Exponential stability of numerical solutions to a stochastic age-structured population system with diffusion

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Abstract

The main aim of this paper is to investigate the exponential stability of the Euler method for a stochastic age-structured population system with diffusion. The definition of exponential mean square stability of numerical method is introduced. It is proved that the Euler scheme is exponentially stable in mean square sense. An example is given for illustration.

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1. Introduction

Stochastic differential equations have been found many applications in areas such as economics, biology, finance, ecology and other sciences [2,7,8]. Recently, one of the most important and interesting problems in the analysis of stochastic differential equations is their numerical solution. For example, Platen [10] gave an introduction to numerical methods for stochastic differential equations. Marion et al. [9] studied the convergence of the form under relaxed linear growth and Lipschitz conditions. Ronghua investigated the Euler method for a class of autonomous stochastic delay differential equations with Markovian switching [13].

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In the present investigation, the random behavior of the birth–death process is carefully incorporated into the continuous-time age-structured population equations to obtain a system of stochastic differential equations that model age-structured dynamics. This age-structured population model is of theoretical interest. However, an application of the stochastic age-structured model is to study how age-structured influences estimated persistence time of a population where extinction is influenced by random fluctuations in the birth–death process. Recently, one of the most important and interesting problems in the analysis of stochastic age-structured population equations is their numerical solution. In this paper, we shall discuss the convergence of stochastic partial differential equations. That is, we consider the convergence of stochastic age-structured population systems with diffusion

\[
\begin{align*}
\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r} - k(r,t)\Delta P + \mu(r,t,x)P &= f(r,t,x,P) + g(r,t,x,P)\frac{\partial W}{\partial t} & \text{in } Q_A = (0, A) \times Q, \\
P(0,t,x) &= \int_0^A \beta(r,t,x)P(r,t,x) \, dr & \text{in } (0, T) \times \Gamma, \\
P(r,0,x) &= P_0(r,x) & \text{in } (0, A) \times \Gamma, \\
P(r,t,x) &= 0 & \text{on } \Sigma_A = (0, A) \times (0, T) \times \partial \Gamma, \\
y(t,x) &= \int_0^A P(r,t,x) \, dr & \text{in } Q,
\end{align*}
\]

where \( t \in (0, T) \), \( r \in (0, A) \), \( x \in \Gamma \), \( P(r,t,x) \) denotes the population density of age \( r \) at time \( t \) and in the location \( x \), \( \beta(r,t,x) \) denotes the fertility rate of females of age \( r \) at time \( t \) and in spatial position \( x \), \( \mu(r,t,x) \) denotes the mortality rate of age \( r \) at time \( t \) and in the location \( x \), \( \Delta \) denotes the Laplace operator with respect to the space variable, \( k(\text{constant})>0 \) is the diffusion coefficient. \( f(r,t,x,P) + g(r,t,x,P)\frac{\partial W}{\partial t} \) denotes effects of external environment for population system, such as emigration and earthquake and so on. The effects of external environment has the deterministic and random parts which depend on \( r, t, x \) and \( P \).

There has been much recent interest in application of deterministic age-structures mathematical models with diffusion (when \( g_1 = 0 \)). For example, Cushing [5] investigated hierarchial age-dependent populations with intra-specific competition or predation when \( g_1 = 0 \). Allen and Thrasher [1] considered vaccination strategies in age-dependent populations in the case of \( g_1 = 0 \). Pollard [11] studied the effects of adding stochastic terms to discrete-time age-dependent models that employ Leslie matrices. Gaston [6] discussed the existence, regularity and localization of the solution of deterministic age-structures mathematical models with diffusion. Renee Fister investigated optimal control of a first order partial differential equation system representing a competitive population model with age structure [12] (e.g., \( k = 0, \beta = 0 \) and \( g = 0 \)).

The effects of the stochastic environmental noise considerations lead to stochastic age-structured population systems, which are more realistic. When the diffusion of the population is not considered \( k(=0) \), Zhang [14] showed the existence, uniqueness and exponential stability for stochastic age-dependent population equation, and numerical analysis for stochastic age-dependent population equation has been studied in [15].

