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NORMALITY OF PRODUCT SPACES AND MORITA'S CONJECTURES

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It is well-known that Z is a perfectly normal space (normal P-space) if and only if $X \times Z$ is perfectly normal (normal) for every metric space X. Conversely, denote by $\boldsymbol{Q}$ (resp. $\boldsymbol{N}$) the class of all spaces X whose products $X \times Z$ with all perfectly normal spaces (all normal P-spaces) Z are normal. It is natural to ask whether $\boldsymbol{Q}$ and $\boldsymbol{N}$ necessarily coincide with the class $\boldsymbol{M}$ of metrizable spaces.

Clearly, $\boldsymbol{M} \subset \boldsymbol{N} \subset \boldsymbol{Q}$. We prove that first countable members of $\boldsymbol{Q}$ are metrizable and that under $V = L$ the classes $\boldsymbol{M}$ and $\boldsymbol{N}$ coincide, thus giving a consistency proof of Morita's conjecture. On the other hand, even though $\boldsymbol{Q}$ contains non-metrizable members, it is quite close to $\boldsymbol{M}$: the class $\boldsymbol{Q}$ is countably productive and hereditary, and all members X of $\boldsymbol{Q}$ are stratifiable and satisfy $c(X) = l(X) = w(X)$. In particular, locally Lindelöf or locally Souslin or locally p-spaces in $\boldsymbol{Q}$ are metrizable.

The above results immediately lead to the consistency proof of another Morita's conjecture, stating that X is a metrizable σ-locally compact space if and only if $X \times Y$ is normal for every normal countably paracompact space Y. No additional set-theoretic assumptions are necessary if X is first countable.

In our investigation, an important role is played by the famous Bing examples of normal, non-collectionwise normal spaces. Answering Dennis Burke's question, we prove that products of two Bing-type examples are always non-normal.

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1. Introduction

It is well-known that Z is a perfectly normal space (normal P-space) if and only if $X \times Z$ is perfectly normal (normal) for every metric space X. Conversely, denote by $\boldsymbol{Q}$ (resp. $\boldsymbol{N}$) the class of all spaces X whose products $X \times Z$ with all perfectly normal spaces (all normal P-spaces) Z are normal. It is natural to ask whether $\boldsymbol{Q}$ and $\boldsymbol{N}$ necessarily coincide with the class $\boldsymbol{M}$ of metrizable spaces.

Clearly, $\boldsymbol{M} \subset \boldsymbol{N} \subset \boldsymbol{Q}$. We prove that first countable members of $\boldsymbol{Q}$ are metrizable and that under $V = L$ the classes $\boldsymbol{M}$ and $\boldsymbol{N}$ coincide, thus giving a consistency proof of Morita's conjecture. On the other hand, even though $\boldsymbol{Q}$ contains non-metrizable members, it is quite close to $\boldsymbol{M}$: The class $\boldsymbol{Q}$ is countably productive and hereditary, and all members X of $\boldsymbol{Q}$ are stratifiable and satisfy $c(X) = l(X) = w(X)$. In particular, locally Lindelöf or locally Souslin or locally p-spaces in $\boldsymbol{Q}$ are metrizable. Our results, therefore, essentially improve those obtained earlier by K. Chiba [2, 3, 4] and by T. and K. Chiba [5]. We also present a simple example of a non-metrizable space of cardinality ω_1 in $\boldsymbol{Q}$.

The above results immediately lead to the consistency proof of another Morita's conjecture, stating that X is a metrizable σ-locally compact space if and only if $X \times Y$ is normal for every normal countably paracompact space Y. No additional set-theoretic assumptions are necessary if X is first countable.

An important role in our investigation is played by the famous Bing examples of normal non-collectionwise normal spaces, and by their generalizations, which we call *Bing-type examples.*

Definition. By a *Bing-type example* we mean any normal space H with the following properties:

(B1) H contains an uncountable, closed and discrete subset D consisting of functions $f: T \to \omega$, for some T;

(B2) basic neighborhoods of elements $f \in D$ have the form $B(f, F, \ldots)$, where F is a finite subset of T and $B(f, F, \ldots) \cap B(g, G, \ldots) \neq \emptyset$ if $f|F \cap G = g|F \cap G$. (The dots in $B(f, F, \ldots)$ indicate that basic neighborhoods in H may also depend on some other parameters, which are however, irrelevant for our study).

We will denote by $H(\kappa)$ any Bing-type example for which $|D| \geqslant \kappa > \omega$.

It is immediate that both examples G and H constructed by Bing [1], as well as many of their later modifications (see e.g. [7] and [6]) are Bing-type examples. In particular, for every $\kappa > \omega$ there exist σ-discrete, perfectly normal and metacompact Bing-type examples $H(\kappa)$ [7]. Applying the Δ-lemma, one easily sees that Bing-type examples are never ω_1-collectionwise Hausdorff. We prove that normality of a product space $X \times H(\kappa)$ imposes strong restrictions on X; in particular, it implies that X is κ-collectionwise Hausdorff. This shows that products of two Bing-type examples are always non-normal, thus answering Dennis Burke's question [13].

Our paper is organized as follows. In Section 2 we prove four theorems and a basic proposition showing how normality of $X \times H(\kappa)$ affects X. In Section 3 we derive from these theorems the properties of the class $\boldsymbol{Q}$ of spaces whose products with all perfectly normal spaces are normal. A simple example of a non-metrizable space in $\boldsymbol{Q}$ is given in Section 4. Section 5 is devoted to the class $\boldsymbol{N}$ and to the consistency proof of Morita's conjectures.

All spaces are assumed to be T_3. For all undefined symbols and notions the reader is referred to [6] and [15]. In particular, by $w(X)$, $d(X)$, $c(X)$, $l(X)$, $hc(X)$, $e(X)$, etc. we denote the weight, the density, the Souslin number, the Lindelöf number, the hereditary Souslin number (spread) and the extent of a space X, respectively. We denote by $T^{<\omega}$ the set of all finite subsets of T, by T^A the set of all functions from A into T and by $f|A$ the restriction of f to A.

2. Products with Bing-type examples

The following lemma shows that normality of $X \times H(\kappa)$ implies that X has a 'σ-cushioned expansion property' for compact families of cardinality $\leqslant \kappa$.

