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On maximal primitive quotients of infinitesimal Cherednik algebras of $\mathfrak{gl}_n$

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ABSTRACT

We prove analogues of some of Kostant's theorems for infinitesimal Cherednik algebras of $\mathfrak{gl}_n$. As a consequence, it follows that in positive characteristic the Azumaya and smooth loci of the center of these algebras coincide.

1. Introduction

Infinitesimal Cherednik algebras (more generally, infinitesimal Hecke algebras) were introduced by Etingof, Gan and Ginzburg [EGG]. Here we will be concerned with infinitesimal Cherednik algebras of $\mathfrak{gl}_n$. Let us recall the definition. Let $\mathfrak{h} = \mathbb{C}^n$ denote the standard representation of $\mathfrak{g} = \mathfrak{gl}_n$. Denote by y_i the standard basis elements of $\mathfrak{h}$, and by x_i the dual basis of $\mathfrak{h}^*$. For the given deformation parameter $b = b_0 + b_1\tau + \cdots + b_m\tau^m \in \mathbb{C}[\tau]$, $b_m \neq 0$, $m \geqslant 0$, one defines the infinitesimal Cherednik algebra of $\mathfrak{gl}_n$ with parameter b, to be denoted by H_b, as the quotient of the semi-direct product $\mathfrak{U}\mathfrak{g} \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x, x'] = 0, \qquad [y, y'] = 0, \qquad [y, x] = b_0 r_0(x, y) + b_1 r_1(x, y) + \cdots + b_m r_m(x, y),$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$, and $r_i(x, y) \in \mathfrak{U}\mathfrak{g}$ are the symmetrizations of the following functions on $\mathfrak{g}$ (thought of as elements in $\operatorname{Sym}\mathfrak{g}$ via the trace pairing):

$$\left(x, (1 - tA)^{-1} y\right) \det(1 - tA)^{-1} = r_0(x, y)(A) + r_1(x, y)(A)t + r_2(x, y)(A)t^2 + \cdots.$$

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The algebras H_b have the following PBW property. If we introduce the filtration on H_b by setting $\deg x = \deg y = 1$, $x \in \mathfrak{h}^*$, $y \in \mathfrak{h}$, $\deg g = 0$, $g \in \mathfrak{g}$, then the natural map: $\mathfrak{U}\mathfrak{g} \ltimes \mathrm{Sym}(\mathfrak{h} \oplus \mathfrak{h}^*) \to \mathrm{gr}\, H_b$ is an isomorphism.

The enveloping algebra $\mathfrak{U}(\mathfrak{sl}_{n+1})$ is an example of H_b for $m = 1$ (Example 4.7 [EGG]). In fact, the algebras H_b have many properties similar to the enveloping algebras of simple Lie algebras. We will introduce a Poisson variety (for each $m := \deg b$) which can be thought of as an analogue of the nilpotent cone of a semi-simple Lie algebra. Our first result shows that it is an irreducible reduced normal variety (Theorem 2.1), an analogue of Kostant's classical result. As an application, we will describe annihilators of Verma modules of H_b, and show that in positive characteristic the Azumaya locus of H_b coincides with the smooth locus of its center.

2. The main results

Since $H_b \simeq H_{ab}$ for any $a \in \mathbb{C}^*$, we will assume from now on that b is monic: $b_m = 1$.

Besides the natural action of $G = GL_n(\mathbb{C})$ on H_b, we also have the action of $\mathfrak{h}$ and $\mathfrak{h}^*$ defined as follows. For any $v \in \mathfrak{h}$, the adjoint action $\mathrm{ad}(v)$ is locally nilpotent on H_b. Thus $\exp(\mathrm{ad}(v))$ gives an automorphism of H_b, and in this way $\mathfrak{h}$ acts on H_b. The action of $\mathfrak{h}^*$ on H_b is defined similarly. Combining these actions with the G-action, we get the actions of $G \ltimes \mathfrak{h}$, $G \ltimes \mathfrak{h}^*$ on H_b.

