Advances in Applied Mathematics 43 (2009) 331–374

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Solutions of Navier equations and their representation structure

Bintao Cao

Institute of Mathematics, Academy of Mathematics & Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

ARTICLE INFO

Article history:
Received 23 March 2009
Accepted 27 April 2009
Available online 8 July 2009

MSC:
primary 17B10, 17B20, 35C99
secondary 35E15

Keywords:
Navier equations
Lamé equations
Polynomial solutions
Bases
Initial value problems

ABSTRACT

Navier equations are used to describe the deformation of a homogeneous, isotropic and linear elastic medium in the absence of body forces. Mathematically, the system is a natural vector $O(n,\mathbb{R})$-invariant generalization of the classical Laplace equation. In this paper, we decompose the space of polynomial solutions of Navier equations into a direct sum of irreducible $O(n,\mathbb{R})$-submodules and construct an explicit basis for each irreducible summand. Moreover, we explicitly solve the initial value problems for Navier equations and their wave-type extension—Lamé equations by Fourier expansion and Xu's method of solving flag partial differential equations. Our work might be counted as a continuation of Olver's important work on the algebraic study of elasticity in a certain sense.

1. Introduction

Navier equations

$$\iota_1\Delta(\vec{u})+(\iota_1+\iota_2)\big(\nabla^T\nabla\big)(\vec{u})=0 \tag{1.1}$$

are used to describe the deformation of a homogeneous, isotropic and linear elastic medium in the absence of body forces (e.g., cf. [1,6]), where $\vec{u}$ is an n-dimensional vector-valued function, $\Delta=\partial_{x_1}^2+\partial_{x_2}^2+\cdots+\partial_{x_n}^2$ is the Laplace operator, $\nabla=(\partial_{x_1},\partial_{x_2},\dots,\partial_{x_n})$ is the divergence operator, acting on the vector-valued functions, ι_1 and ι_2 are Lamé constants with $\iota_1>0$, $2\iota_1+\iota_2>0$ and $\iota_1+\iota_2\neq 0$. Here we do not use the traditional notations λ and μ because in Lie theory they usually stand for the weights of modules. To avoid the confusion with time variable t, we use superscript "T" to denote

E-mail address: caobintao@amss.ac.cn.

0196-8858/$ – see front matter
doi:10.1016/j.aam.2009.04.004

the transpose of a matrix (vector) throughout this paper. In fact, $\nabla^T \nabla = (\partial^2_{x_k x_l})_{n\times n}$ is the well-known Hessian operator.

The wave-type extensions of Navier equations are the well-known Lamé equations

$$\vec{u}_{tt} = \frac{\iota_1}{\iota_1 + \iota_2} \Delta(\vec{u}) + \left(\nabla^T \nabla\right)(\vec{u}). \tag{1.2}$$

The most important systematic algebraic study on Navier equations was done by Olver [7] (1984), where he found the symmetry group (including the variational symmetries and Lie–Bäcklund symmetries) and conservation laws for the equations. The complete classification of all first-order conservation laws and their explicit expressions were also given. Moreover, he found some very interesting divergence identities. Özer [8] (2003) obtained some exact solutions of Lamé equations with $n = 3$ by means of Lie point transformations. Rodionov [9] (2006) got some finite form solutions in given compact domain of Navier equations by holomorphic expansions. In this paper, we give a systematic algebraic study on the solutions of the above equations, which might be counted as a continuation of Olver's work [7] in a certain sense.

Classical Laplace equation

$$u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n} = 0 \tag{1.3}$$

is one of the most fundamental partial differential equations in mathematics and physics. Its solutions are called *harmonic functions*, which physically represent the density of some physical quantity in equilibrium. The more general form of Laplace equation on Riemannian manifolds is the main object in harmonic analysis. A fundamental algebraic characteristic of the above equation is its invariance under the action of the orthogonal group $O(n, \mathbb{R})$, that is, the space of harmonic functions forms an $O(n, \mathbb{R})$-module. Denote by $\mathcal{A}_k$ the space of polynomials of degree k and by $\mathcal{H}_k$ the space of harmonic polynomials of degree k. The classical harmonic analysis says that

$$\mathcal{A}_k = \mathcal{H}_k \oplus \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)\mathcal{A}_{k-2} \tag{1.4}$$

and $\mathcal{H}_k$ forms an irreducible $O(n, \mathbb{R})$-submodule. By induction, the above conclusion gives a decomposition of the polynomial algebra into a direct sum of irreducible $O(n, \mathbb{R})$-submodules. Luo [5] generalized the result (1.4) to certain noncanonical oscillator representations of $O(n, \mathbb{R})$.

Mathematically, Navier equations is a natural vector $O(n, \mathbb{R})$-invariant generalization of the classical Laplace equation. Our first objective in this paper is to decompose the space of polynomial vectors into a direct sum of irreducible $O(n, \mathbb{R})$-submodules in terms of the irreducible $O(n, \mathbb{R})$-submodules included in the subspace of homogeneous polynomial solutions of Navier equations. Unlike the case of the classical Laplace equation, the subspaces of homogeneous polynomial solutions of Navier equations are not irreducible. From an algebraic point of view, it is more convenient to deal with the orthogonal Lie algebra $o(n, \mathbb{R})$ than the orthogonal Lie group $O(n, \mathbb{R})$. They are equivalent under the exponential map from the Lie algebra to the Lie group in our case.