Lemma 1. *Suppose that $\{C_\alpha\colon \alpha < \kappa\}$ is a family of compact subsets of X and $G_\alpha \supset C_\alpha$ are open. If $X \times H(\kappa)$ is normal, then there exists a family $\{U_\alpha\colon \alpha < \kappa\}$ of open subsets of X and a family $\{S_n\colon n < \omega\}$ of subsets of κ such that $C_\alpha \subset U_\alpha$, $\bigcup_{n<\omega} S_n = \kappa$ and*

$$\overline{\bigcup \{U_\alpha\colon \alpha \in S\}} \subset \bigcup \{G_\alpha\colon \alpha \in S\} \text{ for every } n < \omega \text{ and every } S \subset S_n. \tag{1}$$

Proof. Let $\{f_\alpha\colon \alpha < \kappa\}$ be a set of mutually distinct elements of $D \subset H(\kappa)$. Define

$$K_0 = \bigcup \{C_\alpha \times \{f_\alpha\}\colon \alpha < \kappa\} \quad \text{and} \quad K_1 = \bigcup \{(X \setminus G_\alpha) \times \{f_\alpha\}\colon \alpha < \kappa\}.$$

Clearly, the sets K_0 and K_1 are closed and disjoint in $X \times H(\kappa)$. There exist therefore open sets $U_\alpha \supset C_\alpha$ in X and basic neighborhoods $B_\alpha = B(f_\alpha, F_\alpha, \ldots)$ of f_α in $H(\kappa)$ such that

$$\bigcup \{U_\alpha \times B_\alpha\colon \alpha < \kappa\} \cap K_1 \neq \emptyset. \tag{2}$$

Let $S_n = \{\alpha \in \kappa\colon |F_\alpha| \leqslant n \text{ and } f_\alpha(F_\alpha) \subset n\}$. Obviously, $\bigcup_{n<\omega} S_n = \kappa$.

Suppose that condition (1) fails for some n and some $S \subset S_n$. There exists therefore an $x_0 \in X$ such that

$$x_0 \in \overline{\bigcup \{U_\alpha\colon \alpha \in S\}} \setminus \bigcup \{G_\alpha\colon \alpha \in S\}. \tag{3}$$

Clearly, there exists an $L \in T^{<\omega}$ such that $x_0 \in \overline{\bigcup \{U_\alpha\colon \alpha \in S, L \subset F_\alpha\}}$ and $x_0 \notin \overline{\bigcup \{U_\alpha\colon \alpha \in S, L^* \subset F_\alpha\}}$, for any $L \subset L^*$, $L \neq L^*$. Since there are only finitely many functions $p\colon L \to n$ there exists an $\alpha_0 \in S$ such that $x_0 \in \overline{\bigcup \{U_\alpha\colon \alpha \in S, L \subset F_\alpha, f_\alpha | L = f_{\alpha_0} | L\}}$. Take an arbitrary $K \in T^{<\omega}$. We claim that

$$x_0 \in \overline{\bigcup \{U_\alpha\colon \alpha \in S, f_\alpha | K \cap F_\alpha = f_{\alpha_0} | K \cap F_\alpha\}}. \tag{4}$$

Indeed, for every $\alpha \in K\backslash L$ we have $x_0 \notin \overline{\bigcup\{U_\alpha\colon \alpha \in S, L \cup \{\alpha\} \subset F_\alpha\}}$, hence $x_0 \in \overline{\bigcup\{U_\alpha\colon \alpha \in S, L \subset F_\alpha, f_\alpha|L = f_{\alpha_0}|L, K\backslash F_\alpha \subset L\}}$, which clearly implies (4).

Consider the point $z = \langle x_0, f_{\alpha_0}\rangle \in X \times H(\kappa)$. Since $\alpha_0 \in S$, $x_0 \notin G_{\alpha_0}$ and $z \in K_1$. Let $U \times B$ be an arbitrary neighborhood of z, where $B = B(f_{\alpha_0}, K, \ldots)$ for some $K \in T^{<\omega}$. By (4) there exists an $\alpha \in \kappa$ such that $U \cap U_\alpha \neq \emptyset$ and $f_\alpha|K \cap F_\alpha = f_{\alpha_0}|K \cap F_\alpha$. By (B2), $B \cap B_\alpha \neq \emptyset$ and therefore $(U \times B) \cap (U_\alpha \times B_\alpha) \neq \emptyset$, which contradicts (2) and completes the proof. $\square$

From Lemma 1 we derive four theorems showing how normality of $X \times H(\kappa)$ affects X.

Theorem 1. *If $X \times H(\kappa)$ is normal, then X is κ-collectionwise normal with respect to compact sets.*

More generally, every discrete collection of $\leqslant \kappa$ subsets of X, each of which is a union of $\leqslant \kappa$ compact sets, can be separated by open sets.

Proof. Let $\{K_\alpha\colon \alpha \in \kappa\}$ be a discrete collection of subsets of X and let $C_{\alpha,\beta}$ be compact sets such that $K_\alpha = \bigcup\{C_{\alpha,\beta}\colon \beta < \kappa\}$, for all $\alpha < \kappa$. Let $G_{\alpha,\beta} = X\backslash\overline{\bigcup\{K_\gamma\colon \gamma < \kappa \text{ and } \gamma \neq \alpha\}}$. Clearly, $C_{\alpha,\beta} \subset K_\alpha \subset G_{\alpha,\beta}$. By Lemma 1, there exists a family $\{U_{\alpha,\beta}\colon \alpha, \beta < \kappa\}$ of open subsets of X and a family $\{S_n\colon n < \omega\}$ of subsets of $\kappa \times \kappa$ such that $C_{\alpha,\beta} \subset U_{\alpha,\beta}$, $\bigcup_{n<\omega} S_n = \kappa \times \kappa$ and $\overline{\bigcup\{U_{\alpha,\beta}\colon \langle\alpha, \beta\rangle \in S\}} \subset \bigcup\{G_{\alpha,\beta}\colon \langle\alpha, \beta\rangle \in S\}$, for all $n < \omega$ and $S \subset S_n$. Define

$$V_\alpha = \bigcup_{n=1}^{\infty} \left(\bigcup\{U_{\alpha,\beta}\colon \langle\alpha, \beta\rangle \in S_n\} \Big\backslash \bigcup_{i=1}^{n} \overline{\bigcup\{U_{\gamma,\delta}\colon \gamma \neq \alpha, \langle\gamma, \delta\rangle \in S_i\}} \right).$$

The sets V_α are clearly open and one easily checks that $K_\alpha \subset V_\alpha$ and $V_\alpha \cap V_\beta = \emptyset$, for $\alpha \neq \beta$. $\square$

Corollary 1. *Products of two Bing-type examples are always non-normal.*

It was essentially proved in [3] that normality of $X \times H(\kappa)$ with $\kappa \geqslant |X|$ implies perfect normality of X. Here, we strengthen this result.

Theorem 2. *If $X \times H(\kappa)$ is normal and $\kappa \geqslant |X|$, then X is perfectly paracompact. More generally, the condition $\kappa \geqslant |X|$ can be replaced by the assumption that X is a union of $\leqslant \kappa$ compact sets.*

Proof. We prove perfect normality of X first. Let F be a closed subset of X and suppose that $X = \bigcup\{C_\alpha\colon \alpha < \kappa\}$ and C_α are compact. Since $H(\kappa)$ is not ω_1-collectionwise normal, the spaces C_α are metrizable [9] and therefore $C_\alpha\backslash F = \bigcup\{C_{\alpha,n}\colon n < \omega\}$, where $C_{\alpha,n}$ are compact. For every $\langle\alpha, n\rangle$ let $G_{\alpha,n} = X\backslash F$. By Lemma 1, there exist open sets $U_{\alpha,n} \supset C_{\alpha,n}$ and subsets S_n of $\kappa \times \omega$ such that $\bigcup\{S_n\colon n < \omega\} = \kappa \times \omega$ and $L_n = \overline{\bigcup\{U_{\alpha,m}\colon \langle\alpha, m\rangle \in S_n\}} \subset X\backslash F$. Clearly, $\bigcup_{n<\omega} L_n = X\backslash F$.