Let $Q_1, \dots, Q_n \in k[\mathfrak{g}]^G$ be defined as follows:

$$\det(t\,\mathrm{Id} - X) = \sum_{j=0}^{n} (-1)^j t^{n-j} Q_j(X).$$

Also let $\alpha_1, \dots, \alpha_n$ be the corresponding elements of $\mathbf{Z}(\mathfrak{U}\mathfrak{g})$ under the symmetrization identification of $\mathbb{C}[\mathfrak{g}]^G$ and $\mathbf{Z}(\mathfrak{U}\mathfrak{g})$. It was shown in [T1] that the following elements generate the center of H_b

$$t_i = \sum_j [\alpha_i, y_j] x_j - c_i = \sum_j y_j [x_j, \alpha_i] - c_i \in \mathbf{Z}(H_b),$$

where $c_i \in \mathbf{Z}(\mathfrak{U}\mathfrak{g})$ are certain elements. The top symbols of c_i are given as follows. Let us consider the following element of $\mathbb{C}[\mathfrak{g}][t, \tau]$ given by

$$c' = \frac{\det(t - A)}{(t\tau - 1)\det(1 - \tau A)}$$

then the top symbol of c_i considered as an element of $\mathbb{C}[\mathfrak{g}]$ is the coefficient of $t^{n-i}\tau^m$ in c'. We have that $\mathbf{Z}(H_b) = \mathbb{C}[t_1, \dots, t_n]$. For a character $\chi : \mathbb{C}[t_1, \dots, t_n] \to \mathbb{C}$, denote by $U_{b,\chi}$ the quotient $H_b / \ker(\chi) H_b$.

From now on we will assume that $m \geqslant 1$. Let us introduce a new filtration on H_b by setting $\deg x_i = m$, $\deg y_i = 1$, $\deg g = 1$, $g \in \mathfrak{g}$. Then, $\mathrm{gr}\, H_b = \mathrm{Sym}(\mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*)$ is a Poisson algebra (and the Poisson bracket depends only on m). We will denote it by A_m. Denote $B_m = \mathrm{gr}\, H_b / (\mathrm{gr}\, t_1, \dots, \mathrm{gr}\, t_n)$. Again, B_m is a Poisson algebra. Variety $\mathrm{Spec}\, B_1$ is the nilpotent cone of $\mathfrak{sl}_{n+1}(\mathbb{C})$. The main result of this paper is the following analogue of some of Kostant's theorems for semi-simple Lie algebras [K].

Theorem 2.1. *The algebra H_b is a free module over its center. B_m is an integral domain, which is a normal, complete intersection ring. Moreover, the smooth locus of* $\mathrm{Spec}\, B_m$ *under the Poisson bracket is symplectic.*

Proof. We will partially follow [BL]. Denote by f_x (resp. f_y) the element $\det(\{\alpha_i, x_j\})_{i,j} \in B_m$ (resp. $\det(\{\alpha_i, y_j\})_{ij} \in B_m$). Then the localization $(B_m)_{f_x}$ is isomorphic to the localized polynomial algebra $\mathrm{Sym}(\mathfrak{g} \oplus \mathfrak{h})_{f_x}$. A similar statement holds for $(B_m)_{f_y}$. We will use the notation $D(f) = \mathrm{Spec}(B_m)_f \subset \mathrm{Spec}\, B_m$, $f \in B_m$. Let us set $U = D(f_x) \cup D(f_y)$. To show that $X = \mathrm{Spec}\, B_m$ is an irreducible, reduced and normal variety, it is enough to show that it is Cohen–Macaulay, U is connected, and $\dim(X \setminus U) \leqslant \dim X - 2$ [BL, Corollary 2.3].

We have an action of the affine group $G \ltimes \mathfrak{h}$ on $\operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h})$. Then f_x is a semi-invariant of this action, i.e., $(g, v) f_x = \det(g) f_x$, $g \in G$, $v \in \mathfrak{h}$. As explained in [R1, Théorème 3.8], the set $D(f_x) \subset \operatorname{Spec}\operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h})$ is the dense orbit under the action of $G \ltimes \mathfrak{h}$ on $\operatorname{Spec}\operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h})$. In fact, this set consists of pairs (A, v) with $A \in \mathfrak{g}$, $v \in \mathfrak{h}$, such that $v, Av, \ldots, A^{n-1}v$, are linearly independent. We have a similar statement about $D(f_y)$, and the action of $G \ltimes \mathfrak{h}^*$ on $\operatorname{Sym}(\mathfrak{g} \oplus \mathfrak{h}^*)$.