In general, stochastic age-structures mathematical models with diffusion rarely has an explicit solution. Thus, numerical approximation schemes are invaluable tools for exploring its properties. In this paper, we will develop an numerical approximation method for stochastic age-structures population system with diffusion of the type described by Eqs. \((1.1)\)–\((1.5)\). The numerical solution is defined by an implicit equation containing partial derivative. So our work differs from these references \([1,5,6,11,12]\) in that \( a \) numerical analysis is considered, and \( b \) random parts is involved. In particular, our results extend those in [15].

This paper is organized as follows: In Section 2, we shall first collect some basic preliminaries that are essential for our analysis, and introduce Euler approximation. In Section 3, we give the main result that the Euler method is exponential stable in mean square sense under some conditions, and the proof of this main result is completed. In Section 4, we provide an example to illustrate our result.
2. Preliminaries and Euler approximation

Let $O = (0, A) \times \Gamma$, and

$$V \equiv \left\{ \varphi | \varphi \in L^2(O), \quad \frac{\partial \varphi}{\partial x_i} \in L^2(O), \quad \text{where } \frac{\partial \varphi}{\partial x_i} \text{ are generalized partial derivatives} \right\}.$$

Then $V'$ the dual space of $V$. We denote by $| \cdot |$ and $\| \cdot \|$ the norms in $V$ and $V'$ respectively; by $\langle \cdot , \cdot \rangle$ the duality product between $V$, $V'$, and by $\langle \cdot , \cdot \rangle$ the scalar product in $H$. For an operator $B \in \mathcal{L}(M, H)$ be the space of all bounded linear operators from $M$ into $H$, we denote by $\| B \|_2$ the Hilbert–Schmidt norm, i.e.,

$$\| B \|^2_2 = \text{tr} (BW B^T).$$

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with a filtrations $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $P$-null sets).

Let $C = C([0, T]; H)$ be the space of all continuous function from $[0, T]$ into $H$ with sup-norm $\| \psi \|_C = \sup_{0 \leq s \leq T} \| \psi (s) \|, L^p_v = L^p([0, T]; V)$ and $L^p_H = L^p([0, T]; H)$.

**Definition 2.1.** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be the stochastic basis and $\omega_t$ a Wiener process. Suppose that $P_0$ is a random variable such that $E[|P_0|^2] < \infty$. A stochastic process $P_t$ is said to be a solution on $\Omega$ to the stochastic age-structured population system for $t \in [0, T]$ if the following conditions are satisfied:

1. $P_t$ is a $\mathcal{F}_t$-measurable random variable;
2. $P_t \in L^p(0, T; V) \cap L^2(\Omega; C(0, T; V)), p > 1, T > 0$, where $L^p(0, T; V)$ denotes the space of all $V$-valued processes $(P_t)_{t \in [0, T]}$ (we will write $P_t$ for short) measurable (from $[0, T] \times \Omega$ into $V$), and satisfying

$$E \int_0^T \| P_t \|^p dt < \infty.$$

Here $C(0, T; V)$ denotes the space of all continuous functions from $[0, T]$ to $V$;

3. It satisfies the equation:

$$\begin{align*}
\left\langle \frac{\partial P}{\partial t}, v \right\rangle + \int_0^t \left\langle \frac{\partial P}{\partial r}, v \right\rangle \, ds - \int_0^t \langle k(r, t) \Delta P, v \rangle \, ds + \int_0^t \langle \mu(r, s, x) P, v \rangle \, ds
\end{align*}$$

$$= \int_0^t \langle f(r, s, x, P), v \rangle \, ds + \int_0^t \langle g(r, s, x, P), v \rangle \, dw(s) \quad (2.1)$$

for all $v \in V, t \in [0, T]$, a.e. $w \in \Omega$, where the stochastic integral is understood in the Ito sense.