Now, we prove paracompactness of X. Let $\boldsymbol{W}=\{W_t: t\in T\}$ be an open covering of X and let $X=\bigcup\{C_\alpha: \alpha<\kappa\}$, where C_α are compact. For every $\alpha<\kappa$ there exists a finite subset $T_0\subset T$ such that $C_\alpha\subset\bigcup\{W_t: t\in T_0\}$, hence, subdividing the sets C_α if necessary, we can assume that for every $\alpha<\kappa$ there is a $t_\alpha\in T$ such that $C_\alpha\subset W_{t_\alpha}$. Define $G_\alpha=W_{t_\alpha}$ and apply Lemma 1. There exist open sets $U_\alpha\supset C_\alpha$ and subsets $S_n\subset\kappa$ such that $\bigcup_{n<\omega} S_n=\kappa$ and

$$\overline{\bigcup\{U_\alpha: \alpha\in S\}}\subset\bigcup\{W_{t_\alpha}: \alpha\in S\}$$

for every $n<\omega$ and $S\subset S_n$. This means that $\boldsymbol{W}$ has a σ-cushioned open refinement and thus proves paracompactness of X [8]. □

Theorem 2 can be greatly strengthened if we assume that $\kappa\geq\max(|X|, w(X))$.

Theorem 3. *If $X\times H(\kappa)$ is normal and $\kappa\geq\max(|X|, w(X))$, then X is stratifiable.*

Proof. Let $\boldsymbol{U}$ be a base of X of cardinality $\leq\kappa$ and let $\{\langle x_\alpha, U_\alpha\rangle: \alpha\in\kappa\}$ be the enumeration of all possible pairs $\langle x, U\rangle$, where $x\in U\in\boldsymbol{U}$. By Lemma 1 there exist open sets G_α and subsets S_n of κ such that $x_\alpha\in G_\alpha$, $\bigcup_{n<\omega} S_n=\kappa$ and for every $n<\omega$ and every $S\subset S_n$ we have

$$\overline{\bigcup\{G_\alpha: \alpha\in S\}}\subset\bigcup\{U_\alpha: \alpha\in S\}. \tag{5}$$

For every open set W in X and every $n<\omega$ define $W(n)=\bigcup\{G_\alpha: U_\alpha\subset W, \alpha\in S_n\}$. Clearly, the sets $W(n)$ are open, $W\subset V$ implies $W(n)\subset V(n)$ and $W=\bigcup_{n<\omega} W(n)$. By (5), we also have $\overline{W(n)}\subset W$, which proves that X is stratifiable. □

In [5] T. and K Chiba essentially proved that normality of $X\times H(\kappa)$ with $\kappa\geq w(X)$ implies $w(X)=d(X)$. Their proof can be modified to show that actually $w(X)=hc(X)$. Here, we give a much simpler proof, using Lemma 1, of a slightly weaker result.

Theorem 4. *If $X\times H(\kappa)$ is normal and $\kappa\geq\max(|X|, w(X))$, then $w(X)=hc(X)$.*

Proof. Suppose otherwise and let $\mu=hc(X)<w(X)$. Denote by $\boldsymbol{U}$ a base of X of cardinality $\leq\kappa$, let $\lambda=\mu^+$ and suppose that

$$D=\{f_{\langle x,U\rangle}: x\in U\in\boldsymbol{U}\}\subset H(\kappa)$$

consists of mutually distinct functions $f_{\langle x,U\rangle}: T\to\omega$. Define

$$K_0=\{\langle x, f_{\langle x,U\rangle}\rangle: x\in U\in\boldsymbol{U}\}$$

and

$$K_1=\bigcup\{(X\setminus U)\times\{f_{\langle x,U\rangle}\}: x\in U\in\boldsymbol{U}\}.$$

The sets K_0 and K_1 are disjoint and closed subsets of $X \times H(\kappa)$ and therefore there exist open sets $V_{\langle x,U\rangle}$ in X containing x and basic neighborhoods $B_{\langle x,U\rangle} = B(f_{\langle x,U\rangle}, F_{\langle x,u\rangle}, \ldots)$ of $f_{\langle x,U\rangle}$ in $H(\kappa)$ such that

$$\overline{\bigcup \{V_{\langle x,U\rangle} \times B_{\langle x,U\rangle}\colon x \in U \in \boldsymbol{U}\}} \cap K_1 = \emptyset.$$

Let $V = \bigcup \{V_{\langle x,U\rangle} \times B_{\langle x,U\rangle}\colon x \in U \in \boldsymbol{U}\}$. By induction, for every $\alpha < \lambda$ find $x_\alpha \in U_\alpha \in \boldsymbol{U}$ and define $V_\alpha = V_{\langle x_\alpha, U_\alpha\rangle}$. Let $x_0 \in U_0 \in \boldsymbol{U}$ be arbitrary. If $\alpha < \lambda$ then the family $\{V_\beta\colon \beta < \alpha\}$ is not a base of X so there exists a point $x_\alpha \in U_\alpha \in \boldsymbol{U}$ such that $x_\alpha \in V_\beta$ and $\beta < \alpha$ imply $V_\beta \setminus U_\alpha \neq \emptyset$. Let $F_\alpha = F_{\langle x_\alpha, U_\alpha\rangle}$ and $f_\alpha = f_{\langle x_\alpha, U_\alpha\rangle}$.

Using a Δ-system argument on λ we can find a subset S of λ of cardinality λ and a finite subset F of T such that $F_\alpha \cap F_{\alpha'} = F$ and $f_\alpha | F = f_{\alpha'} | F$, for all $\alpha, \alpha' \in S$, $\alpha \neq \alpha'$. Without loss of generality, we can assume that $D = \{f_\alpha\colon \alpha \in \lambda\}$ and $\lambda = S$.

Let $A = \{\beta \in \lambda\colon V_\beta \subset \bigcup_{\alpha \in \lambda} U_\alpha\}$. Suppose that $|A| = \lambda$. Without loss of generality we can assume that $A = \lambda$ and use Lemma 1 on $x_\alpha \in V_\alpha$ to choose for all $\alpha < \lambda$ an open W_α in X with $x_\alpha \in W_\alpha \subset V_\alpha$ so that $\lambda = \bigcup_{n<\omega} \lambda_n$ and $L \subset \lambda_n$ implies $\overline{\bigcup_{\alpha \in L} W_\alpha} \subset \bigcup_{\alpha \in L} V_\alpha$. There is an $n < \omega$ such that $|\lambda_n| = \lambda$. The set $\{x_\alpha\colon \alpha \in \lambda_n\}$ is discrete in X and consists of mutually distinct points. Indeed, if $\beta < \alpha$, then $V_\beta \subset U_\alpha$ and $x_\alpha \notin V_\beta$, thus $x_\alpha \notin \overline{\{x_\beta\colon \beta \in \lambda_n, \beta < \alpha\}}$. If $\beta > \alpha$, then $x_\beta \notin V_\alpha$, thus $x_\alpha \notin \overline{\{x_\beta\colon \beta \in \lambda_n, \beta \neq \alpha\}}$. This contradicts $hc(X) < \lambda$.