It was shown in [T1] that the algebra $\mathbb{C}[\alpha_1, \ldots, \alpha_n]$ is finite over $\mathbb{C}[c_1, \ldots, c_n]$. In particular, $\mathbb{C}[c_1, \ldots, c_n]$ is isomorphic to the polynomial algebra in n variables. Hence, $\mathbb{C}[\alpha_1, \ldots, \alpha_n]$ being a Cohen–Macaulay algebra, it must be a finitely generated projective module over $\mathbb{C}[c_1, \ldots, c_n]$. Therefore, by the Quillen–Suslin theorem, $\mathbb{C}[\alpha_1, \ldots, \alpha_n]$ is a finitely generated free module over $\mathbb{C}[c_1, \ldots, c_n]$.

Let us introduce a filtration on A_m, where $\deg g = 1$, $g \in \mathfrak{g}$, $\deg x_i = \deg y_j = 0$. Since $\operatorname{Sym}\mathfrak{g}$ is a free $\mathbb{C}[\alpha_1, \ldots, \alpha_n]$-module (by Kostant's theorem for $\mathfrak{g}$ [BL]), we conclude that $\operatorname{Sym}\mathfrak{g}$ is a free $\mathbb{C}[c_1, \ldots, c_n]$-module. This implies that $(t_1, \ldots, t_n)$ is a regular sequence (since $\operatorname{gr} c_j = \operatorname{gr} t_j$) and $\operatorname{gr} A_m$ is a free module over $\mathbb{C}[\operatorname{gr} t_1, \ldots, \operatorname{gr} t_n]$. Therefore A_m is a free module over $\mathbb{C}[\operatorname{gr} t_1, \ldots, \operatorname{gr} t_n]$, where gr refers to the first filtration on H_b. In particular, H_b is a free $\mathbf{Z}(H_b)$-module. Also, we obtain that B_m is a complete intersection ring. In particular, $X = \operatorname{Spec} B_m$ is a Cohen–Macaulay variety.

Let us put $Y = X \setminus U$. The latter filtration on A_m induces the corresponding filtration on its quotient B_m. We will denote the degeneration of X (resp. Y) under this filtration by X' (resp. Y'). Then X' is given by equations $c_i = 0$, $i = 1, \ldots, n$. Similarly, Y' is given by $f_x = f_y = 0$, $c_i = 0$, $i = 1, \ldots, n$. Therefore, we get that $X' = \mathfrak{h} \times \mathfrak{h}^* \times N$ and $Y' = \mathfrak{h} \times \mathfrak{h}^* \times N \cap (f_x = 0 = f_y)$, where N denotes the nilpotent cone of $\mathfrak{g}$. We need to prove that $\dim Y \leqslant \dim X - 2$. Since $\dim Y = \dim Y'$, it is equivalent to showing that $\dim Y' \leqslant \dim X - 2 = \dim X' - 2 = \dim N + 2\dim\mathfrak{h} - 2$. Consider the projection map $p : Y' \to N$. Let $W \subset N$ denote the open subset of regular nilpotent matrices. Then, by the before mentioned result of [R1], we have $\dim p^{-1}(W) \leqslant \dim N + 2\dim\mathfrak{h} - 2$, and $p^{-1}(N \setminus W) = (N \setminus W) \times \mathfrak{h} \times \mathfrak{h}^*$, whose dimension is $\dim N + 2\dim\mathfrak{h} - 2$. This proves the desired inequality.

Denote by U' the smooth locus of X. We have $U \subset U'$. It is obvious that $D(f_x) \cap D(f_y)$ is nonempty. It is also clear that $D(f_x) \cup D(f_y)$ is in the orbit of any element of $D(f_x) \cap D(f_y)$ under the actions of $G \ltimes \mathfrak{h}, G \ltimes \mathfrak{h}^*$. Since both $G \ltimes \mathfrak{h}, G \ltimes \mathfrak{h}^*$ are connected algebraic groups preserving the Poisson structure of X, it follows that U lies in a single symplectic leaf of U'. Therefore, U is a symplectic variety and since its complement in U' has the codimension $\geqslant 2$, it follows that U' is also symplectic. $\square$

We will use the following standard simple

Lemma 2.1. *Let A be a nonnegatively filtered $\mathbf{k}$-algebra (where $\mathbf{k}$ is a field) such that $\operatorname{gr} A$ is commutative. Suppose that $z_1, \ldots, z_n \in \mathbf{Z}(A)$ are central elements such that $\operatorname{gr} z_1, \ldots, \operatorname{gr} z_n$ is a regular sequence in $\operatorname{gr} A$. Then $\operatorname{gr}(A/(z_1, \ldots, z_n)) = \operatorname{gr} A/(\operatorname{gr} z_1, \ldots, \operatorname{gr} z_n)$.*

