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Virasoro irregular conformal block and beta deformed random matrix model

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A B S T R A C T

Virasoro irregular conformal block is presented as the expectation value of Jack-polynomials of the beta-deformed Penner-type matrix model and is compared with the inner product of Gaiotto states with arbitrary rank. It is confirmed that there are non-trivial modifications of the Gaiotto states due to the normalization of the states. The relation between the two is explicitly checked for rank 2 irregular conformal block.

1. Introduction

Virasoro irregular module appears in connection with the $N = 2$ super-Yang–Mills theory [1]. The irregular module so-called Gaiotto state or Whittaker state [2] is the simultaneous eigenstate of the positive Virasoro generators. The irregular module is constructed as the superposition of one primary state and its descendents [1,3].

On the other hand, the irregular module is also constructed as the colliding limit of primary operators as shown in [4]. The colliding limit is the fusion of primary vertex operators with the addition of Heisenberg-coherent modes. As a result, the state becomes the simultaneous eigenstate of positive Virasoro operators, *i.e.* the irregular module.

Will the two different approaches produce the same result? In this paper we like to answer this question. We will confine ourselves to the case with the Gaiotto sate $|I_n\rangle$ of rank $n \geq 1$, simultaneous eigenstate of $L_n, L_{n+1}, \cdots, L_{2n}$. In Section 2, Gaiotto state of rank n constructed in [5] is summarized and its inner product is investigated. The inner product is important since it contains all the information of descendents in the Gaiotto state. In Section 3, a differently looking form of the inner product is provided using the colliding limit of the regular conformal correlation. The result is given in terms of the beta-deformed Penner-type matrix model. Since the random matrix model is the result of the fusion of primary operators, the partition function should produce the colliding limit of the conformal block, which we call the (two-point) irregular conformal block (ICB). A simple and clear way to obtain ICB is presented with the help of the loop equation and ICB is compared with the inner product of Gaiotto states. We pinpoint the non-trivial modification from the Gaiotto state in [5]. The summary and discussion are given in Section 4.

2. Virasoro irregular module and its inner product

The irregular state is explicitly constructed for rank 1 in [1,3] and for rank n in [5]. We will use the convention $|\widetilde{G_{2n}}\rangle$ for Gaiotto state with rank n following [5] (another form is also found in [6]), whereas we reserve $|I_n\rangle$ for the state obtained from the colliding limit given in [4]

$$|\widetilde{G_{2n}}\rangle = \sum_{\ell, Y, \ell_p} \Lambda^{\ell/n} \left\{ \prod_{i=1}^{n-1} a_i^{\ell_{2n-i}} b_i^{\ell_i} \right\} m^{\ell_n} Q_\Delta^{-1} \times \left(1^{\ell_1} 2^{\ell_2} \cdots (2n-1)^{\ell_{2n-1}} (2n)^{\ell_{2n}}; Y\right) L_{-Y} |\Delta\rangle, \qquad (2.1)$$

where $L_{-Y} = L_Y^+$ represents the product of lowering operators and $L_Y = L_1^{\ell_1} L_2^{\ell_2} \cdots L_s^{\ell_s}$. $|\Delta\rangle$ is the primary state with conformal dimension Δ and $Q_\Delta(Y; Y')$ is the shorthand notation of $\langle\Delta|L_{Y'} L_{-Y}|\Delta\rangle$. The summation ℓ runs from 0 to ∞, Y and ℓ_p maintaining $|Y| = \ell$ and $\sum p\ell_p = \ell$.

One can confirm that $|\widetilde{G_{2n}}\rangle$ is the simultaneous eigenstate; $L_k|\widetilde{G_{2n}}\rangle = \Lambda^{k/n} a_{2n-s}$ for $n < k \leq 2n$ and $L_n|\widetilde{G_{2n}}\rangle = \Lambda m|\widetilde{G_{2n}}\rangle$ from the expectation values for $W = 1^{\ell_1} 2^{\ell_2} \cdots (2n)^{\ell_{2n}}$,

$$\langle\Delta|L_W L_{2n-s}|\widetilde{G_{2n}}\rangle = \Lambda^{2n-s/n} a_s \langle\Delta|L_W|\widetilde{G_{2n}}\rangle \quad \text{for } 0 \leq s < n,$$
$$\langle\Delta|L_W L_n|\widetilde{G_{2n}}\rangle = \Lambda m \langle\Delta|L_W|\widetilde{G_{2n}}\rangle, \qquad (2.2)$$