Let $P = \{\beta \in \lambda\colon$ there exists a $\beta^* < \lambda$ with $V_\beta \setminus U_{\beta^*} \neq \emptyset\}$. We have $|P| = \lambda$. For each $\beta \in P$ choose $p_\beta \in V_\beta \setminus U_{\beta^*}$ and observe that $\langle p_\beta, f_{\beta^*}\rangle \in K_1$. There exist open sets $W_\beta < V_\beta$ with $p_\beta \in W_\beta$ and basic open neighborhoods $B'_{\beta^*} = B(f_{\beta^*}, F'_{\beta^*}, \ldots)$ of f_{β^*} such that

$$V \cap (W_\beta \times B'_{\beta^*}) = \emptyset.$$

The family $\{p_\beta\colon \beta \in P\}$ is discrete and no infinite number of points p_β are the same, contradicting $hc(X) < \lambda$, again.

Indeed, otherwise there would be a p_α and infinitely many β's say $\{\beta_i\colon i \in \omega\}$ such that $W_\alpha \cap V_{\beta_i} \neq \emptyset$ for all $i \in \omega$. Since $(W_\alpha \times B'_{\alpha^*}) \cap V = \emptyset$ and $V_{\beta_i} \times B_{\beta_i} \subset V$ we must have

$$B'_{\alpha^*} \cap B_{\beta_i} = \emptyset,$$

but $f_{\alpha^*} | F = f_{\beta_i} | F$, therefore for each $i < \omega$ there must exist a $t_i \in F'_{\alpha^*} \cap (F_{\beta_i} \setminus F)$ such that $f_{\alpha^*}(t_i) \neq f_{\beta_i}(T_i)$, but the family $\{F_{\beta_i} \setminus F\}_{i<\omega}$ is disjoint and infinite, and F'_{α^*} is finite, which is impossible. □

Corollary 2. *If $X \times H(\kappa)$ is normal and $\kappa \geqslant \max(|X|, w(X))$, then X is stratifiable and $c(X) = l(X) = w(X)$.*

Proof. It follows from Theorems 3 and 4 that X is stratifiable and $hc(X) = w(X)$. Always, $e(X) \leqslant l(X) \leqslant w(X)$, $e(X) \leqslant hc(X)$ and $c(X) \leqslant hc(X)$ (see [6, Problem

3.12.7]). It is easy to see that in any perfectly paracompact space, $hc(X) = c(X) = e(X)$. □

Remark. It follows, that under the above assumptions also cardinal functions $e(X)$, $d(X)$, $hd(X)$, $hc(X)$, $hl(X)$ and $nw(X)$ are all equal to $w(X)$ and $w(X) \leq |X|$.

Corollary 2 gives necessary conditions for a space X to have the property that $X \times H(\kappa)$ is normal if $\kappa \geq \max(|X|, w(X))$. These conditions are by no means sufficient as the following example shows.

Example 1 [2, Example 1]. There exists a first countable and stratifiable space X of cardinality c (= continuum) such that $w(Y) = hc(Y)$ for every subspace Y of X, but $X \times H(c)$ is not normal.

In fact, the following basic result holds, which plays a fundamental role in the consistency proof of Morita's conjectures.

Proposition 1 (basic). *Suppose that $\langle Y, U\rangle$ is a non-metrizable topological space and $\kappa = |U|$. There is a perfectly normal Bing-type example $H(\kappa)$ such that if $Y \times H(\kappa)$ is normal then there exists an increasing sequence $\{Y_\alpha\colon \alpha < \omega_1\}$ of subsets of Y such that $\overline{\bigcup\{Y_\alpha\colon \alpha < \omega_1\}} \backslash \bigcup\{\bar{Y}_\alpha\colon \alpha < \omega_1\} \neq \emptyset$. In particular, Y has uncountable tightness.*

Proof. Suppose otherwise and let $H(\kappa)$ be a modified 'Bing's G on κ'. Let $T = \{\text{subsets of } \kappa\}$ and, for $x \in U \in U$, let f_{xU} be the function $f\colon T \to 2$ defined by $f(A) = 1$ if and only if $\langle x, U\rangle \in A$. Let $F = \{f_{xU} \mid x \in U \in U\}$. Let $\Sigma = \{\sigma\colon D \to 2 \mid D$ is a finite subset of $T\}$ and, for $\sigma \in \Sigma$, $B_\sigma = \{f \in F \cup \Sigma \mid f$ extends $\sigma\}$. If $H(\kappa) = F \cup \Sigma$ is topologized by using $\{\{\sigma\} \mid \sigma \in \Sigma\} \cup \{B_\sigma \mid \sigma \in \Sigma\}$ as a basis, then $H(\kappa)$ is a perfectly normal Bing-type example.

Assume $Y \times H(\kappa)$ is normal and define $K_0 = \{(x, f_{xU}) \mid x \in U \in U\}$ and $K_1 = \bigcup\{(Y - U) \times \{f_{xU}\} \mid x \in U \in U\}$. As usual there is an open V containing K_0 whose closure misses K_1. For all $y \in Y$, define $\Gamma_y = \{\gamma \in \Sigma \mid (\{y\} \times B_\gamma) \cap \bar{V} = \emptyset\}$ and, for $\sigma \in \Sigma$, let V_σ be the interior of $V'_\sigma = \{y \in Y \mid$ for every $\gamma \in \Gamma_y$, there exists an $A \in (\operatorname{dom}\sigma \cap \operatorname{dom}\gamma)$ with $\sigma(A) \neq \gamma(A)\}$.

Suppose $x \in U \in U$. Then $(x, f_{xU}) \in V$ and $(y, f_{xU}) \notin \bar{V}$ for any $y \in Y - U$. So there are σ and σ_y in Σ for all $y \in (Y - U)$, which are all extended by f_{xU}, and there are open W and W_y for each $y \in (Y - U)$ with $x \in W$ and $y \in W_y$ such that $(W \times B_\sigma) \subset V$ and $(W_y \times B_{\sigma y}) \cap \bar{V} = \emptyset$. If $z \in W_y$, $\sigma_y \in \Gamma_z$ and σ and σ_y agree on their common domain; so $\overline{V'_\sigma} \subset U$. If $z \in W$, $(\{z\} \times B_\sigma) \subset V$ so there can be no $\gamma \in \Sigma$ with $(\{z\} \times B_\gamma) \cap \bar{V} = \emptyset$ for which γ and σ agree on their common domain. So $W \subset V'_\sigma$ and W is open and $x \in V_\sigma \subset \overline{V'_\sigma} \subset U$. Let $\Sigma_x = \{\sigma \in \Sigma \mid x \in V_\sigma$ but $x \notin V_\tau$ for any $\tau \subset \sigma$, $\tau \neq \sigma\}$.

Since σ is finite and $\tau \subset \sigma$ implies $V_\tau \subset V_\sigma$, if $x \in U \in \mathcal{U}$ there is $\sigma \in \Sigma_x$ with $x \in V_\sigma \subset V'_\sigma \subset U$ and we let σ_{xU} denote one such σ.