Proof. We need to show that $\operatorname{gr}(\sum_i x_i z_i) \in (\operatorname{gr} z_1, \ldots, \operatorname{gr} z_n)$ for all $x_i \in A$. We may assume that $\sum \operatorname{gr} x_i \operatorname{gr} z_i = 0$. We proceed by the induction on $\sum \deg(x_i)$. It follows from the regularity of the sequence $(\operatorname{gr} z_1, \ldots, \operatorname{gr} z_n)$ that there exist $a_1, \ldots, a_n \in A$, such that $\operatorname{gr} a_i = \operatorname{gr} x_i$, $1 \leqslant i \leqslant n$ and $\sum_i a_i z_i = 0$. Now replacing x_i by $x_i - a_i$, we are done by the inductive assumption. $\square$

As a consequence of the proof of Theorem 2.1 and Lemma 2.1, we get that $\operatorname{gr} U_{b,\chi} = B_m$ is a domain, so $U_{b,\chi}$ is also a domain.

In analogy with semi-simple Lie algebras, one defines an analogue of the category $\mathcal{O}$, and Verma modules for H_b [T1]. Let us recall their definition. Denote by n_+ (resp. n_-) the Lie subalgebra of $\mathfrak{g}$ consisting of upper (resp. lower) triangular matrices. Then we have a triangular decomposition $H_b = H_- \otimes \mathfrak{U}(C) \otimes H_+$, where H_+ (resp. H_-) denotes the subalgebra of H_b generated by n_+ and $\mathfrak{h}$ (resp. n_- and $\mathfrak{h}^*$), and $C \subset \mathfrak{g}$ is the Cartan subalgebra of all diagonal matrices. Denote by L_+ (resp. L_-) the Lie algebra $n_+ \ltimes \mathfrak{h}$ (resp. $n_- \ltimes \mathfrak{h}^*$). Thus, H_+ (resp. H_-) is the enveloping algebra $\mathfrak{U}L_+$ (resp. $\mathfrak{U}L_-$).

For a weight $\lambda \in C^*$, the corresponding Verma module $M(\lambda)$ is defined as $H_b \otimes_{\mathfrak{U}(C) \otimes H_+} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the 1-dimensional representation of $\mathfrak{U}(C) \otimes H_+$ on which C acts by λ and L_+ acts by 0.

The category $\mathcal{O}$ (analogue of the BGG category $\mathcal{O}$ of semi-simple Lie algebras) is defined as the full subcategory of the category of finitely generated left H_b-modules whose objects are modules on which C acts semi-simply and L_+ acts locally nilpotently.

We have the following analogue of a theorem of Duflo [D].

Theorem 2.2. *The annihilator of a Verma module $M(\lambda)$ is generated by* $\mathrm{Ann}(M(\lambda)) \cap \mathbf{Z}(H_b)$.

Proof. The following lemma and its proof is directly analogous to [J, Corollary 2.8]. We present it for the completeness sake. In what follows $GK(-)$ denotes the Gelfand–Kirillov dimension.

Lemma 2.2. $GK(H_b/Ann(M(\lambda))) = 2GK(M(\lambda))$.