Another closely related fundamental equation is the wave equation:

$$u_{tt} = u_{x_1 x_1} + u_{x_2 x_2} + \cdots + u_{x_n x_n}. \tag{1.5}$$

Physically it describes the vibration of strings and membranes. The solutions of the initial value problems for the Laplace equation (1.3) and the wave equation (1.5) are elementary known facts appeared in many textbooks of partial differential equations.

A *partial differential equation of flag type* is a linear differential equation of the form

$$(d_1 + f_1 d_2 + f_2 d_3 + \cdots + f_{n-1} d_n)(u) = 0, \tag{1.6}$$

where $d_1, d_2, \dots, d_n$ are certain commuting locally nilpotent differential operators on the polynomial algebra $\mathbb{R}[x_1, x_2, \dots, x_n]$ and $f_1, \dots, f_{n-1}$ are polynomials satisfying

$$d_r(f_s) = 0 \quad \text{if } r > s. \tag{1.7}$$

Flag partial differential equations naturally appear in the problem of decomposing the polynomial algebra (symmetric tensor) over an irreducible module of a Lie algebra into the direct sum of its irreducible submodules. Many important linear partial differential equations in physics and geometry are also of flag type. Xu [10] used the grading technique in algebra to develop methods of solving such equations. In particular, he found new special functions by which we are able to explicitly give the solutions of the initial value problems of a large family of constant-coefficient linear partial differential equations in terms of their coefficients.

Our second objective in this paper is to solve the initial value problems for Navier equations and Lamé equations by Xu's method of solving flag partial differential equations in [10] and his matrix-differential-operator approach in [11]. Below we give a more detailed technical introduction.

To state our results in this paper, we denote

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \qquad \vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix}, \tag{1.8}$$

$$\hat{\mathcal{A}} = \bigoplus_{k=0}^{\infty} \hat{\mathcal{A}}_k \quad \text{with } \hat{\mathcal{A}}_k = \{\vec{f} \mid f_j \in \mathcal{A}_k\}, \tag{1.9}$$

where $\mathcal{A}_k$ is the space of polynomials in $x_1, \dots, x_n$ of degree k. Moreover, we define

$$\hat{\mathcal{H}}_k = \{\vec{f} \in \hat{\mathcal{A}}_k \mid \iota_1 \Delta(\vec{f}) + (\iota_1 + \iota_2)(\nabla^T \nabla)(\vec{f}) = 0\}. \tag{1.10}$$

The action of the orthogonal group $O(n, \mathbb{R})$ on $\hat{\mathcal{A}}$ is defined by

$$\mathcal{T}(\vec{f}(\vec{x})) = (\mathcal{T}\vec{f})(\mathcal{T}^{-1}(\vec{x})). \tag{1.11}$$

Furthermore, we denote $\varsigma_r = (0, \dots, 0, \overset{r}{1}, 0, \dots, 0)^T$ and $b = (\iota_1 + \iota_2)/\iota_1$. Then $\vec{f} = \sum_{r=1}^n f_r \varsigma_r$. We define linear maps $\psi, \varphi_1, \varphi_2 : \mathbb{R}[x_1, \dots, x_n] \to \hat{\mathcal{A}}$ by

$$\psi(x_{i_1} x_{i_2} \cdots x_{i_r}) = \sum_{s=1}^{r} x_{i_1} \cdots x_{i_{s-1}} x_{i_{s+1}} \cdots x_{i_r} \varsigma_s, \tag{1.12}$$

$$\varphi_1(x_{i_1} x_{i_2} \cdots x_{i_r}) = \sum_{s=1}^{r} x_{i_1} \cdots x_{i_{s-1}} x_{i_{s+1}} \cdots x_{i_r} \left[r\left(\sum_{l=1}^{n} x_l^2\right) \varsigma_{i_s} - (2r + n - 2) x_{i_s} \sum_{l=1}^{n} x_l \varsigma_l \right] \tag{1.13}$$

and $\varphi_2 = (x_1^2 + \cdots + x_n^2)\psi$.