We now forget all about $H(\kappa)$ and only use the following facts about Y obtained from the fact that $Y \times H(\kappa)$ is normal:

(1) (Theorem 2) Y is perfectly paracompact.

(2) Σ is a set of functions into 2 with finite domains, and, for each $y \in Y$, Γ_y is a subset of Σ.

(3) For $\sigma \in \Sigma$ define V_σ to be the interior of $V'_\sigma = \{y \in Y \mid \text{for every } \gamma \in \Gamma_y, \text{ there exists an } A \in (\operatorname{dom} \sigma \cap \operatorname{dom} \gamma) \text{ with } \sigma(A) \neq \gamma(A)\}$ and, for $x \in Y$, $\Sigma_x = \{\sigma \in \Sigma \mid x \in V_\sigma$ but $x \notin V_\tau$ for any $\tau \subset \sigma$, $\tau \neq \sigma$, in $\Sigma\}$. Then, for $x \in U \in \mathcal{U}$ there is $\sigma = \sigma_{xU} \in \Sigma_x$ such that $x \in V_\sigma \subset V'_\sigma \subset U$.

(4) (See the proof of Lemma 1 and Theorem 3). If $n \in N$, $y \in Y$, $Q \subset \{(x, U) \mid x \in U \in \mathcal{U}\}$, and, for all $(x, U) \in Q$, $|\operatorname{dom} \sigma_{xU}| \leq n$ and $y \notin U$, then $y \notin \operatorname{closure}(\bigcup \{V_{\sigma_{xU}} \mid (x, U) \in Q\})$.

Claim. For each $y \in Y$, Σ_y is countable.

Suppose that Σ_y is not countable. For each $\alpha < \omega_1$, we will define a set Y_α such that $Y_\beta \subset Y_\alpha$ for all $\beta < \alpha$, and $y \notin \overline{Y_\alpha}$, but $y \in \operatorname{closure}(\bigcup \{Y_\alpha \mid \alpha < \omega_1\})$.

By a Δ-system argument we choose a set D and, for each $\alpha < \omega_1$, a $\sigma_\alpha \in \Sigma_y$ such that, for all $\alpha \neq \beta$ in ω_1,

(i) $\operatorname{dom} \sigma_\alpha \neq D$ and $\sigma_\alpha \neq \sigma_\beta$,

(ii) $(\operatorname{dom} \sigma_\alpha) \cap (\operatorname{dom} \sigma_\beta) = D$,

(iii) $\sigma_\alpha(A) = \sigma_\beta(A)$ for all $A \in D$.

For $\alpha < \omega_1$, let $E_\alpha = \bigcup \{\operatorname{dom} \sigma_\beta \mid \beta < \alpha\}$ and let $E = \bigcup \{E_\alpha \mid \alpha < \omega_1\}$. Define $Y_\alpha = \{x \in Y \mid \text{there exists a } \gamma \in \Gamma_x \text{ with } \gamma(A) = \sigma_0(A) \text{ if } A \in (\operatorname{dom} \gamma \cap D) \text{ and } ((\operatorname{dom} \gamma) \cap E) \subset E_\alpha\}$.

Clearly $Y_\beta \subset Y_\alpha$ if $\beta < \alpha$ and $y \notin \overline{Y_\alpha}$ since $y \in V_{\sigma_\alpha}$ and $V_{\sigma_\alpha} \cap Y_\alpha = \emptyset$. But every neighborhood of y intersects some Y_α. For suppose $y \in U \in \mathcal{U}$. By (i) we can choose $A \in (\operatorname{dom} \sigma_0 - D)$; let $\sigma = \sigma_0 | (\operatorname{dom} \sigma_0 - \{A\})$. Since $\sigma_0 \in \Sigma_y$, $y \in V_{\sigma_0} - V_\sigma$. So there must exist $x \in U \cap V_{\sigma_0}$ and $\gamma \in \Gamma_x$ such that $A \in \operatorname{dom} \gamma$ and $\gamma(A) \neq \sigma_0(A)$ but γ agrees with σ on their common domain. There is an $\alpha < \omega_1$ such that $((\operatorname{dom} \gamma) \cap E) \subset E_\alpha$; thus $x \in U \cap Y_\alpha$; and the proof of the claim is complete.

Continuing the proof of Proposition 1 we begin our construction of a σ-locally finite basis for Y, thus proving that Y is metrizable.

By the Claim we can index $\Sigma_y = \{\sigma_{iy} \mid i \in N\}$.

If $j \in N$ and $U \in \mathcal{U}$, define $U(j) = \bigcup \{V_\sigma \mid \sigma = \sigma_{xW}$ for some open $W \subset U$ and $|\operatorname{dom} \sigma| \leq j\}$.

For all i and j in N and $y \in Y$ we define $W_{ij}(y) = V_{\sigma_{iy}} - \operatorname{closure}\{\bigcup \{V_{\sigma_{ix}}(j) \mid y \notin V_{\sigma_{ix}}\}\}$; then, by (4), $W_{ij}(y)$ is an open neighborhood of y.

By induction we construct a set S_n of sequences $S = \langle U_1, i_1, j_1, U_2, y_2, i_2, j_2, U_3, y_3, i_3, j_3, \ldots, U_n, y_n, i_n, j_n \rangle$ where $U_1 = Y$, for all $m < n$, U_m is open in Y, $i_m \in N$, $j_m \in N$, and, for $1 < m \leq n$, $y_m \in Y$ and $U_m \subset W_{i_1 j_1}(y_m)$.

If $n=1$, $\boldsymbol{S}_1=\{\langle Y, i, j\rangle \mid i, j\in N\}$.

If $S\in \boldsymbol{S}_n$ is as above, define $X_S=U_n-\bigcup\{V_{\sigma_{i_m y_m}} \mid 1<m\leq n\}$, and $W_S=U_n\cap(\bigcup\{W_{i_1 j_1}(x)\mid x\in X_S\})$. Since Y is perfectly paracompact, for each $j\in N$ we can define a locally finite refinement $\boldsymbol{L}_{Sj}$ of $\{W_{i_1 j_1}(x)\mid x\in X_S\}$ such that $\bigcup\{U\boldsymbol{L}_{Sj}\mid j\in N\}=W_S$. For each $U\in \boldsymbol{L}_{Sj}$ choose $y_U\in X_S$ with $U\subset W_{i_1 j_1}(y_U)$. (Observe that y_U actually depends on S and j as well as on U.) For later use, if $i\in N$, define $\boldsymbol{L}_{Sij}=\{U\cap V_{\sigma_{i y_U}}\mid U\in \boldsymbol{L}_{Sj}\}$. Define $\boldsymbol{S}_{n+1}(S)=\{\langle U_1, i_1, j_1, \ldots, U_n, y_n, i_n, j_n, U, y_U, i, j\rangle \mid i, j\in N$ and $U\in \boldsymbol{L}_{Sj}\}$.

Let $\boldsymbol{S}_{n+1}=\bigcup\{\boldsymbol{S}_{n+1}(S)\mid S\in \boldsymbol{S}_n\}$.