Proof. At first, we will show that $GK(H_b/AnnL(\lambda)) = 2GK(L(\lambda))$, where $L(\lambda)$ is the simple module with the highest weight λ. Since L_- acts locally nilpotently on H_b, we get an imbedding $H_b/AnnL(\lambda) \to D(L_-, L(\lambda))$ where $D(L_-, L(\lambda))$ is the subalgebra of $End_{\mathbb{C}}(L(\lambda))$ consisting of elements annihilated by some power of $\mathrm{ad}(L_-)$. According to [J, Lemma 2.6] $GK(D(L_-, L(\lambda))) \leqslant 2GK(L(\lambda))$, thus $GK(H_b/AnnL(\lambda)) \leqslant 2GK(L(\lambda))$. Let v_λ be a maximal weight vector of $L(\lambda)$. Thus $L(\lambda) = H_- v_\lambda$. Let us choose $\delta \in C$ so that $\mathrm{ad}(\delta)$ has positive (negative) eigenvalues on $L_+(L_-)$. For $a \in \mathbb{C}$, and H_b-module M, we will denote by $M_a \subset M$ the space of eigenvectors of δ with the eigenvalue a. In particular, $L(\lambda)_{\lambda(\delta)} = \mathbb{C}v_\lambda$, and $L(\lambda) = \bigoplus_{l \geqslant 0} L(\lambda)_{\lambda(\delta)-l}$. Then for any $a \in L(\lambda)_\mu$, there is an element $c \in H_b$, such that $ca = v_\lambda$. But since λ is the maximal weight of $L(\lambda)$, using the triangular decomposition $H = H_- \otimes \mathfrak{U}(C) \otimes H_+$ we may choose $c \in H_+$. Thus, for any $a \in L(\lambda)_\mu$, $b \in L(\lambda)_{\mu'}$ there exist $\alpha \in (H_+)_{\lambda(\delta)-\mu}$, $\alpha' \in (H_-)_{\mu'-\lambda(\delta)}$ such that $b = \alpha'\alpha a$. Let us denote by ρ the quotient map $H_b \to H_b/AnnL(\lambda)$. Thus, $\dim Hom_{\mathbb{C}}(L(\lambda)_\mu, L(\lambda)_{\mu'}) \leqslant \dim \rho((H_-)_{\lambda(\delta)-\mu}(H_+)_{\mu'-\lambda(\delta)})$. Let $F_l = \sum_{i \leqslant l}(\mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*)^i \subset H_b$, $l \geqslant 0$. Then it follows that there is a positive integer $k > 0$ such that $F_l v_\lambda \subset \sum_{i \leqslant kl} L(\lambda)_{(\lambda(\delta)-i)}$ and $(H_-)_{-l} \subset F_{kl} \cap H_-$, $(H_+)_l \subset F_{kl} \cap H_+$, for all $l > 0$. Now it follows that $\dim \rho(F_{2kl}) \geqslant (\dim F_{l/k} v_\lambda)^2$. This implies that $GK(H_b/AnnL(\lambda)) \geqslant 2GK(L(\lambda))$, and so $GK(H_b/AnnL(\lambda)) = 2GK(L(\lambda))$.

Suppose that $L(\lambda_i)$, $i = 1, \dots, l$ are the elements of the Jordan–Holder series of $M(\lambda)$ ($M(\lambda)$ has a finite length [T1, Theorem 4.1]). Then,

$$\begin{aligned} GK\big(H_b/AnnM(\lambda)\big) &= Max_i\big\{GK\big(H_b/AnnL(\lambda_i)\big)\big\} \\ &= 2Max_i\big\{GK\big(L(\lambda_i)\big)\big\} = 2GK\big(M(\lambda)\big). \qquad \square \end{aligned}$$

Now since $H_b/\mathrm{Ann}(M(\lambda)) \cap \mathbf{Z}(H_b)$ is a domain and its quotient $H_b/(AnnM(\lambda))$ has the same GK-dimension as $2GK(M(\lambda)) = GK(H_b/\mathrm{Ann}(M(\lambda)) \cap \mathbf{Z}(H_b))$, we conclude using [BK, 3.5] that

$$\mathrm{Ann}\big(M(\lambda)\big) = \big(\mathrm{Ann}\big(M(\lambda)\big) \cap \mathbf{Z}(H_b)\big)H_b. \qquad \square$$

This implies that maximal primitive quotients of H_b are precisely algebras $U_{b,\chi}$, $\chi \in \mathrm{Spec}\,\mathbf{Z}(H_b)$. Indeed, by [T1] every primitive quotient of H_b has the form $H_b/\mathrm{Ann}\,L(\lambda), \lambda \in C^*$. Let $M(\lambda')$ be an irreducible Verma module which belongs to the same block of the BGG category $\mathcal{O}$ as $L(\lambda)$. Thus $\mathrm{Ann}(M(\lambda')) \cap \mathbf{Z}(H_b) = \mathrm{Ann}(L(\lambda)) \cap \mathbf{Z}(H_b)$. Therefore, using Theorem 2.2, we conclude that $\mathrm{Ann}(M(\lambda')) \subset \mathrm{Ann}(L(\lambda))$. Thus $H_b/\mathrm{Ann}(L(\lambda))$ is a quotient of $U_{b,\chi}$, where χ is the character of $\mathbf{Z}(H_b)$ corresponding to $M(\lambda')$.