Again we let $\mathcal{H}_k$ be the space of harmonic polynomials of degree k. Set

$$\hat{\mathcal{H}}_{k,1} = \psi(\mathcal{H}_{k+1}), \qquad \hat{\mathcal{H}}_{k,2} = \left\{ \sum_{l=1}^{n} f_l \varsigma_l \;\middle|\; f_l \in \mathcal{H}_k,\ \sum_{l=1}^{n} x_l f_l = 0 \right\}, \tag{1.14}$$

$$\hat{\mathcal{H}}_{k,3} = \left(\varphi_1 + \frac{(2k + n - 2)(k + n - 3)(k - 1)}{2(b^{-1}(2k + n - 4) + k - 1)} \varphi_2 \right)(\mathcal{H}_{k-1}). \tag{1.15}$$

Denote $d_{r,s} = x_s\partial_{x_r} - x_r\partial_{x_s}$ and set

$$\mathcal{D} = \begin{pmatrix} 0 & d_{3,4} & d_{4,2} & d_{2,3} \\ d_{4,3} & 0 & d_{1,4} & d_{3,1} \\ d_{2,4} & d_{4,1} & 0 & d_{1,2} \\ d_{3,2} & d_{1,3} & d_{2,1} & 0 \end{pmatrix}. \tag{1.16}$$

With $n = 4$, we define

$$\hat{\mathcal{H}}_{k,2\pm} = \{\vec{f} \in \hat{\mathcal{H}}_{k,2} \mid \mathcal{D}\vec{f} = \pm(k+1)\vec{f}\}. \tag{1.17}$$

Main Theorem 1. *Let $n \geqslant 3$ be an integer. The subspace of polynomial solutions $\hat{\mathcal{H}}_k = \hat{\mathcal{H}}_{k,1} \oplus \hat{\mathcal{H}}_{k,2} \oplus \hat{\mathcal{H}}_{k,3}$ and*

$$\hat{\mathcal{A}}_k = \hat{\mathcal{H}}_k \oplus (x_1^2 + \cdots + x_n^2)\hat{\mathcal{A}}_{k-2}. \tag{1.18}$$

Moreover, the subspaces $\hat{\mathcal{H}}_{k,1}$, $\hat{\mathcal{H}}_{k,2}$ ($n \neq 4$) and $\hat{\mathcal{H}}_{k,3}$ are irreducible $O(n,\mathbb{R})$-submodules. When $n = 4$, $\hat{\mathcal{H}}_{k,2} = \hat{\mathcal{H}}_{k,2+} \oplus \hat{\mathcal{H}}_{k,2-}$ and $\hat{\mathcal{H}}_{k,2+}$, $\hat{\mathcal{H}}_{k,2-}$ are irreducible $O(4,\mathbb{R})$-submodules.

Furthermore, an explicit basis is constructed for each of the above irreducible submodules. Xu's Lemma in [10] on the polynomial solutions of constant-coefficient partial differential equation of certain type is used to prove the decomposition $\hat{\mathcal{H}}_k = \hat{\mathcal{H}}_{k,1} \oplus \hat{\mathcal{H}}_{k,2} \oplus \hat{\mathcal{H}}_{k,3}$. Our above result is exactly a natural vector generalization of those scalar one in (1.4). Indeed, it does not only reveal an internal symmetry of Navier equations but also gives the explicit realizations of certain irreducible $O(n,\mathbb{R})$-modules.

We remark that the map ψ in (1.12) is nothing but the gradient operator. Thus the polynomials in $\hat{\mathcal{H}}_{k,1}$ represent the irrotational part, with respect to the well-known Lamé scalar potentials [4], of the displacement field in the 3-dimensional case. However, it is difficult to get a basis of $\hat{\mathcal{H}}_k$ (cf. (1.10)) by just using Lamé potentials. Another difficulty is that the Lamé vector potentials are valid only in the 3-dimensional case.

Our second main result is obtaining explicit exact solutions of Navier equations (1.1) subject to the initial conditions

$$\vec{u}(0, x_2, \ldots, x_n) = \vec{g}_0(x_2, \ldots, x_n), \qquad \vec{u}_{x_1}(0, x_2, \ldots, x_n) = \vec{g}_1(x_2, \ldots, x_n) \tag{1.19}$$

for $x_r \in [-a_r, a_r]$, and explicit exact solutions of Lamé equations (1.2) subject to the initial conditions

$$\vec{u}(0, x_1, \ldots, x_n) = \vec{h}_0(x_1, \ldots, x_n), \qquad \vec{u}_t(0, x_1, \ldots, x_n) = \vec{h}_1(x_1, \ldots, x_n) \tag{1.20}$$

for $x_r \in [-b_r, b_r]$, where $\vec{g}_1$, $\vec{g}_2$, $\vec{h}_1$, $\vec{h}_2$ are vector-valued continuous functions, and a_r, b_r are positive real constants. Xu's method of solving the initial value problems of flag partial differential equations in [10] and his matrix-differential-operator approach [11] play fundamental roles in obtaining our solutions. Moreover, Fourier expansions are used.

Section 2 is devoted to the proof of Main Theorem 1. In Section 3, we construct an explicit basis for each irreducible submodule included in the solution space of Navier equations. Moreover, a uniform explicit basis of the homogeneous polynomial solutions of Navier equations is obtained by a method of Xu in [10], whose cardinality was pre-used in Section 2 to prove the completeness of polynomial solutions for a technical reason. In Section 4, we solve the above mentioned initial value problems.

2. Polynomial solutions and representations

In this section, we will study the homogeneous polynomial solutions of (1.1) in detail.