For $n\in N$, let $\boldsymbol{J}_n$ be the set of all sequences $J=\langle i_1, j_1, i_2, j_2, \ldots, i_n, j_n\rangle$ of terms of N; if $m\leq n$ we use J_m to denote $\langle i_1, j_1, \ldots, i_m, j_m\rangle$. If $J\in \boldsymbol{J}_n$, let $\boldsymbol{L}_J=\bigcup\{\boldsymbol{L}_{Si_{m+1}j_{m+1}} \mid$for some $m<n$, $S\in \boldsymbol{S}_m$ and extends $J_m\}$. We make two claims for $J=\bigcup\{\boldsymbol{J}_n\mid n\in N\}$:

(I) For each $J\in \boldsymbol{J}$, $\boldsymbol{L}_J$ is a locally finite family of open sets.

(II) $\bigcup\{\boldsymbol{L}_J\mid J\in \boldsymbol{J}\}$ is a basis for the topology of Y.

When we prove these claims we will have proved Proposition 1.

Proof of (I). Suppose $p\in Y$ and $J=\langle i_1, j_1, \ldots, i_n, j_n\rangle$.

If $S_1\in \boldsymbol{S}_1$ and extends J_1, then $S_1=\langle Y, i_1, j_1\rangle$ and $\boldsymbol{L}_{S_1 j_2}$ is a locally finite open cover of Y. Thus there is an open neighborhood O_1 of p such that $\boldsymbol{U}_1=\{U\in \boldsymbol{L}_{S_1 j_2}\mid U\cap O_1\neq\emptyset\}$ is finite. Thus $\boldsymbol{L}_{S_1 i_2 j_2}$ has only finitely many members which intersect O_1.

If $S_2\in \boldsymbol{S}_2$ and S_2 extends J_2, then $S_2=\langle Y, i_1, j_1, U_2, y_2, i_2, j_2\rangle$ where $U_2\in \boldsymbol{L}_{S_1 j_2}$. Since $O_1\cap U_2=\emptyset$ unless $U_2\in \boldsymbol{U}_1$ and $\boldsymbol{L}_{S_2 j_3}$ is a locally finite family of subsets of U_2, we can assume that $U_2\in \boldsymbol{U}_1$ and we can find an open neighborhood O_2 of p with $\boldsymbol{U}_2=\{U\in \boldsymbol{L}_{S_{j_3}}\mid S\in \boldsymbol{S}_2$, S extends J_2, and $U\cap O_2\neq\emptyset\}$ being finite.

Continuing in this way, for each $m<n$ we choose a neighborhood O_m of p such that $\boldsymbol{U}_m=\{U\in \boldsymbol{L}_{Sj_{m+1}}\mid S\in \boldsymbol{S}_m$, S extends J_m, and $U\cap O_m\neq\emptyset\}$ is finite. Thus $\bigcap\{O_m \mid m<n\}$ is a neighborhood of p meeting only finitely many members of $\boldsymbol{L}_j$.

Proof of (II). Since by (3) $\{V_{\sigma_{ip}}\mid i\in N\}$ is a local basis at $p\in Y$, it suffices to show that for every p and i there is $J\in \boldsymbol{J}$ and $U\in \boldsymbol{L}_j$ with $p\in U\subset V_{\sigma_{ip}}$.

Let $i_1=i$ and choose $j_1\in N$ sufficiently large that $p\in V_{\sigma_{ip}}(j_1)$; (if $W=V_{\sigma_{ip}}$, then $|\sigma_{pW}|$ would clearly be 'sufficiently large'.)

Let $S_1=\langle Y, i, j_1\rangle$; since $Y=W_{S_1}$, $\bigcup\{\boldsymbol{L}_{S_1 j}\mid j\in N\}$ covers Y and there is $j_2\in N$ and $U_2\in \boldsymbol{L}_{S_1 j_2}$ such that $p\in U_2$. Let $y_2=y_{U_2}$. Then $U_2\subset W_{i, j_1}(y_2)\subset(V_{iy_2}-\bigcup\{V_{\sigma_{ix}}(j_1')\mid y_2\notin V_{\sigma_{ix}}\})$. Since $p\in U_2$, $y_2\in V_{\sigma_{ip}}$. We have two cases:

Case a$_1$. $\sigma_{ip}=\sigma_{i_2 y_2}$ for some $i_2\in N$. In this case $p\in(U_2\cap V_{\sigma_{i_2 y_2}})\subset V_{\sigma_{ip}}$ and $(U_2\cap V_{\sigma_{i_2 y_2}})\in \boldsymbol{L}_{S_1 i_2 j_2}\subset \boldsymbol{L}_{J_2}$ where $J_2=\langle i, j_1, i_2, j_2\rangle$. So we are done.

Case b$_1$. $\sigma_{ip}\neq\sigma_{i_2 y_2}$ for any $i_2\in N$. Since $y_2\in V_{\sigma_{ip}}$ there must be some $i_2\in N$ such that $\sigma_{i_2 y_2}\subsetneqq\sigma_{ip}$. Let $S_2=\langle Y, i, j_1, U_2, y_2, i_2, j_2\rangle$. By the definition of σ_{ip}, $p\notin V_\tau$ for any $\tau\subsetneqq\sigma_{ip}$; so $p\notin V_{\sigma_{i_2 y_2}}$; but $p\in U_2$ so $p\in X_{S_2}\subset W_{S_2}$. Thus there is $j_3\in N$ and $U_3\in \boldsymbol{L}_{S_2 j_3}$ such that $p\in U_3$. Let $y_3=y_{U_3}$. Then $U_3\subset W_{ij_1}(y_3)$. As with y_2, $y_3\in V_{\sigma_{ip}}$. But observe that also $y_3\notin V_{\sigma_{i_2 y_2}}$ since $X_{S_2}\cap V_{\sigma_{i_2 y_2}}=\emptyset$. Again we have two cases:

Case a$_2$. $\sigma_{ip} = \sigma_{i_3y_3}$ for some $i_3 \notin N$. As in Case a$_1$, since $y_3 \in V_{\sigma_{ip}}$ and $(U_3 \cap V_{\sigma_{i_3y_3}}) \in L_{S_2 i_3 j_3} \subset L_{J_3}$ where $J_3 = \langle i, j_1, i_2, j_2, i_3, j_3\rangle$, we are done and we can assume:

Case b$_2$. $\sigma_{ip} \neq \sigma_{i_3y_3}$ for any $i_3 \in N$. As in (b$_1$), there is $i_3 \in N$ with $\sigma_{i_3y_3} \subsetneq \sigma_{ip}$; but this time, since $y_3 \notin V_{\sigma_{i_2y_2}}$, also $\sigma_{i_3y_3} \neq \sigma_{i_2y_2}$. Defining $S_3 = \langle Y, i, j_1, U_2, y_2, i_2, j_2, U_3, y_3, i_3, j_3\rangle$ we continue this process, building for each $n > 1$ an S_n with the properties that $\sigma_{i_ny_n} \subsetneq \sigma_{ip}$ and $\sigma_{i_ny_n} \neq \sigma_{i_my_m}$ for any $m < n$. Since σ_{ip} is finite and can have only finitely many distinct subsets, this contradiction proves Proposition 1. □

3. Products with perfectly normal spaces

Results obtained in Section 2 enable us to investigate more closely the class $\boldsymbol{Q}$ of spaces, whose products with all perfectly normal spaces are normal.

