left\lfloor \frac{x_3 - x_1 + 2j + k}{2} \right\rfloor + 1 \right),\end{aligned}$$

which can be further modified to yield

$$\begin{aligned}\eta_{\mathbb{X}_3}(x) = \sum_{j=0}^{\lfloor x_1/2 \rfloor} \Bigg(\sum_{k=0}^{(1/2)(x_1 + x_2 - x_3 - 2j)} \left(\left\lfloor \frac{x_3 - x_1 + 2j + k}{2} \right\rfloor + 1 \right) \Bigg) \\ + \sum_{j=0}^{(1/2)(x_1 - x_2 + x_3 - 2)} \Bigg(\sum_{k=(1/2)(x_1 + x_2 - x_3 - 2j) + 1}^{x_1 - 2j} \left(\left\lfloor \frac{x_2 - k}{2} \right\rfloor + 1 \right) \Bigg) \quad \blacksquare\end{aligned}$$

Now we find a recursive form for $\eta_{\mathbb{X}_m}(x)$. Let $x = (x_1, ..., x_m, x_{m+1}) \in \mathbb{X}_m$. We first list all the irreducibles in $\mathbb{X}_m$, with nonzero first entry:

$$\mathbf{i} = (2, 0, ..., 0, 1)$$

$$\mathbf{j}_1 = (1, 1, 0, ..., 0, 1)$$

$$\mathbf{j}_2 = (1, 0, 1, 0, ..., 0, 1)$$

$$\vdots$$

$$\mathbf{j}_k = (1, \underbrace{0, ..., 0, 1}_{k}, 0, ..., 0, 1)$$

$$\vdots$$

$$\mathbf{j}_{m-1} = (1, \underbrace{0, ..., 0, 1}_{m-1}, 1).$$

Since $\mathbf{i}, \mathbf{j}_1, ..., \mathbf{j}_{m-1}$ are the irreducibles that affect x_1, we will be concerned with their coefficients, $a, b_1, ..., b_{m-1}$ (respectively) in an irreducible factorization of x. The coefficient a may take on the values $0 \leqslant a \leqslant \lfloor x_1/2 \rfloor$. We choose an allowable value for a, and let

$$x' = x - a\mathbf{i} = (x_1 - 2a, x_2, x_2, ..., x_m, x_{m+1} - a).$$

Now b_1 has a range of $0 \leqslant b_1 \leqslant x'_1 = x_1 - 2a$. We choose an allowable value for b_1, and let

$$x'' = x' - b_1\mathbf{j}_1 = (x_1 - 2a - b_1, x_2 - b_1, x_3, ..., x_m, x_{m+1} - a - b_1).$$

We continue in this fashion, choosing allowable coefficients of relevant irreducibles, to obtain for each $1 \leqslant k \leqslant m-2$,

$$\begin{aligned} x^{(k+1)} &= x^{(k)} - b_k\mathbf{j}_k \\ &= (x_1 - 2a - b_1 - \cdots - b_k, x_2 - b_1, ..., x_{k+1} - b_k, x_{k+2}, ..., x_m, \\ &\quad x_{m+1} - a - b_1 - \cdots - b_k) \\ &= \left(x_1 - 2a - \sum_{l=1}^{k} b_l, x_2 - b_1, x_3 - b_2, ..., x_{k+1} - b_k, x_{k+2}, ..., x_m, \right. \\ &\quad \left. x_{m+1} - a - \sum_{l=1}^{k} b_l \right). \end{aligned}$$

Letting $x_i^{(k)}$ represent the ith coordinate of $x^{(k)}$, we are now forced to set $b_{m-1} = x_1^{(m-1)}$, and so we have

$$x^{(m)} = \left(x_1^{(m-1)} - b_{m-1}, x_2 - b_1, ..., x_m - b_{m-1}, x_{m+1} - a - \sum_{l=1}^{m-1} b_l\right)$$
$$= \left(0, x_2 - b_1, ..., x_m - b_{m-1}, x_{m+1} - a - \sum_{l=1}^{m-1} b_l\right)$$
$$= (0, x_2^{(m)}, ..., x_{m+1}^{(m)}).$$

Letting $y = (x_2^{(m)}, ..., x_{m+1}^{(m)}) \in \mathbb{X}_{m-1}$, we obtain the following.