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with $a_0 \equiv 1$. Here, the eigenvalues are given in terms of Λ, a_i's and m only. The other coefficients b_i's are not fixed by the eigenvalues but enter in the inner product since

$$\langle \Delta | L_W | \widetilde{G_{2n}} \rangle = \Lambda^{\ell/n} \left\{ \prod_{i=1}^{n-1} a_i^{\ell_{2n-i}} b_i^{\ell_i} \right\} m^{\ell_n}. \tag{2.3}$$

Note that the inner product contains all the information on the descendents. Thus, one may assume that b_i's are related with the contribution of descendents. To find out further information of b_i's, we need to resort to other procedures.

3. Irregular conformal block and colliding limit

The inner product can be evaluated using the idea of colliding limit of the multi-point regular conformal correlation introduced in [7,4,6]. We follow the procedure appeared in [9]. Let us consider the conformal part of $n+2$ primary operator correlation with N screening operators. If one fuses $n+1$ operators at the origin with the colliding limit, one ends up with the β-deformed Penner-type partition function

$$Z_{(0:n)}\big(c_0; \{c_k\}\big) = \int \prod_{i=1}^{N} d\lambda_i \Delta(\lambda)^{2\beta} e^{-\frac{\sqrt{\beta}}{g} \sum_i V(\lambda_i; c_0, \{c_k\})}, \tag{3.1}$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$ is the Vandermonde determinant and $\beta = -b^2$ (or $b = i\sqrt{\beta}$) with the screening charge b. The Penner-type potential is given as the sum of logarithmic and inverse power terms

$$\frac{1}{\hbar} V_{(0:n)}\big(z; c_0, \{c_k\}\big) = -c_0 \log z + \sum_{k=1}^{n} \frac{c_k}{k z^k}. \tag{3.2}$$

(One may identify $c_k = \sum_{r=1}^{n} \alpha_r (z_r)^k$ where α_r is the Liouville charge of the primary operator at z_r. Since the colliding limit corresponds to $z_r \to 0$ and $\alpha_r \to \infty$ so that c_k is ensured finite, one may consider the limit as the ideal multi-pole expansion. In addition, we use the notation $g = i\hbar/2$ so that $\sqrt{\beta}/g = -2b/\hbar$.)

We remark by passing that the integration range of the partition function is naturally given as 0 to ∞. Before the colliding limit one usually chooses the integration range between the positions of the primary operators. For example, one may choose the position of the primary operators as $(0,\ z_1 = z,\ z_2 = 1,\ \infty)$ and chooses the integration ranges from 0 to z_1 or from z_2 to ∞. However, to have the proper colliding limit, one needs to choose the integration range from z_1 to z_2 and take the limit $z_1 \to 0$ and $z_2 \to \infty$.

Let us introduce the primary state $|\Delta\rangle$ with conformal dimension $\Delta = c_0(Q - c_0)$ in the presence of the background charge Q. Then $\langle \Delta|$ is the primary state with the conformal dimension $\Delta = c_\infty(Q - c_\infty)$ where c_∞ is fixed by the neutrality condition $c_0 + c_\infty + bN = Q$. The colliding limit introduces the irregular state $|I_n\rangle$ and the partition function is identified with the inner product $Z_{(0:n)}(c_0; \{c_k\}) = \langle \Delta | I_n \rangle$. This ensures that the irregular state $|I_n\rangle$ is dependent on the set of coefficients $\{c_1, \cdots c_n\}$. In fact, it is demonstrated in [4] that the coefficient c_k is the coherent coordinate of Heisenberg mode a_k, $a_k |I_n\rangle = c_k |I_n\rangle$.

Since $|I_n\rangle$ is the simultaneous eigenstate of $L_n, L_{n+1}, \cdots, L_{2n}$ generators, their eigenvalues can be parametrized as $\Lambda_k = (k+1) Q c_k - \sum_{p=0}^{k} c_p c_{k-p}$ with $k = n, \cdots, 2n$. However, the eigenstate condition is not enough to fix $|I_n\rangle$ as seen in (2.3) and needs the information on the descendents in $|I_n\rangle$. Note that the lower positive generators L_k $(k = 1, \cdots, n-1)$ obeying $[L_k, L_n] = (k-n) L_{k+n}$. An easy way to realize this non-commutative properties is to represent L_k as the differential form of the coherent coordinates c_k's.