As we mentioned in the introduction, studying $O(n,\mathbb{R})$-representation structure of the polynomial solutions is equivalent to studying their $o(n,\mathbb{R})$-representation structure via the exponential map from the Lie algebra to the Lie group. Recall that $E_{r,s}$ is the square matrix with 1 as its (r,s)-entry and 0 as the others. The orthogonal Lie algebra

$$o(n,\mathbb{R})=\sum_{r,s=1}^{n}\mathbb{R}(E_{r,s}-E_{s,r}). \tag{2.1}$$

Its action on the space $\hat{\mathcal{A}}$ of polynomial vectors (cf. (1.9)) is given by

$$(E_{r,s}-E_{s,r})(\vec{f})=(x_r\partial_{x_s}-x_s\partial_{x_r})(\vec{f})+f_s\varsigma_r-f_r\varsigma_s, \tag{2.2}$$

where $\varsigma_r=(0,\dots,0,\overset{r}{1},0,\dots,0)^T$ as in the introduction. Moreover, the elements of $o(n,\mathbb{R})$ map solutions of (1.1) into solutions. That is,

$$\xi(\hat{\mathcal{H}}_k)\subset\hat{\mathcal{H}}_k\quad\text{for }\xi\in o(n,\mathbb{R}),\ k\in\mathbb{N} \tag{2.3}$$

(cf. (1.10)). Note that $\vec{f}+\vec{g}i$ is a complex solution of Navier equations (1.1) if and only if $\vec{f}$ and $\vec{g}$ are real solutions. In order to use the representation theory of the Lie algebras over the complex field $\mathbb{C}$, we need to complexify our vector spaces, indicated by the subscript of $\mathbb{C}$. The complexification of $o(n,\mathbb{R})$ is $o(n,\mathbb{C})$. We extend the representation (2.2) of $o(n,\mathbb{R})$ on $\hat{\mathcal{A}}$ to that of $o(n,\mathbb{C})$ on $\hat{\mathcal{A}}_{\mathbb{C}}$ $\mathbb{C}$-bilinearly.

Denote by $M_{m\times m}(\mathbb{F})$ the algebra of $m\times m$ matrices with entries in the field $\mathbb{F}$. Set

$$o'(2m+1,\mathbb{C})=\left\{\begin{pmatrix}0&-\vec{b}^T&-\vec{a}^T\\ \vec{a}&A&A_1\\ \vec{b}&A_2&-A^T\end{pmatrix}\ \middle|\ A\in M_{m\times m}(\mathbb{C});\ A_1,A_2\in o(m,\mathbb{C});\ \vec{a},\vec{b}\in\mathbb{C}^m\right\} \tag{2.4}$$

and

$$o'(2m,\mathbb{C})=\left\{\begin{pmatrix}A&A_1\\ A_2&-A^T\end{pmatrix}\ \middle|\ A\in M_{m\times m}(\mathbb{C});\ A_1,A_2\in o(m,\mathbb{C})\right\}. \tag{2.5}$$

Note that $o'(3,\mathbb{C})$ is the standard form of complex simple Lie algebra of type A_1, $o'(4,\mathbb{C})$ is the standard form of complex semisimple Lie algebra of type $A_1\oplus A_1$ and $o'(6,\mathbb{C})$ is the standard form of complex simple Lie algebra of type A_3 (or D_3). Moreover, $o'(2m+1,\mathbb{C})$ with $m\geqslant 2$ is the standard form of the complex simple Lie algebra of type B_m and $o'(2m,\mathbb{C})$ with $m\geqslant 4$ is the standard form of the complex simple Lie algebra of type D_m. If $n=2m+1$ is odd, we take

$$H_{B_m}=\sum_{r=1}^{m}\mathbb{C}(E_{r+1,r+1}-E_{m+r+1,m+r+1}) \tag{2.6}$$

as a Cartan subalgebra of $o'(2m+1,\mathbb{C})$,

$$\{E_{r+1,s+1}-E_{m+s+1,m+r+1},\,E_{r+1,m+s+1}-E_{s+1,m+r+1}E_{r_1,1}-E_{1,m+r_1}\mid 1\leqslant r<s\leqslant m;\ 1\leqslant r_1\leqslant m\} \tag{2.7}$$

as positive root vectors, and

$$\{E_{s+1,r+1} - E_{m+r+1,m+s+1}, E_{m+s+1,r+1} - E_{m+r+1,s+1} E_{m+r_1,1} - E_{1,r_1} \mid 1 \leqslant r < s \leqslant m;\ 1 \leqslant r_1 \leqslant m\} \tag{2.8}$$

as negative root vectors.