Corollary 3. *If $X \times Z$ is normal for every perfectly normal space Z, then X is stratifiable and $w(X) = c(X) = l(X)$.*

Proof. Immediate consequence of Corollary 2. □

Lemma 2. *Suppose that spaces $X_i \times Z$ are normal for every $i < \omega$ and every perfectly normal space Z. Then, the product space $\prod_{i<\omega} X_i \times Z$ is perfectly normal for every perfectly normal space Z.*

Proof. Let Z be an arbitrary perfectly normal space. It suffices to show by induction that $\prod_{i\leq n} X_i \times Z$ is perfectly normal, for every n. This is certainly true for $n = 0$. If $\prod_{i\leq n} X_i \times Z$ is perfectly normal, then the space $P = X_{n+1} \times \prod_{i\leq n} X_i \times Z$ is normal, by assumption. By Corollary 3, the space X_{n+1} is stratifiable and therefore has a σ-discrete network, which easily implies that P is perfectly normal. □

Corollary 4. *If $X \times Z$ is normal for every perfectly normal space Z, then $X^\omega \times Z$ is perfectly normal, for every perfectly normal Z.*

Corollary 5. *The class $\boldsymbol{Q}$ of spaces, whose products with all perfectly normal spaces are normal, is countably productive and hereditary.*

There exist non-metrizable spaces, whose products with all perfectly normal spaces are normal (see [5] or Section 4). On the other hand, we have the following:

Corollary 6. *Every locally ccc or locally Lindelöf or locally p-space, whose products with all perfectly normal spaces are normal, is metrizable.*

Proof. Corollary 3 implies that every locally *ccc* or locally Lindelöf or locally p-space is locally metrizable and paracompact, and therefore metrizable. □

Corollary 7. *Every space with countable tightness (e.g. first countable), whose products with all perfectly normal spaces are normal, is metrizable.*

Proof. Immediate consequence of Proposition 1. □

Corollary 8. *Every k-space whose products with all perfectly normal spaces are normal, is metrizable.*

Proof. By Lemma 2 of [4], such a space has to have countable tightness. □

The class $\boldsymbol{Q}$ is countably productive, hereditary and appears to be quite close to the class of metrizable spaces.

Problem 1. Characterize the class $\boldsymbol{Q}$ of spaces, whose products with all perfectly normal spaces are normal.

4. A non-metrizable space whose product with every perfectly normal space is normal

The first example of a non-metrizable space whose product with every perfectly normal space is normal was given in [5]. Here, we present a simple example of cardinality ω_1. We call a cover $\{V_F: F \in \omega_1^{<\omega}\}$ *increasing* if $V_F \subset V_G$ for $F \subset G$ and by a *shrinking* of a cover $\{U_s: s \in S\}$ we mean any covering $\{A_s: s \in S\}$ such that $A_s \subset U_s$.

Example 2. A non-metrizable space Ω of cardinality ω_1 such that, for any space Z, $\Omega \times Z$ is normal if and only if Z is normal and every increasing open covering $\{V_F: F \in \omega_1^{<\omega}\}$ of Z has a closed shrinking.

In particular, $\Omega \times Z$ is normal for every perfectly normal space Z and for every normal space Z being the union of $\leqslant \omega_1$ compact sets.

Remark. In Section 5 we prove that under $V = L$ there are no non-metrizable spaces whose product with every normal P-space is normal (Morita's conjecture). This does not preclude the possibility that under suitable set-theoretic assumptions Example 2 could be a counter example to this conjecture.

It is easy to verify that for a countably paracompact Z the following conditions are equivalent:

(i) $\Omega \times Z$ is normal;

(ii) Z is normal and every increasing open covering $\{W_\alpha: \alpha < \omega_1\}$ of Z has a closed shrinking:

(iii) Z is normal and for every increasing open covering $\{W_\alpha: \alpha < \omega_1\}$ of Z there exist closed subsets $F_{\alpha\beta}$ of Z, α, $\beta < \omega_1$, such that $F_{\alpha\beta} \subset W_\alpha$ and $\bigcup_{\alpha,\beta} F_{\alpha\beta} = Z$.

Construction of the Example. We define the space Ω to be as in Ohta [12]. Let $\Omega = \{\omega_1\} \cup \omega_1^{<\omega}$. Points different from ω_1 are isolated and the point ω_1 has a base of neighborhoods consisting of all sets of the form

$$B_F = \{x \in \Omega \colon F \subset x\},$$

for $F \in \omega_1^{<\omega}$. One easily checks that Ω is a regular non-metrizable, σ-discrete space with one non-isolated point.

Suppose that $\Omega \times Z$ is normal and let $\{V_F \colon F \in \omega_1^{<\omega}\}$ be an increasing open covering of Z. Define $V = \bigcup\{B_F \times V_F \colon F \in \omega_1^{<\omega}\}$. Clearly, V is an open neighborhood of $S = \{\omega_1\} \times Z$, hence there exists an open set $G \supset S$ such that $\bar{G} \subset V$. Let

$$K_F = \overline{\bigcup\{W \subset Z \colon W \text{ is open and } B_F \times W \subset G\}}.$$

The family $\{K_F \colon F \in \omega_1^{<\omega}\}$ is a closed covering of Z, so it remains to show that $K_F \subset V_F$. Suppose that $z \in K_F$. Then, $B_F \times \{z\} \subset \bar{G} \subset V$, hence there exists an $F_0 \in \omega_1^{<\omega}$ such that $\langle F, z\rangle \in B_{F_0} \times V_{F_0}$. This implies that $F_0 \subset F$ and therefore $z \in V_{F_0} \subset V_F$.

Suppose now that the condition is satisfied. We have to show that $\Omega \times Z$ is normal. Since ω_1 is the only non-isolated point of Ω and Z is normal it clearly suffices to prove that for every open neighborhood V of S there exists an open $G \supset S$ such that $\bar{G} \subset V$. Let $V_F = \bigcup\{W \subset Z \colon W \text{ is open and } B_F \times W \subset V\}$. The family $\{V_F \colon F \in \omega_1^{<\omega}\}$ is an increasing open covering of Z. There exists therefore an open covering $\{G_F \colon F \in \omega_1^{<\omega}\}$ of Z such that $\bar{G}_F \subset V_F$. Define $G = \bigcup\{B_F \times G_F \colon F \in \omega_1^{<\omega}\}$. Clearly, $G \supset S$ and $\overline{B_F \times G_F} \subset V$ for every F. This implies that $\bar{G} \subset V$, because the base $\{B_F \colon F \in \omega_1^{<\omega}\}$ is locally finite at every point $x \in \Omega$ different from ω_1.