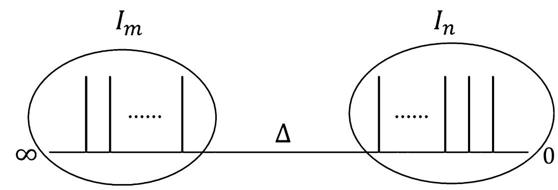

Fig. 1. Schematic diagram of $\langle I_m | I_n \rangle$ from the colliding limit.

Putting $\mathcal{L}_k = \Lambda_k + v_k$, one has

$$v_k \equiv \sum_{\ell \in \mathbb{N}} \ell c_{\ell+k} \frac{\partial}{\partial c_\ell} \tag{3.3}$$

and the consistency condition

$$[v_k, v_\ell] \langle \Delta | I_n \rangle = (\ell - k) v_{\ell+k} \langle \Delta | I_n \rangle. \tag{3.4}$$

It should be noted that the Gaiotto state $|\widetilde{G_{2n}}\rangle$ in (2.1) satisfies the consistence condition trivially since $v_k \langle \Delta | \widetilde{G_{2n}} \rangle = 0$.

One can find the parameter dependence for rank 1 simply by scaling the integration variable $\lambda_i \to c_1 \lambda_i$ to get $Z_{(0:1)}(c_0; c_1) = c_1^{-bN(bN+2c_0-Q)} Z_{(0:1)}(c_0; 1)$. However, for the rank higher than 1, one needs more complicated process. The easiest way to find the parameter dependence is to use the loop equation of the matrix model [8]. The loop equation has the form [9]

$$\sum_{k=0}^{n-1} \frac{v_k(\log(Z_{(0:n)}))}{z^{2+k}} = -\frac{\xi(z)}{\hbar^2}, \tag{3.5}$$

where v_0 conforms to the notation of (3.3), $v_0 \equiv \sum_{\ell \in \mathbb{N}} \ell c_\ell \frac{\partial}{\partial c_\ell}$ and $\xi(z) = 4W(z)^2 - 4W(z)V'(z) + 2\hbar Q\, W'(z) - \hbar^2 W(z,z)$. Here $W(z)$ is the resolvent $W(z) = \hbar b/2 \langle 1/(z - \lambda_i) \rangle$, and $W(z,z)$ is the connected two-point resolvent $W(z,z) = -b^2 \langle \sum_{i,j} 1/(z-\lambda_i)(z-\lambda_j) \rangle_c$. The prime stands for the differentiation.

One may view that the loop equation provides the energy momentum expectation value $\varphi_2(z)$, which encodes the Seiberg–Witten curve [10–12,1]. Putting $\varphi_2(z) = \sum_{n \le k \le 2n} \Lambda_k / z^{2+k} + \sum_{0 \le kn-1} \mathcal{L}_k / z^{2+k}$, one has the relation with the resolvent according to the loop equation: $\varphi_2(z) = (2W - V')^2 + \hbar Q (2W - V')' - \hbar^2 W(z,z)$. Large z expansion of the loop equation eventually reduces to the flow equation

$$v_k(\log Z_{(0:n)}) = d_k^{(0:n)}\big(\{c_k\}\big), \tag{3.6}$$

where $d_k^{(0:n)}$ is the moment of $\xi(z)$; $\oint dz z^{1+k} \xi(z)/(-\hbar^2 2\pi i)$. The flow equation satisfies the consistency condition (3.4) automatically whose explicit solutions can be found in [9,13].

The idea can be extended to find the inner product $\langle I_m | I_n \rangle$ from the colliding limit of $(m+n+2)$-point correlation (see Fig. 1). Fusing $n+1$ primary operators at the origin and $m+1$ operators at infinity, one has the partition function $Z_{(m:n)}$

$$Z_{(m:n)}\big(c_0; \{c_k\}; \{c_{-\ell}\}\big) = \int \prod_{i=1}^{N} d\lambda_i \Delta(\lambda)^{2\beta} e^{-\frac{\sqrt{\beta}}{g} \sum_i V_{(m:n)}(\lambda_i; c_0, \{c_k\}, \{c_{-\ell}\})}$$

$$\frac{1}{\hbar} V_{(m:n)}\big(z; c_0, \{c_k\}, \{c_{-\ell}\}\big) = -c_0 \log z + \sum_{k=1}^{n} \left(\frac{c_k}{k z^k} \right) + \sum_{\ell=1}^{m} \left(\frac{c_{-\ell} z^\ell}{\ell} \right). \tag{3.7}$$