For $1 \leqslant r \leqslant m$, we define the linear function ε_r on $H^*_{B_m}$ by

$$\varepsilon_r(E_{s+1,s+1} - E_{m+s+1,m+s+1}) = \delta_{r,s}. \tag{2.9}$$

Then $H^*_{B_m} = \text{span}\{\varepsilon_1, \dots, \varepsilon_m\}$ and $\Phi = \{\pm\varepsilon_r, \pm(\varepsilon_r \pm \varepsilon_s) \mid r \neq s\}$ forms the root system of $o'(2m+1, \mathbb{C})$. Denote by $\alpha_r = \varepsilon_r - \varepsilon_{r+1}$ for $r = 1, \dots, m-1$ and $\alpha_m = \varepsilon_m$. As we know, $\{\alpha_1, \dots, \alpha_m\}$ is the simple root system. Moreover, we define a symmetric bilinear form $(\cdot,\cdot)$ on $H^*_{B_m}$ by

$$(\varepsilon_r, \varepsilon_s) = \delta_{r,s}. \tag{2.10}$$

Furthermore, we denote by $\lambda_1, \dots, \lambda_m$ the fundamental dominant weights in $H^*_{B_m}$, i.e.

$$\langle \lambda_r, \alpha_s \rangle = \frac{2(\lambda_r, \alpha_s)}{(\alpha_s, \alpha_s)} = \delta_{r,s}. \tag{2.11}$$

In particular,

$$\varepsilon_1 = \lambda_1, \quad \varepsilon_2 = -\lambda_1 + \lambda_2, \quad \dots, \quad \varepsilon_{m-1} = -\lambda_{m-2} + \lambda_{m-1}, \quad \varepsilon_m = -\lambda_{m-1} + 2\lambda_m. \tag{2.12}$$

When $n = 2m$ is even, we take

$$H_{D_m} = \sum_{r=1}^{m} \mathbb{C}(E_{r,r} - E_{m+r,m+r}) \tag{2.13}$$

as a Cartan subalgebra of $o'(2m, \mathbb{C})$,

$$\{E_{r,s} - E_{m+s,m+r}, E_{r,m+s} - E_{s,m+r} \mid 1 \leqslant r < s \leqslant m\} \tag{2.14}$$

as positive root vectors, and

$$\{E_{s,r} - E_{m+r,m+s}, E_{m+s,r} - E_{m+r,s} \mid 1 \leqslant r < s \leqslant m\} \tag{2.15}$$

as negative root vectors. Moreover, we define $\varepsilon_r \in H^*_{D_m}$ by

$$\varepsilon_r(E_{s,s} - E_{m+s,m+s}) = \delta_{r,s}. \tag{2.16}$$

Similarly, $H^*_{D_m} = \text{span}\{\varepsilon_1, \dots, \varepsilon_m\}$ and $\Phi = \{\pm(\varepsilon_r \pm \varepsilon_s) \mid r \neq s\}$ forms the root system of $o'(2m, \mathbb{C})$. Moreover, we define a symmetric bilinear form $(\cdot,\cdot)$ on $H^*_{D_m}$ by (2.10). Denote by $\alpha_r = \varepsilon_r - \varepsilon_{r+1}$ for $r = 1, \dots, m-1$ and $\alpha_m = \varepsilon_{m-1} + \varepsilon_m$. Then $\{\alpha_1, \dots, \alpha_m\}$ is the simple root system. Moreover, $\lambda_1, \dots, \lambda_m$ are the fundamental dominant weights in $H^*_{D_m}$ defined by (2.11). In this case,

$$\varepsilon_1 = \lambda_1, \quad \varepsilon_2 = -\lambda_1 + \lambda_2, \quad \dots, \quad \varepsilon_{m-2} = -\lambda_{m-3} + \lambda_{m-2}, \tag{2.17}$$

$$\varepsilon_{m-1} = -\lambda_{m-2} + \lambda_{m-1} + \lambda_m, \quad \varepsilon_m = -\lambda_{m-1} + \lambda_m. \tag{2.18}$$

Denote by I_m the $m \times m$ identity matrix. Set

$$K = \frac{1-i}{2} \begin{pmatrix} 1+i & 0 & 0 \\ 0 & iI_m & I_m \\ 0 & I_m & iI_m \end{pmatrix} \tag{2.19}$$

if $n = 2m+1$, and

$$K = \frac{1-i}{2} \begin{pmatrix} iI_m & I_m \\ I_m & iI_m \end{pmatrix} \tag{2.20}$$

when $n = 2m$. Then K is a symmetric matrix,

$$K^{-1} = \frac{1+i}{2} \begin{pmatrix} 1-i & 0 & 0 \\ 0 & -iI_m & I_m \\ 0 & I_m & -iI_m \end{pmatrix} \tag{2.21}$$

if $n = 2m+1$, and

$$K^{-1} = \frac{1+i}{2} \begin{pmatrix} -iI_m & I_m \\ I_m & -iI_m \end{pmatrix} \tag{2.22}$$

when $n = 2m$. Furthermore, we have a Lie algebra isomorphism $\sigma : o'(n, \mathbb{C}) \to o(n, \mathbb{C})$ given by

$$\sigma(X) = K^{-1} X K \quad \text{for } X \in o'(n, \mathbb{C}). \tag{2.23}$$

We define a representation ρ of $o'(n, \mathbb{C})$ on $\hat{\mathcal{A}}$ by

$$\rho(X)(\vec{f}) = \sigma(X)(\vec{f}) \quad \text{for } X \in o'(n, \mathbb{C}). \tag{2.24}$$

So $\hat{\mathcal{A}}$ forms an $o'(n, \mathbb{C})$-module.