Every open covering $\{U_s \colon s \in S\}$ of a perfectly normal space Z has a closed shrinking. Indeed, there exist open sets $U_{s,n}$ such that $\bigcup_{n<\omega} U_{s,n} = U_s$ and $\overline{U_{s,n}} \subset U_s$, $U_{s,n} \subset U_{s,n+1}$. Since Z is sountably paracompact, there exists an increasing locally finite open refinement $\{G_n \colon n < \omega\}$ of the open covering $\{V_n \colon n < \omega\}$, where $V_n = \bigcup\{U_{s,n} \colon s \in S\}$. It suffices to define $W_s = \bigcup\{U_{s,n} \cap G_n \colon n < \omega\}$. Clearly, $\bar{W}_s \subset U_s$ and $\bigcup\{W_s \colon s \in S\} = Z$.

Suppose now that Z is a union of $\leqslant\omega_1$ compact sets $\{C_\alpha \colon \alpha < \omega_1\}$ and $V = \{V_F \colon F \in \omega_1^{<\omega}\}$ is an increasing open covering of Z. Since V is increasing, for every $\alpha < \omega_1$ there exists an $F_\alpha \in \omega_1^{<\omega}$ such that $C_\alpha \subset V_{F_\alpha}$. Since the union of countably many finite sets is countable, we can also assume that $F_\alpha \neq F_\beta$, for $\alpha \neq \beta$. Define

$$K_F = \begin{cases} C_\alpha & \text{if } F = F_\alpha, \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, $\{K_F \colon F \in \omega_1^{<\omega}\}$ is a closed shrinking of V.

5. Morita's conjectures

K. Morita proved that a space Z is a normal P-space if and only if $X \times Z$ is normal for every metrizable space X (see [15, Corollary 4.11]). He also conjectured that the converse of this result is true:

Morita's conjecture I. *A space X is metrizable if and only if $X \times Z$ is normal for every normal P-space Z.*

As we have seen in Corollary 7, the above characterization holds for all spaces X having countable tightness. Below we will show, using Proposition 1 and a recent consistency result due to Beslagic and Rudin [10], that Morita's conjecture I holds under the assumption of the Gödel's Axiom of Constructibility $V = L$.

Theorem 5. ($V = L$) *Morita's conjecture* I *holds.*

A related result of K. Morita states that a *metrizable* space X is σ-locally compact if and only if $X \times Z$ is normal for every normal and countably paracompact space Z (see [15, Theorem 4.13]). K. Morita conjectured that the assumption of metrizability of X can be omitted:

Morita's conjecture II. *A space X is metrizable and σ-locally compact if and only if $X \times Z$ is normal for every normal and countably paracompact space Z.*

Since every normal P-space is countably paracompact, whenever the characterization in Conjecture I holds for a given space X, the corresponding characterization in Conjecture II holds for that space, too. In particular, we have the following two results:

Corollary 8. *A space X having countable tightness is metrizable and σ-locally compact if and only if $X \times Z$ is normal for every normal and countably paracompact space Z.*

Theorem 6. ($V = L$) *Morita's conjecture* II *holds.*

The proof of Theorem 5 is an immediate consequence of Proposition 1 and Example 3 and Lemma 3 below. Notice, that $V = L$ is only used in Example 3.

Example 3 [10]. ($V = L$) *For every cardinal λ there exists a collectionwise normal P-space X_λ having an increasing open cover $\{U_\alpha: \alpha < \omega_1\}$ with the property that, if for each $\alpha < \omega_1$ and $\beta < \lambda$, $C_{\alpha\beta}$ is a closed in X_λ subset of U_α, then $\bigcup\{C_{\alpha\beta}: \alpha < \omega_1, \beta < \lambda\} \neq X_\lambda$.*

Lemma 3. *If Y is a space, $\kappa \geq |Y|$, $y \in Y$, and $\{Y_\alpha: \alpha < \omega_1\}$ is an increasing sequence of subsets of Y, such that $y \in \overline{\bigcup\{Y_\alpha: \alpha < \omega_1\}} \setminus \bigcup\{\bar{Y}_\alpha: \alpha < \omega_1\}$, then, $X_\kappa \times Y$ is not normal.*

Proof. Assume $X_\kappa \times Y$ is normal. For $\alpha < \omega_1$ let $L_\alpha = X_\kappa - U_\alpha$. Let $K_0 = X_\kappa \times \{y\}$ and $K_1 = \bigcup\{L_\alpha \times Y_\alpha \mid \alpha < \omega_1\}$. Since K_0 and K_1 are closed and disjoint there is an open V containing K_0 whose closure misses K_1. Index $\bigcup\{Y_\alpha \mid \alpha < \omega_1\} = \{y_\beta \mid \beta < \kappa\}$ and define $C_{\alpha\beta} = \{x \in X_\kappa \mid (x, y_\beta) \in \bar{V}$ and $y_\beta \in Y_\alpha\}$; $C_{\alpha\beta}$ is a closed subset of U_α. By Example 3 there is an $x \in X_\kappa$ not in any $C_{\alpha\beta}$. Since $(x, y) \in V$, there is an open W in Y with $y \in W$ and $(\{x\} \times W) \subset V$. There is $y_\beta \in W$; so $(x, y_\beta) \in V$. But $y_\beta \in Y_\alpha$ for some α; contradicting $x \notin C_{\alpha\beta}$. □

Remark. To show that the Ω of Section 4 would do for a counter example to Morita's Conjecture I, and thus that Morita's Conjecture I is undecidable, it would suffice to show that in some model for set theory every increasing open covering $\{U_\alpha : \alpha < \omega_1\}$ of a normal P-space has a closed shrinking.

Problem 2. Are Morita's conjectures independent of the axioms of set theory?

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References

[1] R.H. Bing, Metrization of topological spaces, Canad. J. Math. 3 (1951) 175-186.
[2] K. Chiba, On products of normal spaces, Rep. Fac. Sci. Shizuoka Univ. 9 (1974) 1-11.
[3] K. Chiba, Two remarks on the normality of product spaces, Rep. Fac. Sci. Shizuoka Univ. 11 (1976) 17-22.
[4] K. Chiba, On the normality of product spaces, Math. Jaonica 25 (1980) 559-565.
[5] T. Chiba and K. Chiba, A note on normality of product spaces, Sci. Rep. Tokyo Kyoiku Daigaku 12 (1974) 165-173.
[6] R. Engelking, General Topology (Polish Scientific Publishers, Warsaw, 1977).
[7] E. Michael, Point-finite and locally finite coverings, Canad. J. Math. 7 (1955) 275-279.
[8] E. Michael, Yet another note on paracompact spaces, Proc. Amer. Math. Soc. 8 (1957) 822-828.
[9] M.E. Rudin, The normality of products with a compact factor, Gen. Topology Appl. 5 (1975) 45-59.
[10] A. Beslagic and M.E. Rudin, Non-shrinking open covers from $V = L$, to appear.
[11] K. Morita, Products of normal spaces with metric spaces, Math. Ann. 154 (1964) 365-382.
[12] H. Ohta, On normal, non-rectangular products, Quart. J. Math. Oxford 32 (1981) 339-344.
[13] D. Burke, personal communication.
[14] T.C. Przymusinski, Products of normal spaces, in: K. Kunen and J. Vaughan, Eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984).