The partition function is related with the inner product $\langle I_m | I_n \rangle$. However, there is a subtlety, so-called $U(1)$ contribution. This factor comes from the limiting procedure: It is noted that as

$z_a \to \infty$ and $z_b \to 0$ one has the finite contribution $\prod_{a,b}(1 - z_b/z_a)^{-2\alpha_a\alpha_b} \to e^{\zeta(m:n)}$, where $\zeta_{(m:n)} = \sum_k^{\min(m,n)} 2c_k c_{-k}/k$. Therefore, one has the inner product of the form $\langle I_m|I_n\rangle = e^{\zeta_{(m:n)}} Z_{(m:n)} \times (c_0; \{c_k\}; \{c_{-\ell}\})$.

The inner product between the two irregular modules inherits the property of the conformal block of the regular multi-correlation. Considering the colliding limit, one may define the irregular conformal block $\mathcal{F}_\Delta^{(m:n)}$ as the inner product of the irregular modules with appropriate normalization: $\mathcal{F}_\Delta^{(m:n)} = \langle I_m|I_n\rangle / (\langle I_m|\Delta\rangle\langle\Delta|I_n\rangle)$ whose conformal dimension is given as $\Delta = (c_0 + N_0)(Q - c_0 - N_0) = (c_\infty + N_\infty)(Q - c_\infty - N_\infty)$ [13].

In this spirit, one may naturally define ICB using the β deformed Penner-type matrix model as the following:

$$\mathcal{F}_\Delta^{(m:n)}\big(\{c_{-\ell}\} : \{c_k\}\big) = \frac{e^{\zeta_{(m:n)}} Z_{(m:n)}(c_0; \{c_k\}; \{c_{-\ell}\})}{Z_{(0:n)}(c_0; \{c_k\}) Z_{(0:m)}(c_\infty; \{c_{-\ell}\})}, \tag{3.8}$$

where $Z_{(0:n)}(c_0; \{c_k\})$ and $Z_{(0:m)}(c_\infty; \{c_{-\ell}\})$ provide the proper normalization for the irregular conformal block. Here we use the change of variable $\lambda_i \to 1/\lambda_i$ to express $\langle I_m|\Delta\rangle$ as $Z_{(0:m)}(c_\infty; \{c_{-\ell}\})$.

To evaluate ICB we note that the potential $V_{(m:n)}$ contains the information of the irregular module at the origin and at infinity at the same time. Therefore, each module can be derived if one views the same potential on a different footing. The information of the irregular module at the origin is obtained if one regards the potential $V_0 = V_{(0:n)}(\{\lambda_i\}; c_0, \{c_k\})$ as the reference one and ΔV_0 as its perturbation:

$$\frac{1}{\hbar} V_0 = \sum_{I=1}^{N_0}\left(-c_0 \log\lambda_I + \sum_{k=1}^{n}\frac{c_k}{k}\lambda_I^{-k}\right);$$
$$\frac{1}{\hbar}\Delta V_0 = \sum_{I=1}^{N_0}\left(\sum_{\ell=1}^{n}\frac{c_{-\ell}}{\ell}\lambda_I^{\ell}\right). \tag{3.9}$$

That is, V_0 is the potential for the partition function $Z_{(0:n)}$ with $N_0(\le N)$ number of screening operators. At infinity one has the reference potential $\sum_{J=1}^{N_\infty}(-c_0 \log\lambda_J + \sum_{\ell=1}^{n} c_{-\ell}\lambda_i^{\ell}/\ell)$ and its perturbation $\sum_{J=1}^{N_\infty}(\sum_{k=1}^{n} c_k\lambda_J^{-k}/k)$. We introduce the number N_∞ of screening operators at infinity so that $N_\infty + N_0 = N$. One may rewrite the potential in a familiar form if one changes the variable $\lambda_J \to 1/\mu_J$ to get the equivalent potential

$$\frac{1}{\hbar} V_\infty = \sum_{J=1}^{N_\infty}\left(-c_\infty \log\mu_J + \sum_{\ell=1}^{m}\frac{c_{-\ell}}{\ell}\mu_J^{-\ell}\right);$$
$$\frac{1}{\hbar}\Delta V_\infty = \sum_{J=1}^{N_\infty}\left(\sum_{k=1}^{n}\frac{c_k}{k}\mu_J^{k}\right). \tag{3.10}$$