We consider the convergence of the following stochastic age-structured population systems with diffusion

$$\begin{align*}
\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r} - k(r, t) \Delta P + \mu(r, t, x) P
\end{align*}$$

$$= f(r, t, x, P) + g(r, t, x, P) \frac{\partial W}{\partial t}, \quad \text{in } Q_A = (0, A) \times Q, \quad (2.2)$$

$$P(0, t, x) = \int_0^A \beta(r, t, x) P(r, t, x) \, dr \quad \text{in } (0, T) \times \Gamma, \quad (2.3)$$

$$P(r, 0, x) = P_0(r, x) \quad \text{in } (0, A) \times \Gamma, \quad (2.4)$$

$$P(r, t, x) = 0 \quad \text{on } \Sigma_A = (0, A) \times (0, T) \times \partial \Gamma, \quad (2.5)$$

$$y(t, x) = \int_0^A P(r, t, x) \, dr \quad \text{in } Q, \quad (2.6)$$
A is the maximal age of the population species, so
\[ P(r, t, x) = 0 \quad \forall r \geq A. \]

Let \( \Delta t = \frac{T}{N} \), for system (2.2)–(2.6) the discrete approximate solution on \( t = 0, \Delta t, \cdots, N \Delta t \) is defined by the iterative scheme
\[
Q_{t}^{n+1} - Q_{t}^{n} - \frac{\partial Q_{t}^{n}}{\partial r} \Delta t - k(r, t) \Delta Q_{t}^{n} \Delta t + \mu(r, t, x) Q_{t}^{n} \Delta t = f(r, t, x, Q_{t}^{n}) \Delta t + g(r, t, x, Q_{t}^{n}) \Delta W_{n}, \tag{2.7}
\]

Here, \( Q_{t}^{n} \) is the approximation to \( P(r, t_{n}, x) \), for \( t_{n} = \frac{n}{N}T \), the time increment is \( \Delta t = \frac{T}{N} < 1 \), and the Brownian motion increment is \( \Delta W_{n} = W(t_{n+1}) - W(t_{n}) \).

For convenience, we shall extend the discrete numerical solution to continuous time. We first define the step function
\[
Z_{t} \equiv Z(r, t, x) = \sum_{k=0}^{N-1} Q_{t}^{k} 1[k \Delta t, (k+1) \Delta t), \tag{2.8}
\]

where \( 1_{G} \) is the indicator function for the set \( G \). Then we define
\[
Q_{t} - P_{0} + \int_{0}^{t} \frac{\partial Q_{s}}{\partial r} ds + \int_{0}^{t} k(r, s) \Delta Q_{s} ds + \int_{0}^{t} \mu(r, s, x) Z_{s} ds
\]
\[
= \int_{0}^{t} f(r, s, x, Z_{s}) ds + \int_{0}^{t} g(r, s, x, Z_{s}) dW_{s}, \tag{2.9}
\]

with \( Q_{0} = P(r, 0, x), Q_{t} = Q(r, t, x) \). It is straightforward to check that \( Z(r, t_{k}, x) = Q_{t_{k}}^{k} = Q(r, t_{k}, x) \). First, we state the assumptions about the stochastic age-dependent population system with diffusion that will be considered:

(i) \( f(r, t, x, 0) = 0, g(r, t, x, 0) = 0; \)

(ii) (Lipschitz condition) there exists a positive constant \( K \) such that \( p_{1}, p_{2} \in C \)
\[
|f(r, t, x, p_{1}) - f(r, t, x, p_{2})| \vee \|g(r, t, x, p_{1}) - g(r, t, x, p_{2})\|_{2} \leq K|p_{1} - p_{2}|, \text{ a.e. } t; \tag{2.10}
\]

(iii) \( \mu(r, t, x) \), and \( \beta(r, t, x) \) are continuous in \( Q \) such that
\[
0 \leq \mu(r, t, x) \leq \bar{\mu} < \infty, \quad 0 \leq \beta(r, t, x) \leq \bar{\beta} < \infty, \quad k_{0} \leq k(r, t) \leq \bar{k}. \tag{2.11}
\]

**Definition 2.2.** Suppose that \( P_{0} \) is a random variable such that \( E|P_{0}|^{2} < \infty \). For a given step size \( \Delta > 0 \), a numerical method is said to be exponentially stable in mean square on Eqs. (2.2)–(2.6) if there is a pair of positive constants \( \gamma \) and \( \bar{N} \), such that with initial data \( P_{0} \),
\[
E|Q_{n}^{n}|^{2} \leq \bar{N} E|P_{n}|^{2} e^{-\gamma n \Delta} \quad \forall n = 0, 1, 2, \ldots . \tag{2.12}
\]

**3. The main results**

In this section, we shall provide some lemmas which are necessary for the proof of our result. Because \( Q_{t} \) is the discrete numerical solution of Eqs. (2.2)–(2.6), we first study properties of \( Q_{t} \).