PROPOSITION 5. *Let $m \geqslant 4$ be a positive integer, $x = (x_1, ..., x_{m+1}) \in \mathbb{X}_m$ with $x_1 \leqslant \cdots \leqslant x_m$ and $a, b_1, ..., b_{m-1}$, y be defined as above. Then*

$$\eta_{\mathbb{X}_m}(x) = \sum_{a=0}^{\lfloor x_1/2 \rfloor} \sum_{b_1=0}^{x_1-2a} \sum_{b_2=0}^{x_1-2a-b_1} \cdots \sum_{b_{m-2}=0}^{x_1-2a-\sum_{k=1}^{m-3} b_k} \eta_{\mathbb{X}_{m-1}}(y).$$

For example, when $m=4$ we obtain $\eta_{\mathbb{X}_4}(x) =$

$$\sum_{a=0}^{\lfloor x_1/2 \rfloor} \sum_{b_1=0}^{x_1-2a} \sum_{b_2=0}^{x_1-2a-b_1} \eta_{\mathbb{X}_3}((x_2 - b_1, x_3 - b_2, x_4 - b_3, x_5 - a - b_1 - b_2 - b_3)).$$

Hence, given a Krull monoid M with divisor class $\mathbb{Z}_2$ whose nontrivial divisor class contains finitely many primes, we have produced a formula for counting the number of different factorizations of any nonzero nonunit $x \in M$. To see this, write

$$x = p_1^{n_1} \cdots p_k^{n_k} q_1^{m_1} \cdots q_t^{m_t},$$

where the p_i's are distinct prime divisors from the trivial divisor class and the q_j's distinct prime divisors from the nontrivial divisor class. Any irreducible factorization of x will contain the product $p_1^{n_1} \cdots p_k^{n_k}$. Hence $\eta(x)$ depends on $\eta(x')$ where $x' = q_1^{m_1} \cdots q_t^{m_t}$. Now, x' is divisible by t distinct nontrivial prime divisors, and if we assume without loss that $m_1 \leqslant \cdots \leqslant m_t$, then

$$\eta(x) = \eta(x') = \eta_{\mathbb{X}_t}\left(\left(m_1, ..., m_t, \frac{m_1 + \cdots + m_t}{2}\right)\right).$$

Notice that the same reasoning works if M has infinitely many primes in the nontrivial class. Thus our formula works on *any* Krull monoid with divisor class group $\mathbb{Z}_2$.

EXAMPLE 6. If $R = \mathbb{R}[\sqrt{-5}]$, then R has class number two. If p is prime in $\mathbb{Z}$, then p ramifies in R if $p = 2$ or 5, splits in R if $p \equiv 1$, 3, 7 or 9 (mod 20), and is inert in R if $p \not\equiv 1$, 3, 7 or 9 (mod 20) (with the exception of 2 and 5 as noted above) [8]. We compute $\eta(70398)$ in R. In $\mathbb{Z}$, $70398 = 2 \cdot 3^2 \cdot 3911$ where 2 ramifies, 3 splits and 3911 is inert. Now, $R - \{0\}$ is a multiplicative Krull monoid with divisor class group $\mathbb{Z}_2$. Hence, there is a prime divisor p_1 in the trivial divisor class and distinct nonzero prime divisors q_1, q_2 and q_3 so that

$$70398 = p_1 q_1^2 (q_2 q_3)^2 = p_1 q_1^2 q_2^2 q_3^2.$$

Thus

$$\eta(70398) = \eta(18) = \eta(q_1^2 q_2^2 q_3^2) = \eta_{\mathbb{X}_3}((2, 2, 2, 3))$$
$$= \sum_{j=0}^{1} \sum_{k=0}^{2-2j} \left(\left\lfloor \frac{\min\{2-j, 2j+k\}}{2} \right\rfloor + 1 \right) = 5.$$

ACKNOWLEDGMENT

Part of this work was completed while the second two authors were at Trinity University participating in an NSF sponsored REU Program. They wish to thank Michael McQuistan for discussions related to this work.

REFERENCES

1. D. F. Anderson, Elasticity of factorization in integral domains: A survey, *in* "Factorization in Integral Domains," Lecture Notes in Pure and Applied Mathematics, Vol. 189, pp. 1–29, Dekker, New York, 1997.
2. L. Carlitz, A characterization of algebraic number fields with class number two, *Proc. Amer. Math. Soc.* **11** (1960), 391–392.
3. S. T. Chapman and A. Geroldinger, Krull domains and monoids, their sets of lengths, and associated combinatorial problems, *in* "Factorization in Integral Domains," Lecture Notes in Pure and Applied Mathematics, Vol. 189, pp. 73–112, Dekker, New York, 1997.
4. S. T. Chapman, U. Krause, and E. Oeljeklaus, Monoids determined by a homogenous linear diophanine equation and the half-factorial property, to appear in *J. Pure Appl. Algebra.*
5. F. Halter-Koch, Factorization problems in class number two, *Colloq. Math.* **65** (1993), 255–265.
6. F. Halter-Koch, On the asymptotic behaviour of the number of distinct factorizations into irreducibles, *Ark. Math.* **31** (1993), 297–305.
7. U. Krause and C. Zahlten, Arithmetic in Krull monoids and the cross number of divisor class groups, *Mitt. Math. Ges. Hamburg* **12** (1991), 681–696.
8. D. Marcus, "Number Fields," Springer-Verlag, New York, 1977.
9. W. Narkiewicz, Numbers with unique factorization in an algebraic number field, *Acta Arith.* **21** (1972), 313–322.