In this way the perturbative potential and the cross terms in the Vandermonde determinant provide ICB:

$$\mathcal{F}_\Delta^{(m:n)}\big(\{c_{-\ell}\} : \{c_k\}\big) = e^{\zeta_{(m:n)}}\left\langle \prod_{I,J}(1-\lambda_I\mu_J)^{2\beta} e^{-\frac{\sqrt{\beta}}{g}(\Delta V_0(\lambda_I)+\Delta V_\infty(\mu_J))}\right\rangle, \tag{3.11}$$

where the bracket denotes the expectation value using the reference partition function:

$$\big\langle\mathcal{O}(\lambda_I)\big\rangle \equiv \langle\mathcal{O}\rangle_+ = \big(Z_{(0:n)}\big(c_0; \{c_k\}\big)\big)^{-1} \times \int \prod_{I=1}^{N_0} d\lambda_I \Delta(\lambda)^{2\beta}\mathcal{O}(\lambda_I) e^{-\frac{\sqrt{\beta}}{g}\sum_i V_0(\lambda_I)},$$

$$\big\langle\mathcal{O}(\mu_J)\big\rangle \equiv \langle\mathcal{O}\rangle_- = \big(Z_{(0:m)}\big(c_\infty; \{c_\ell\}\big)\big)^{-1} \times \int \prod_{J=1}^{N_\infty} d\mu_J \Delta(\mu)^{2\beta}\mathcal{O}(\mu_J) e^{-\frac{\sqrt{\beta}}{g}\sum_i V_\infty(\mu_J)}, \tag{3.12}$$

which can be regarded as the generalization of Selberg integral [14,15]. One may put ICB in (3.11) compactly in terms of Jack polynomial [16–18]. Putting $p_k = \sum_I \lambda_I^k$ and $p'_k = \sum_J \mu_J^k$, one has the identity

$$\prod_{I,J}(1-\lambda_I\mu_J)^{2\beta} e^{-\frac{\sqrt{\beta}}{g}(\Delta V_0(\lambda_I)+\Delta V_\infty(\mu_J))} = \exp\left\{-\beta\sum_{k=1}^{\infty}\frac{1}{k}p_k\big(p'_k - \tilde{c}_{-k}\big)\right\} \times \exp\left\{-\beta\sum_{k=1}^{\infty}\frac{1}{k}p'_k(p_k - \tilde{c}_k)\right\}, \tag{3.13}$$

where $\tilde{c}_{\pm k} = i2c_{\pm k}/\sqrt{\beta} = -2c_{\pm k}/b$ (with $\tilde{c}_k = 0$ for $k > n$ and $\tilde{c}_{-k} = 0$ for $k > m$). Using the Cauchy–Stanley identity [19,20] $e^{\beta\sum_{k\ge 1}\frac{1}{k}p_k p'_k} = \sum_R j_R^{(\beta)}(p) j_R^{(\beta)}(p')$, one has ICB as

$$\mathcal{F}_\Delta^{(m:n)} = e^{\zeta_{(m:n)}}\sum_{Y,W}\big\langle j_Y^{(\beta)}(p_k) j_W^{(\beta)}(-p_k+\tilde{c}_k)\big\rangle_+ \times \big\langle j_Y^{(\beta)}\big(-p'_k+\tilde{c}_{-k}\big) j_W^{(\beta)}\big(p'_k\big)\big\rangle_-. \tag{3.14}$$