**Lemma 3.1.** Under assumptions (i)–(iii), for any \( T > 0 \),
\[
\sup_{0 \leq t \leq T} E|Q_{t}|^{2} \leq C_{1T}, \tag{3.1}
\]

where \( C_{1T} \) is a positive constant independent of \( \Delta \), but it depends on \( Q_{0} \) and \( T \).
Proof. From Eqs. (2.1) and (2.9), applying Itô formula to $|Q_t|^2$ yields

$$
|Q_t|^2 = |Q_0|^2 + 2 \int_0^t \int_O \left[ \left( -\frac{\partial Q_s}{\partial r} + k(r, t)\Delta Q_s Q_s \right) \right] \, dx \, dr \, ds \\
- 2 \int_0^t \int_O \mu(r, s, x) Z_s Q_s \, dx \, dr \, ds \\
+ 2 \int_0^t \int_O f(r, s, x, Z_s) Q_s \, dx \, dr \, ds \\
+ 2 \int_0^t \int_O Q_s g(r, s, x, Z_s) \, dx \, dW_s \\
+ \int_0^t \|g(r, s, x, Z_s)\|^2 \, ds.
$$

Since

$$
- \int_0^t \int_O \frac{\partial Q_s}{\partial r} Q_s \, dx \, dr \, ds \\
= \frac{1}{2} \int_0^t \int_O \left[ Q^2(A, s, x) - Q^2(0, s, x) \right] \, dx \, ds \\
= \frac{1}{2} \int_0^t \int_O \left( \int_0^A \beta(r, s, x) Q(r, s, x) \, dr \right)^2 \, dx \, ds,
$$

by (iii) and Hölder inequality, we have

$$
- \int_0^t \int_O \frac{\partial Q_s}{\partial r} Q_s \, dx \, dr \, ds \leq \frac{1}{2} A \tilde{\beta}^2 \int_0^t |Q_s|^2 \, ds.
$$

However, by (iii), we have

$$
\int_0^t \int_O k(r, s)\Delta Q_s Q_s \, dx \, dr \, ds \\
= - \int_0^t \int_O k(r, s)\Delta Q_s \cdot \nabla Q_s \, dx \, dr \, ds \\
\leq - k_0 \int_0^t \|Q_s\|^2.
$$

Therefore, we get that

$$
|Q_t|^2 \leq |Q_0|^2 + \frac{1}{2} A \tilde{\beta}^2 \int_0^t |Q_s|^2 \, ds - k_0 \int_0^t \|Q(s)\| \, ds \\
+ \int_0^t |f(r, s, x, Z_s)|^2 \, ds + 2\tilde{\mu} \int_0^t |Q_s||Z_s| \, ds + \int_0^t |Q_s| \, ds \\
+ \int_0^t \|g(r, s, x, Z_s)\|^2 \, ds + 2 \int_0^t \int_Q g(r, s, x, Z_s) \, dx \, dW_s \, dr.
$$
Now, it follows that for any $t \in [0, T]$

\[
E \sup_{0 \leq s \leq t} |Q_s|^2 \leq E|Q_0|^2 + \left( \frac{1}{2} \beta^2 A + \bar{\mu} + 1 \right) \int_0^t E \sup_{0 \leq s \leq t} |Q_s|^2 \, ds + \tilde{\mu} \int_0^t E|Z_s|^2 \, ds
\]

\[+ \int_0^t E|f(r, s, x, Z_s)|^2 \, ds + \int_0^t E\|g(r, s, x, Z_s)\|_2^2 \, ds
\]

\[+ 2E \sup_{0 \leq s \leq t} \int_0^s \int_O Q_r g(r, \tau, x, Z_\tau) \, dr \, dx \, dW_\tau.\] (3.2)