In order to study the $o'(n, \mathbb{C})$-module structure of $\mathcal{A}_{\mathbb{C}}$, we let

$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \frac{1+i}{2} K^{-1} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \tag{2.25}$$

Define two representations ρ_1 and ρ_2 of $gl(n, \mathbb{C})$ on $\mathcal{A}_{\mathbb{C}} = \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[y_1, \ldots, y_n]$ by

$$\rho_1(E_{r,s}) = y_r \partial_{y_s}, \qquad \rho_2(E_{r,s}) = x_r \partial_{x_s}. \tag{2.26}$$

Writing $K = (b_{r,s})_{n\times n}$ and $K^{-1} = (c_{r,s})_{n\times n}$, we have

$$K^{-1} E_{p,q} K = \sum_{r,s=1}^{n} c_{s,p} b_{q,r} E_{s,r} \tag{2.27}$$

and

$$y_p \partial_{y_q} = \sum_{r,s=1}^{n} \frac{\partial y_p}{\partial x_s} \frac{\partial x_r}{\partial y_q} x_s \partial_{x_r} = \sum_{r,s=1}^{n} c_{s,p} b_{q,r} x_s \partial_{x_r}. \tag{2.28}$$

Thus

$$\rho_1(X)=\rho_2\big(K^{-1}XK\big)\quad \text{for } X\in gl(n,\mathbb{C}). \tag{2.29}$$

Recall $\varsigma_r=(0,\dots,0,\overset{r}{1},0,\dots,0)^T$. Furthermore, we set

$$\begin{pmatrix}\kappa_1\\ \vdots\\ \kappa_n\end{pmatrix}=\frac{1+i}{2}K^{-1}\begin{pmatrix}\varsigma_1\\ \vdots\\ \varsigma_n\end{pmatrix}. \tag{2.30}$$

Then

$$\hat{\mathcal{A}}_{\mathbb{C}}=\sum_{r=1}^{n}\mathcal{A}_{\mathbb{C}}\varsigma_r=\sum_{s=1}^{n}\mathcal{A}_{\mathbb{C}}\kappa_s. \tag{2.31}$$

We define two representations $\hat\rho_1$ and $\hat\rho_2$ of $gl(n,\mathbb{C})$ on $\hat{\mathcal{A}}_{\mathbb{C}}$ by

$$\hat\rho_1(X)\left(\sum_{r=1}^{n}f_r\kappa_r\right)=\sum_{r=1}^{n}\rho_1(X)(f_r)\kappa_r+(f_1,\dots,f_n)X^T\begin{pmatrix}\kappa_1\\ \vdots\\ \kappa_n\end{pmatrix}, \tag{2.32}$$

$$\hat\rho_2(X)\left(\sum_{r=1}^{n}g_r\varsigma_r\right)=\sum_{r=1}^{n}\rho_2(X)(g_r)\varsigma_r+(g_1,\dots,g_n)X^T\begin{pmatrix}\varsigma_1\\ \vdots\\ \varsigma_n\end{pmatrix} \tag{2.33}$$

for $f_r,g_r\in\mathcal{A}_{\mathbb{C}}$ and $X\in gl(n,\mathbb{C})$. By (2.27)–(2.31), we have

$$\hat\rho_1(X)=\hat\rho_2\big(K^{-1}XK\big)\quad \text{for } X\in gl(n,\mathbb{C}). \tag{2.34}$$

According to (2.2), (2.23), (2.24) and (2.34), we obtain:

Lemma 2.1. *For $X\in o'(n,\mathbb{C})$, $\rho(X)=\hat\rho_1(X)$.*

Denote by $\mathcal{A}_{\mathbb{C},k}$ the subspace of the polynomials with degree k in $\mathcal{A}_{\mathbb{C}}$. Set

$$V=\mathbb{R}^n=\bigoplus_{s=1}^{n}\mathbb{R}\varsigma_s,\qquad V_{\mathbb{C}}=\mathbb{C}^n=\bigoplus_{r=1}^{n}\mathbb{C}\kappa_r=\bigoplus_{s=1}^{n}\mathbb{C}\varsigma_s, \tag{2.35}$$

$$\mathcal{H}'_k=\left\{f\in\mathcal{A}_{\mathbb{C},k}\;\middle|\;\left(\partial_{y_1}^2+2\sum_{r=1}^{m}\partial_{y_{r+1}}\partial_{y_{m+r+1}}\right)(f)=0\right\} \tag{2.36}$$

if $n=2m+1$, and

$$\mathcal{H}'_k=\left\{f\in\mathcal{A}_{\mathbb{C},k}\;\middle|\;\left(\sum_{r=1}^{m}\partial_{y_r}\partial_{y_{m+r}}\right)(f)=0\right\} \tag{2.37}$$

if $n=2m$. It can be verified that

$$\mathcal{H}'_k=\mathbb{C}\otimes_{\mathbb{R}}\mathcal{H}_k=\left\{f\in\mathcal{A}_{\mathbb{C},k}\;\Big|\;\left(\sum_{r=1}^{n}\partial_{x_r}^2\right)(f)=0\right\} \tag{2.38}$$