The explicit form of the general ICB is not available yet. Here we check a few non-trivial terms using the resolvent in the loop equation of the reference partition function. Each term can be obtained from the large z expansion of the resolvent and ICB in power of $\eta_0 \equiv c_1 c_{-1}$, which is compatible with the Young diagram expansion. For rank 1, up to order $\mathcal{O}(\eta_0^2)$ one has

$$\mathcal{F}_\Delta^{(1:1)} = 1 + \eta_0\frac{2\bar{c}_0\bar{c}_\infty}{\Delta} + \eta_0^2\frac{4\bar{c}_0^2\bar{c}_\infty^2 c/\Delta + 4\Delta + 2 + 12(\bar{c}_0^2 + \bar{c}_\infty^2) + 32\bar{c}_0^2\bar{c}_\infty^2}{c + 2c\Delta + 2\Delta(8\Delta - 5)}, \tag{3.15}$$

where $\bar{c}_0 = Q - c_0$, $\bar{c}_\infty = Q - c_\infty$, and $c = 1 + 6Q^2$. Comparing this with the Gaiotto inner product up to $\mathcal{O}(\Lambda\Lambda')^2$ (using (2.1) with $\langle\widetilde{G}_2|$ under the primed notation)

$$\langle\widetilde{G}_2|\widetilde{G}_2\rangle = 1 + \Lambda\Lambda'\frac{mm'}{2\Delta} + \big(\Lambda\Lambda'\big)^2\frac{m^2m'^2c/4\Delta + 4\Delta + 2 - 3(m^2+m'^2) + 2m^2m'^2}{c + 2c\Delta + 2\Delta(8\Delta - 5)}, \tag{3.16}$$

we find $\Lambda^2 = -c_1^2$ ($\Lambda'^2 = -c_{-1}^2$) and $m\Lambda = 2c_1\bar{c}_0$ ($m'\Lambda' = 2c_{-1}\bar{c}_\infty$), consistent with the eigenvalues of L_2 and L_1.

Non-trivial checks are given for rank 2. Matrix model provides $\mathcal{F}_\Delta^{(1:2)}$ up to $\mathcal{O}(\eta_0^2)$

$$\mathcal{F}_\Delta^{(1:2)} = 1 + \eta_0\frac{\bar{b}_1\bar{c}_\infty}{\Delta} + \eta_0^2\frac{c\bar{c}_\infty^2\bar{b}_2/\Delta + (2 + 12\bar{c}_\infty^2 + 4\Delta)(1 - c_2(Q + 2\bar{c}_0)/c_1^2) + (3 + 8\bar{c}_\infty^2)\bar{b}_2}{c + 2c\Delta + 2\Delta(8\Delta - 5)}, \tag{3.17}$$

where $c_1\bar{b}_1=(d_1^{(0:2)}+2\bar{c}_0c_1)$, $c_1^2\bar{b}_2=(d_1^{(0:2)}+2\bar{c}_0c_1)^2+c_2\partial d_1^{(0:2)}/\partial c_1$. The explicit form of $d_1^{(0:2)}$ is given in powers of $\eta_1\equiv c_2/c_1^2$ [9,13]

$$\frac{d_1^{(0:2)}}{c_1}=-2bN_{(2)}c_0+\big[-bN_0(bN_0+2c_0-Q)+bN_{(2)}(3bN_{(2)}+4c_0-3Q)\big]\eta_1+\mathcal{O}(\eta_1^2), \tag{3.18}$$

where $N_{(1)}$ and $N_{(2)}$ are filling fractions of two cuts, $N_0=N_{(1)}+N_{(2)}$. On the other hand, one has the Gaiotto inner product up to $\mathcal{O}(\Lambda'\sqrt{\Lambda})^2$

$$\langle\widetilde{G_2}|\widetilde{G_4}\rangle=1+\Lambda'\sqrt{\Lambda}\frac{m'b_1}{2\Delta}+\big(\Lambda'\sqrt{\Lambda}\big)^2\times\frac{cm'^2b_1^2/(4\Delta)+(2-3m'^2+4\Delta)m+(-3+2m'^2)b_1^2}{c+2c\Delta+2\Delta(8\Delta-5)}. \tag{3.19}$$

Comparing the two we obtain the parameter relation $\Lambda m=-c_1^2+2c_2(Q/2+\bar{c}_0)$, the eigenvalues of L_2. However in (3.17) $\bar{b}_2$ is not $\bar{b}_1^2$, which is different from (3.19). Therefore, $(\sqrt{\Lambda}b_1)^\ell$ cannot be considered as a simple constant but should be of the form $(c_1\bar{b}_1)^\ell=\frac{1}{Z_{(0:2)}}(\Lambda_1+\nu_1)^\ell Z_{(0:2)}$.