Using condition (ii) yields

\[
E \sup_{0 \leq s \leq t} |Q_s|^2 \leq E|Q_0|^2 + \left( \frac{1}{2} \beta^2 A + \bar{\mu} + 1 \right) \int_0^t E \sup_{0 \leq s \leq t} |Q_s|^2 \, ds
\]

\[+ (\bar{\mu} + 2K^2) \int_0^t |Z_s|^2 \, ds + 2E \sup_{0 \leq s \leq t} \int_0^s \int_O Q_r g(r, \tau, x, Z_\tau) \, dr \, dx \, dW_\tau.\] (3.3)

By Burkholder–Davis–Gundy’s inequality (see, for example, [4]), we have

\[
E \left[ \sup_{0 \leq s \leq t} \int_0^s \int_O Q_\tau g(r, \tau, x, Z_\tau) \, dW_\tau \, dx \, dr \right]
\]

\[\leq 3E \left[ \sup_{0 \leq s \leq t} |Q_s| \left( \int_0^t \|g(r, s, x, Z_s)\|_2^2 \, ds \right)^{1/2} \right]
\]

\[\leq \frac{1}{4} E[ \sup_{0 \leq s \leq t} |Q_s|^2 ] + K_1 \int_0^t \|g(r, s, x, Z_s)\|_2^2 \, ds
\]

\[\leq \frac{1}{4} E[ \sup_{0 \leq s \leq t} |Q_s|^2 ] + K_1 \cdot K^2 \int_0^t E|Z_s|^2 \, ds,\] (3.3)

for some positive constant $K_1 > 0$. Thus, it follows from (3.2) and (3.3)

\[
E \sup_{0 \leq s \leq t} |Q_s|^2 \leq 2 \left( \frac{1}{2} \beta A + 2\bar{\mu} + 1 + 2K^2 + 2K_1 K^2 \right) \int_0^t E \sup_{0 \leq r \leq s} |Q_r|^2 \, ds
\]

\[+ 2E|Q_0|^2, \quad \forall t \in [0, T].\]

Now, Gronwall’s lemma obviously implies the required result. The proof is complete. \qed

**Lemma 3.2.** Under the assumptions (i)–(iii), for any $T > 0$,

\[
E \sup_{0 \leq t \leq T} |Q_t - Z_t|^2 \leq C_2 A t \sup_{t \in [0, T]} E|Q_s|^2.\] (3.4)
Proof. For $\forall t \in [0, T]$, there exists an integer $k$ such that $t \in [k\Delta t, (k+1)\Delta t)$. We have

\[
Q_t - Z_t = Q_t - Q_t^k = -\int_{k\Delta t}^{t} \frac{\partial Q_s}{\partial r} \, ds + \int_{k\Delta t}^{t} k(r, s)\Delta Q_s \, ds - \int_{k\Delta t}^{t} \mu(r, s, x)Z_s \, ds + \int_{k\Delta t}^{t} f(r, s, x, Z_s) \, ds + \int_{k\Delta t}^{t} g(r, s, x, Z_s) \, dW_s.
\]

Thus,

\[
|Q_t - Z_t|^2 \leq 5 \int_{k\Delta t}^{t} \left| \frac{\partial Q_s}{\partial r} \right|^2 \, ds + 5k^2 \Delta t \int_{k\Delta t}^{t} |\Delta Q_s|^2 \, ds + 5\mu^2 \Delta t \int_{k\Delta t}^{t} |Z_s|^2 \, ds + 5\mu^2 \Delta t \int_{k\Delta t}^{t} |f(r, s, x, Z_s)|^2 \, ds + 5 \int_{k\Delta t}^{t} |g(r, s, x, Z_s)|^2 \, dW_s.
\]