3. The Azumaya locus

Let us discuss the case of a field $\mathbf{k} = \bar{\mathbf{k}}$ of positive characteristic. As before, let $b \in \mathbf{k}[\tau]$, $\deg b = m > 1$, be a monic polynomial. If p is large enough (with respect to m) then the definition of H_b over $\mathbf{k}$ makes sense. One checks easily that $\mathfrak{h}^p, \mathfrak{h}^{*p}, g^p - g^{[p]} \in \mathbf{Z}(H_b)$, $g \in \mathfrak{g}$, where $g^{[p]} \in \mathfrak{g}$ denotes the p-th power of g as a matrix [T1]. We will denote by $\mathbf{Z}_0(H_b)$ the algebra generated by the above elements. Also, for $p \gg 0$ central elements $t_1, \dots, t_n \in \mathbf{Z}(H_b)$ are defined.

We have the following result which was conjectured in [T1].

Theorem 3.1. *The smooth and Azumaya loci of $\mathbf{Z}(H_b)$ coincide, and $\mathbf{Z}(H_b)$ is generated by $t_1, \dots, t_n$ over $\mathbf{Z}_0(H_b)$. The PI-degree of H_b is $p^{\frac{1}{2}(n^2+n)}$.*

Proof. Algebra B_m can be defined over $\mathbb{Z}[\frac{1}{d!}] = R$ for large enough d. Call this algebra $\tilde{B}_m$. Thus, $B_m = \tilde{B}_m \otimes_R \mathbb{C}$. Since by Theorem 2.1 $\operatorname{Spec} \tilde{B}_m \otimes_R \mathbb{C}$ is an irreducible normal Poisson variety whose regular locus is symplectic, it follows that for large enough $p = \mathit{char}\,\mathbf{k}$, a similar statement holds for $\bar{B}_m = \tilde{B}_m \otimes_R \mathbf{k}$. Since $\operatorname{gr} H_b/(t_1 - a_1, \dots, t_n - a_n) = \bar{B}_m$, $a_1, \dots, a_n \in \mathbf{k}$ (by Lemma 2.1), the claim now follows from [T2, Theorem 2.3] and the following simple lemma. $\square$

Lemma 3.1. *Let S be an affine Poisson algebra over $\mathbf{k}$, and let $(f_1, \dots, f_n)$ be a regular sequence of Poisson central elements. Let $S/(f_1, \dots, f_n)$ be a normal domain such that its smooth locus is symplectic. Then the Poisson center of S is generated as an algebra by $S^p, f_1, \dots, f_n$.*

Proof. Let us denote the ideal $(f_1, \dots, f_n)$ by I. It follows immediately that the Poisson center of S lies in $S^p + I$ [T2, proof of Lemma 2.4]. Let $f \notin S^p[f_1, \dots, f_n]$ be in the Poisson center of S. Then there is k such that $f \in (S^p[f_1, \dots, f_n] + I^k) \setminus (S^p[f_1, \dots, f_n] + I^{k+1})$. Let us write $f = g + h$, where $g \in S^p[f_1, \dots, f_n]$, $h \in I^k \setminus (S^p[f_1, \dots, f_n] + I^{k+1})$. But I^k/I^{k+1} is a free Poisson S/I-module. Indeed, since $f_1, \dots, f_n$ is a regular sequence, it follows that I^k/I^{k+1} is a free S/I-module with the basis $f_1^{m_1} f_2^{m_2} \cdots f_n^{m_n}$, $\sum_{l=1}^n m_l = k$. Since $f_1, \dots, f_n$ are Poisson central elements, it follows that I^k/I^{k+1} is a free Poisson S/I-module with the basis consisting of monomials $f_1^{m_1} \cdots f_n^{m_n}$. Thus, $I^k/I^{k+1} = \bigoplus_{m_1, \dots, m_n} (S/I) f_1^{m_1} \cdots f_n^{m_n}$. Let us denote by $\bar{h}$ the image of h in I^k/I^{k+1}. Let us write $\bar{h} = \sum a_{m_1, \dots, m_n} f_1^{m_1} \cdots f_n^{m_n}$, $a_{m_1, \dots, m_n} \in S/I$. Since $\{S, h\} = 0$, it follows that $a_{m_1, \dots, m_n} \in (S/I)^p$ (since the Poisson center of S/I is $(S/I)^p$). Therefore, the image of h in I^k/I^{k+1} must lie in $S^p[f_1, \dots, f_n]/I^{k+1}$, a contradiction. $\square$

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