One can further check this relation holds for $\mathcal{F}_\Delta^{(2:2)}$ if the Gaiotto inner product

$$\langle\widetilde{G_4}|\widetilde{G_4}\rangle=1+\big(\Lambda'\Lambda\big)^{1/2}\frac{b_1b_1'}{2\Delta}+\frac{\Lambda'\Lambda}{c+2c\Delta+2\Delta(8\Delta-5)}\bigg[\frac{c\,b_1^2b_1'^2}{4\Delta}+2\big(b_1^2b_1'^2+mm'\big)-3\big(mb_1'^2+3m'b_1^2\big)+4\Delta mm'\bigg]+\big(\Lambda'\Lambda\big)^{3/2} \tag{3.20}$$

is compared with its matrix result

$$\begin{aligned}\mathcal{F}_\Delta^{(2:2)}=1&+\frac{\bar{b}_1\bar{b}_{-1}}{2\Delta}\eta_0+\frac{\eta_0^2}{c+2c\Delta+2\Delta(8\Delta-5)}\bigg[\frac{c\bar{b}_1^2\bar{b}_{-1}^2}{4\Delta}\\&+2\big(\bar{b}_1^2\bar{b}_{-1}^2+\big(1-c_2(Q+2\bar{c}_0)/c_1^2\big)\\&\times\big(1-c_{-2}(Q+2\bar{c}_\infty)/c_{-1}^2\big)\big)\\&+3\big(\big(1-c_2(Q+2\bar{c}_0)/c_1^2\big)\bar{b}_{-1}^2\\&+\big(1-c_{-2}(Q+2\bar{c}_\infty)/c_{-1}^2\big)\bar{b}_1^2\big)\\&+4\Delta\big(1-c_2(Q+2\bar{c}_0)/c_1^2\big)\\&\times\big(1-c_{-2}(Q+2\bar{c}_\infty)/c_{-1}^2\big)\bigg]+\mathcal{O}\big(\eta_0^3\big),\end{aligned} \tag{3.21}$$

where $c_{-1}\bar{b}_{-1}=(d_{-1}^{(0:2)}+2\bar{c}_\infty c_{-1})$ and $c_{-1}^2\bar{b}_{-1}^2=(d_{-1}^{(0:2)}+2\bar{c}_\infty c_{-1})^2+c_{-2}\partial d_{-1}^{(0:2)}/\partial c_{-1}$. From this we confirm the parameter relation at infinity in the same fashion at the origin; $\Lambda'm'=-c_{-1}^2+2c_{-2}(Q/2+\bar{c}_\infty)$ and b_1' with $\bar{b}_{-1}$ with $(c_{-1}\bar{b}_{-1})^\ell=\frac{1}{Z_{(0:2)}}(\Lambda_{-1}+\nu_{-1})^\ell Z_{(0:2)}$.

4. Summary and discussion

We found the Virasoro irregular conformal block using the beta deformed Penner type matrix model and presented the result in terms of the expectation values of the Jack polynomial (3.14). We checked ICB explicitly for a few ranks and compare with the inner product of Gaiotto state proposed by [5]. There is a non-trivial modification between the two results due to the difference of the normalization as is suggested in [5].

Referring to the explicit check given for the rank 1 and 2, we can clearly see that the Gaiotto state needs to be modified to represent the colliding limit of the conformal correlation. Note that the expectation value $\langle\Delta|L_k^{\ell_k}|\widetilde{G_{2n}}\rangle$ is $(\Lambda^{k/n}b_k)^{\ell_k}$ according to (2.3). On the other hand, $|I_n\rangle$ has the $L_k^{\ell_k}$ expectation value $((\Lambda_k+\nu_k)^{\ell_k}Z_{(0:n)})/Z_{(0:n)}$. Therefore, one may conclude that the Gaiotto state should be modified by using the coefficient relation $(\Lambda^{k/n}b_k)^{\ell_k}\to((\Lambda_k+\nu_k)^{\ell_k}Z_{(0:n)})/Z_{(0:n)}$. Considering $\langle\Delta|L_1^{\ell_1}L_2^{\ell_2}|I_n\rangle=(\Lambda_2+\nu_2)^{\ell_2}(\Lambda_1+\nu_1)^{\ell_1}Z_{(0:n)}$, one has

$$\langle\Delta|L_W|I_n\rangle=\Lambda^{\ell/n}m^{\ell_n}\left\{\prod_{i=1}^{n-1}a_i^{\ell_{2n-i}}\right\}\times\left\{(\Lambda_{n-1}+\nu_{n-1})^{\ell_{n-1}}\cdots(\Lambda_1+\nu_1)^{\ell_1}Z_{(0:n)}\right\} \tag{4.1}$$

with proper ordering. The case of rank 1 is trivial since there is no b_k's.