Now, the Cauchy–Schwarz inequality and the assumptions (i)–(iii) give

\[
|Q_t - Z_t|^2 \leq 5\Delta t \int_{k\Delta t}^{t} \left| \frac{\partial Q_s}{\partial r} \right|^2 \, ds + 5k^2 \Delta t \int_{k\Delta t}^{t} |\Delta Q_s|^2 \, ds + 5\mu^2 \Delta t \int_{k\Delta t}^{t} |Z_s|^2 \, ds + 5\mu^2 \Delta t \int_{k\Delta t}^{t} |f(r, s, x, Z_s)|^2 \, ds + 5 \int_{k\Delta t}^{t} |g(r, s, x, Z_s)|^2 \, dW_s,
\]

whence applying the Burkholder–Davis–Gundy inequality and condition (ii) leads to

\[
E \sup_{t \in [0, T]} |g(r, s, x, Z_s) d W_s|^2 \leq C_3 \int_{k\Delta t}^{t} E \sup_{t \in [0, T]} |Z_s|^2 \, ds,
\]

where $C_3$ is a constant. Because the differential operator $\frac{\partial}{\partial r}$ and Laplace operator $\Delta$ are bounded linear operator, we obtain

\[
E \sup_{t \in [0, T]} |Q_t - Z_t|^2 \leq 5C_4 \Delta t \sup_{t \in [0, T]} E|Q_s|^2 + 5(\mu^2 \Delta t + K^2 \Delta t + C_3) \Delta t \sup_{t \in [0, T]} E|Q_s|^2,
\]

where $C_4$ is a constant. The result (3.4) is obtained.

We are now in a position to prove a strong convergence result. □
Lemma 3.3. Under assumptions (i)–(iii), for any $T > 0$,

\[
\sup_{0 \leq t \leq T} E|Q_t - P_t|^2 \leq C_T \Delta t \sup_{t \in [0, T]} E|Q_t|^2,
\]

where $C_T$ is independent of $\Delta t$, but it depends on $T$.

**Proof.** Combining (2.2) with (2.9) has

\[
P_t - Q_t
= - \int_0^t \frac{\partial(P_s - Q_s)}{\partial r} \, ds + \int_0^t k(r, t)(\Delta P_s - \Delta Q_s) \, ds
- \int_0^t \mu(r, s, x)(P_s - Z_s) \, ds
+ \int_0^t (f(r, s, x, P_s) - f(r, s, x, Z_s)) \, ds
+ \int_0^t (g(r, s, x, P_s) - g(r, s, x, Z_s)) \, dW_s.
\]

Therefore using Itô formula, along with the Cauchy–Schwarz inequality and (ii) yields,

\[
d|P_t - Q_t|^2
= -2\left(P_t - Q_t, \frac{\partial(P_t - Q_t)}{\partial r}\right) \, dt + 2\langle k(r, t)(P_t - Q_t), \Delta P_t - \Delta Q_t \rangle \, dt
- 2\langle P_t - Q_t, \mu(r, t, x)(P_t - Z_t) \rangle \, dt
+ 2\langle P_t - Q_t, f(r, t, x, P_t) - f(r, t, x, Z_t) \rangle \, dt
+ \|g(r, t, x, P_t) - g(r, t, x, Z_t)\|_2^2 \, dt
+ 2\langle P_t - Q_t, (g(r, t, x, P_t) - g(r, t, x, Z_t)) \, dW_t \rangle
\leq \bar{\beta}^2 \int_{\mathcal{O}} (P_t - Q_t)^2 \, dx \, dr \, dt - k_0 \int_{\mathcal{O}} \nabla(P_t - Q_t) \cdot \nabla(P_t - Q_t) \, dx \, dr \, dt
- 2\int_{\mathcal{O}} \mu(r, t, x)(P_t - Z_t)(P_t - Q_t) \, dx \, dr \, dt
+ 2\int_{\mathcal{O}} (P_t - Q_t)(f(r, t, x, P_t) - f(r, t, x, Z_t)) \, dx \, dr \, dt
+ 2\int_{\mathcal{O}} (P_t - Q_t)(g(r, t, x, P_t) - g(t, x, Z_t)) \, dx \, dW_t
+ \|g(r, t, x, P_t) - g(r, t, x, Z_t)\|_2^2 \, dt
\leq \bar{\beta}^2 A|P_t - Q_t|^2 \, dt - k_0 \|P_t - Q_t\|^2 \, dt + 2\bar{\mu}|P_t - Q_t||P_t - Z_t| \, dt
+ K|P_t - Q_t||P_t - Z_t| \, dt + K^2|P_t - Z_t|^2 \, dt
+ 2(P_t - Q_t, (g(r, t, x, P_t) - g(r, t, x, Z_t)) \, dW_t).
Hence, for any \( t \in [0, T] \),