(also see (2.68)). Now V forms an $o(n,\mathbb{R})$-submodule of $\hat{\mathcal{A}}$ that gives the basic representation with the canonical basis $\{\varsigma_1,\dots,\varsigma_n\}$ (cf. (2.2)). Moreover, $V_{\mathbb{C}}$ forms an $o'(n,\mathbb{C})$-submodule of $\hat{\mathcal{A}}_{\mathbb{C}}$ that gives the basic representation with the canonical basis $\{\kappa_1,\dots,\kappa_n\}$ (cf. (2.30)). Furthermore, $\mathcal{A}$ forms an $o(n,\mathbb{R})$-module with the representation $\rho_2|_{o(n,\mathbb{R})}$, and $\mathcal{A}_{\mathbb{C}}$ forms an $o'(n,\mathbb{C})$-module with the representation $\rho_1|_{o'(n,\mathbb{C})}$ (cf. (2.26)). As a real $o(n,\mathbb{R})$-module, $\hat{\mathcal{A}}_k=\mathcal{A}_kV\cong\mathcal{A}_k\otimes_{\mathbb{R}}V$ (cf. (2.2)). On the other hand, $\hat{\mathcal{A}}_{\mathbb{C},k}=\mathcal{A}_{\mathbb{C},k}V_{\mathbb{C}}\cong\mathcal{A}_{\mathbb{C},k}\otimes_{\mathbb{C}}V_{\mathbb{C}}$ as a complex $o'(n,\mathbb{C})$-module by the above lemma. We want to decompose $\hat{\mathcal{A}}_k$ as a direct sum of real irreducible $o(n,\mathbb{R})$-submodules via decomposing $\hat{\mathcal{A}}_{\mathbb{C},k}$ as a direct sum of complex irreducible $o'(n,\mathbb{C})$-submodules. To study the complex $o'(n,\mathbb{C})$-module $\mathcal{A}_{\mathbb{C},k}\otimes_{\mathbb{C}}V_{\mathbb{C}}$, we recall a fact about the tensor product of modules for a finite-dimensional complex semisimple Lie algebra $\mathcal{G}$.

Denote by $V(\lambda)$ a finite-dimensional irreducible $\mathcal{G}$-module with highest weight λ, by Λ the weight lattice of $\mathcal{G}$ and by Λ^+ the set of dominant weights. Moreover, $V_\nu(\lambda)$ stands for the weight space of $V(\lambda)$ with the weight ν.

Lemma 2.2. *(E.g., cf. [2].) The irreducible representations occurring in $V(\lambda)\otimes V(\mu)$ have highest weights of the form $\mu+\nu\in\Lambda^+$, where ν is a weight in $V(\lambda)$. Moreover, the multiplicity of $V(\mu+\nu)$ in $V(\lambda)\otimes V(\mu)$ is less than or equal to* $\dim V_\nu(\lambda)$.

The weights of $o'(2m,\mathbb{C})$-module $V_{\mathbb{C}}\cong V(\lambda_1)$ with $m\geqslant 3$ are

$$\varepsilon_1\succ\varepsilon_2\succ\cdots\succ\varepsilon_{m-2}\succ\varepsilon_{m-1}\begin{matrix}\succ\varepsilon_m\\ \succ-\varepsilon_m\end{matrix}\succ-\varepsilon_{m-1}\succ\cdots\succ-\varepsilon_1. \tag{2.39}$$

Moreover, the weights of $o'(2m+1,\mathbb{C})$-module $V_{\mathbb{C}}\cong V(\lambda_1)$ with $m\geqslant 2$ are

$$\varepsilon_1\succ\varepsilon_2\succ\cdots\succ\varepsilon_{m-2}\succ\varepsilon_{m-1}\succ\varepsilon_m\succ 0\succ-\varepsilon_m\succ-\varepsilon_{m-1}\succ\cdots\succ-\varepsilon_2\succ-\varepsilon_1. \tag{2.40}$$

The weights of $o'(4,\mathbb{C})$-module $V_{\mathbb{C}}\cong V(\lambda_1+\lambda_2)$ are

$$\lambda_1+\lambda_2\begin{matrix}\succ-\lambda_1+\lambda_2\\ \succ\lambda_1-\lambda_2\end{matrix}\succ-\lambda_1-\lambda_2. \tag{2.41}$$

The weights of $o'(3,\mathbb{C})$-module $V_{\mathbb{C}}\cong V(2\lambda_1)$ are

$$2\lambda_1\succ 0\succ-2\lambda_1. \tag{2.42}$$

Each weight in $V_{\mathbb{C}}$ is with multiplicity 1 (cf. (2.9) and (2.16)).