In the paper we consider mainly the two-point ICB. One may extend the result to N-point ICB $\langle\prod_{A=1}^N I_{m_A}(z_A)\rangle$ by generalizing the potential in (3.7):

$$\frac{1}{\hbar}V_{(\{m_A\})}\big(z;\{c_0^{(A)}\},\{c_k^{(A)}\}\big)=\sum_{A=1}^N\left\{-c_0^{(A)}\log(z-z_A)+\sum_{k=1}^{n_A}\left(\frac{c_k^{(A)}}{k(z-z_A)^k}\right)\right\}. \tag{4.2}$$

ICB will be given with the appropriate normalization at each point, *i.e.*, by treating the potential as the sum of the reference potential and perturbation at each point.

Noting the Penner-type matrix model provides ICB, one may wonder if there exists another systematic way of obtaining the irregular conformal block of arbitrary rank from regular conformal block, as seen in the rank 1 case [3] or for $SU(N)$ in [21] by decoupling a certain large mass limit. However, such a decoupling limit is not achieved yet for rank greater than 1. It will be interesting to find the limit using the relation of the Selberg integral with the Jack polynomials to have (3.14).

In addition, one may have the colliding limit for W-algebraic symmetry as done in [5] using $SU(N)$ Toda theories. The corresponding matrix model is straight-forward generalization of the Virasoro symmetric case for $SU(N)$. Making use of [22], we have

$$\begin{aligned}Z_{(m:n)}^{SU(N)}&=\int\prod_{a=1}^{N-1}\prod_{i=1}^{N_a}d\lambda_i^{(a)}\,\Delta\big(\lambda^{(a)}\big)^{2\beta}\prod_{a=1}^{N-2}\Delta\big(\lambda^{(a)},\lambda^{(a+1)}\big)^{-\beta}e^{-\frac{\sqrt{\beta}}{g}V},\\ \frac{1}{\hbar}V&=\sum_{a=1}^{N-1}\sum_{i=1}^{N_a}\left[-c_0^{(a)}\log\lambda+\sum_{k=1}^{n}\left(\frac{c_k^{(a)}}{k\lambda^k}\right)+\sum_{\ell=1}^{m}\left(\frac{c_{-\ell}^{(a)}\lambda^\ell}{\ell}\right)\right],\end{aligned} \tag{4.3}$$

with $c_k^{(a)}=\sum_{r=1}^n(\alpha_r,e_a)(z_r)^k$, and $c_{-\ell}^{(a)}=\sum_{r=1}^m(\tilde{\alpha}_r,e_a)(\tilde{z}_r)^{-\ell}$. Here e_a are the simple roots of $SU(N)$. This leads to the $SU(N)$ ICB:

$$\mathcal{F}_\Delta^{(m:n)}=e^{\zeta_{(m:n)}}\sum_{\vec{Y}}\left\langle\prod_{a=1}^N j_{Y_a}^{(\beta)}\big(p_k^{(a-1)}-p_k^{(a)}+\tilde{c}_k^{(a)}\big)\right\rangle_+\times\left\langle\prod_{a=1}^N j_{Y_a}^{(\beta)}\big(p_k'^{(a)}-p_k'^{(a-1)}+\tilde{c}_{-k}^{(a)}\big)\right\rangle_-, \tag{4.4}$$

with $\tilde{c}_{-k}^{(a)}=2\sum_{s=1}^{a-1}c_{-k}^{(s)}/b$, $\tilde{c}_k^{(N-a)}=-2\sum_{s=1}^{a-1}c_k^{(N-s)}/b$, $p_k^{(0)}=p_k^{(N)}=p_k'^{(0)}=p_k'^{(N)}\equiv 0$, and $\langle\mathcal{O}\rangle_\pm$ are generalizations of A_{N-1} Selberg integral, expectation values of the matrix model with the reference potential, part of (4.3). $U(1)$ factor $\zeta_{(m:n)}$ is also summed over $SU(N)$ index a.

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