\[
E \sup_{s \in [0, T]} |P_t - Q_t|^2 \leq (\tilde{\beta}^2 A + \tilde{\mu} + K) \int_0^T E \sup_{r \in [0, T]} |P_r - Q_r|^2 \, dt
+ (\tilde{\mu} + K + K^2) E \int_0^T |P_t - Z_t|^2 \, dt
+ 2E \sup_{s \in [0, T]} \int_0^T (P_t - Q_t, (g(\cdot, t, x, P_t) - g(\cdot, t, x, Z_t)) \, dW_t).
\]  

(3.6)

By Burkholder–Davis–Gundy’s inequality, we have

\[
E \sup_{s \in [0, T]} \int_0^s \left( P_s - Q_s, (g(\cdot, s, x, P_s) - g(\cdot, s, x, Z_s)) \, dW_s \right)^2 \leq k_1 E \left[ \sup_{0 \leq s \leq T} |P_t - Q_t| \left( \int_0^T \|g(\cdot, s, x, P_s) - g(\cdot, s, x, Z_s)\|^2 \, ds \right)^{1/2} \right]
\leq \frac{1}{4} E \left[ \sup_{0 \leq s \leq T} |P_t - Q_t|^2 + K_1 \int_0^T E |P_t - Z_t|^2 \, dt \right],
\]

(3.7)

where \( k_1 \) and \( K_1 \) are two positive constants. Therefore inserting (3.7) into (3.6) has

\[
E \sup_{s \in [0, T]} |P_s - Q_s|^2 \leq (\tilde{\beta}^2 A + \tilde{\mu} + K) \int_0^T E \sup_{r \in [0, s]} |P_r - Q_r|^2 \, ds
+ (\tilde{\mu} + K + K^2 + 2K_1) \int_0^T |P_t - Z_t|^2 \, dt
+ \frac{1}{2} E \sup_{s \in [0, T]} |P_s - Q_s|^2.
\]

Hence,

\[
E \sup_{s \in [0, T]} |P_s - Q_s|^2 \leq 2(\tilde{\beta}^2 A + \tilde{\mu} + K) \int_0^T E \sup_{r \in [0, s]} |P_r - Q_r|^2 \, ds
+ 2(\tilde{\mu} + K + K^2 + 2K_1) \int_0^T |P_t - Z_t|^2 \, dt
\leq 2(\tilde{\beta}^2 A + \tilde{\mu} + K) \int_0^T E \sup_{r \in [0, s]} |P_r - Q_r|^2 \, ds
+ 4(\tilde{\mu} + K + K^2 + 2K_1) \int_0^T (|Q_t - Z_t|^2 + |P_t - Q_t|^2) \, dt
\leq 2(\tilde{\beta}^2 A + 3\tilde{\mu} + 3K + 2K^2 + 4K_1) \int_0^T E \sup_{r \in [0, s]} |P_r - Q_r|^2 \, ds
+ 4(\tilde{\mu} + K + K^2 + 2K_1) \int_0^T |Q_t - Z_t|^2 \, dt.
\]
Applying Lemma 3.2 we obtain a bound of the form

\[ E \sup_{s \in [0,T]} |P_s - Q_s|^2 \leq D_1 \Delta t + D_2 \int_0^T E \sup_{r \in [0,s]} |P_r - Q_r|^2 \, ds, \]

where \( D_1 = 4(\mu + K + K^2 + 2K_1)TC_2 \) \( \sup_{r \in [0,T]} E|Q_s|^2 \), and \( D_2 = 2(\beta^2 A + 3\mu + 3K + 2K^2 + 4K_1) \). By applying the Gronwall inequality, we have the following inequality:

\[ E \left( \sup_{s \in [0,T]} |P_s - Q_s|^2 \right) \leq D_1 \Delta t \exp(D_2 T). \]

By Lemma 3.1, (3.5) is obtained. The proof is proved. \( \square \)

**Lemma 3.4.** Under assumptions (i)–(iii), the trivial solution of Eqs. (2.2)–(2.6) is exponentially stable in mean square. That is, there is a pair of positive constants \( \lambda \) and \( M \) such that, for any \( P_0 \)

\[ E|P(t)|^2 \leq ME|P_0|^2 e^{-\lambda t} \quad \forall t \geq 0. \]  

(3.8)

The proof of this lemma is analogous to that of Theorem in [3].

Now we are in a position to prove the main result: Theorem 3.5

**Theorem 3.5.** Under assumptions (i)–(iii), the Euler method applied to Eqs. (2.2)–(2.6) is exponentially stable in mean square.

The proof of this theorem is analogous to that of Theorem 2.2 in [13].

4. An example

Consider the following stochastic age-structured population system with diffusion

\[
\begin{align*}
\frac{\partial P}{\partial t} + \frac{\partial P}{\partial r} - r^2 \Delta P + x \frac{1}{(1-r)^2} P = & -txP + P \, dBi \quad \text{in} \ (0, 1) \times (0, T) \times (0, 1), \\
P(0, t, x) = & \int_0^1 \frac{x}{(1-r)^2} P(r, t, x) \, dr \quad \text{in} \ (0, T) \times (0, 1), \\
P(r, 0, x) = & \exp\left(-\frac{1}{1-r}\right) \quad \text{in} \ (0, 1) \times (0.1), \\
P(r, t, x) = & 0 \quad \text{on} \ \Sigma = (0, 1) \times (0, T) \times \{0, 1\}, \\
y(t, x) = & \int_0^1 P(r, t, x) \, dr \quad \text{in} \ Q.
\end{align*}
\]

Here \( B_i \) is a real standard Brownian motion (so, \( M = R \) and \( W = 1 \)). We can set this problem in our formulation by taking \( H = L^2(\{0, 1\} \times [0, 1]) \), \( V = W_0^1(\{0, 1\} \times [0, 1]) \) (a Sobolev space with elements satisfying the boundary conditions above), \( M = R \), \( k(t, x) = x^2 \), \( \mu(r, t, x) = k(r, t, x) = \frac{x^2}{(1-r)^2} \), \( f(r, t, x, P) = -txP \), and \( g(r, t, P) = P, \)

\( P(r, 0, x) = \exp\left(-\frac{1}{1-r}\right) \).

Clearly, the operators \( f \) and \( g \) satisfy conditions (i) and (ii), \( k(t, x) \) and \( \mu(r, t, x) \) and \( \beta(r, t, x) \) satisfy condition (iii). Consequently, the approximate solution will converge to the true solution of (4.1)–(4.5) for any \( (r, t, x) \in (0, 1) \times (0, T) \times (0, 1) \) in the sense of Theorem 3.5.

Take \( T = 1, \Delta t = 0.005, \Delta x = 0.05 \). In (Fig. 1), the upper in the left is the explicit solution to Eqs. (4.1)–(4.5) without perturbation, that is \( P(r, t, 1/2) = \exp\left(-\frac{1}{1-r} - \frac{t^2}{4}\right) \). The other three pictures are numerical simulations of the stochastic age-dependent population equations with 1000 and 10 000 experiments respectively, where \( E\sum_{k=1}^n Q_k(r, t, x) \). It clearly reveals the fact that the numerical approximation will tend to the true solution in the mean sense.
Fig. 1. Numerical simulations of stochastic population equation.

Fig. 2. Error simulation of stochastic population equation.
It is difficult to obtain the true explicit solution to Eqs. (4.1)–(4.5), so the explicit solution $P(r,t,1/2)$ to Eqs. (4.1)–(4.5) can be replaced by $\exp\left(-\frac{r^2}{1 + \Delta B_t}\right)$. Fig. 2 shows the computer simulation for the difference between $\exp\left(-\frac{r^2}{1 + \Delta B_t}\right) (1 + \Delta B_t)$ and the Euler approximation solution $Q(r,t,1/2)$. The maximum value of the error square is not greater than 0.05. Clearly the numerical approximation will tend to the true solution in the mean square sense.

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