It is well known that

$$\mathcal{A}_{\mathbb{C},k}=\mathcal{H}'_k\oplus\left(y_1^2+2\sum_{r=1}^{2m}y_{r+1}y_{m+r+1}\right)\mathcal{A}_{\mathbb{C},k-2} \tag{2.43}$$

if $n=2m+1$, and

$$\mathcal{A}_{\mathbb{C},k} = \mathcal{H}'_k \oplus \left(\sum_{r=1}^{2m} y_r y_{m+r}\right)\mathcal{A}_{\mathbb{C},k-2} \tag{2.44}$$

if $n = 2m$. Moreover, $y_1^2 + 2\sum_{r=1}^{2m} y_{r+1}y_{m+r+1}$ is an $o'(2m+1,\mathbb{C})$-invariant and $\sum_{r=1}^{2m} y_r y_{m+r}$ is an $o'(2m,\mathbb{C})$-invariant. Furthermore, $\mathcal{H}'_k \cong V(k\lambda_1)$ if $n > 4$, $\mathcal{H}'_k \cong V(2k\lambda_1)$ if $n = 3$ and $\mathcal{H}'_k \cong V(k\lambda_1 + k\lambda_2)$ if $n = 4$.

Lemma 2.3.

(i) *As $o'(n,\mathbb{C})$-modules with $n \geqslant 7$,*

$$\mathcal{H}'_k V_{\mathbb{C}} \cong \mathcal{H}'_k \otimes_{\mathbb{C}} V_{\mathbb{C}} \cong V\big((k+1)\lambda_1\big) \oplus V\big((k-1)\lambda_1 + \lambda_2\big) \oplus V\big((k-1)\lambda_1\big). \tag{2.45}$$

(ii) *For $o'(6,\mathbb{C})$,*

$$\mathcal{H}'_k V_{\mathbb{C}} \cong \mathcal{H}'_k \otimes_{\mathbb{C}} V_{\mathbb{C}} \cong V\big((k+1)\lambda_1\big) \oplus V\big((k-1)\lambda_1 + \lambda_2 + \lambda_3\big) \oplus V\big((k-1)\lambda_1\big). \tag{2.46}$$

(iii) *For $o'(5,\mathbb{C})$,*

$$\mathcal{H}'_k V_{\mathbb{C}} \cong \mathcal{H}'_k \otimes_{\mathbb{C}} V_{\mathbb{C}} \cong V\big((k+1)\lambda_1\big) \oplus V\big((k-1)\lambda_1 + 2\lambda_2\big) \oplus V\big((k-1)\lambda_1\big). \tag{2.47}$$

(iv) *For $o'(4,\mathbb{C})$,*

$$\begin{aligned}\mathcal{H}'_k V_{\mathbb{C}} &\cong \mathcal{H}'_k \otimes_{\mathbb{C}} V_{\mathbb{C}} \\ &\cong V\big((k+1)(\lambda_1+\lambda_2)\big) \oplus V\big((k-1)\lambda_1 + (k+1)\lambda_2\big) \oplus V\big((k+1)\lambda_1 + (k-1)\lambda_2\big) \\ &\quad \oplus V\big((k-1)(\lambda_1+\lambda_2)\big). \end{aligned} \tag{2.48}$$

(v) *For $o'(3,\mathbb{C})$,*

$$\mathcal{H}'_k V_{\mathbb{C}} \cong \mathcal{H}'_k \otimes_{\mathbb{C}} V_{\mathbb{C}} \cong V\big(2(k+1)\lambda_1\big) \oplus V(2k\lambda_1) \oplus V\big(2(k-1)\lambda_1\big). \tag{2.49}$$

Proof. (i) First we consider the case that $n = 2m$ is even with $m \geqslant 4$. Lemma 2.2, (2.17), (2.18) and (2.39) tell us that the irreducible modules may occur in $V(k\lambda_1) \otimes V(\lambda_1)$ are $V((k+1)\lambda_1)$, $V((k-1)\lambda_1 + \lambda_2)$ and $V((k-1)\lambda_1)$. The multiplicities of them are all less than or equal to 1. According to the dimension formula of finite-dimensional irreducible modules for a complex finite-dimensional simple Lie algebra (e.g., cf. p. 139 in [3]), we have

$$\begin{aligned}\dim V\big((k-1)\lambda_1 + \lambda_2\big) &= \frac{\prod_{\alpha \succ 0}((k-1)\lambda_1 + \lambda_2 + \delta, \alpha)}{\prod_{\alpha \succ 0}(\delta, \alpha)} \\ &= \frac{k(k+2m-2)\prod_{s=3}^m (k+2m-1-s)(s-1)\prod_{j=3}^m (2m-1-j)(j-1)}{(2m-3)\prod_{s=3}^m (2m-1-s)(s-1)\prod_{j=3}^m (2m-2-j)(j-2)} \\ &= \frac{2k(k+2m-2)(k+m-1)(k+2m-4)!}{(2m-3)!(k+1)!} \\ &= \frac{k(k+n-2)(2k+n-2)(k+n-4)!}{(n-3)!(k+1)!}, \end{aligned} \tag{2.50}$$

where $\delta = \sum_{r=1}^m \lambda_r$. Moreover,

$$\begin{aligned}
&\dim\big(V(k\lambda_1)\otimes V(\lambda_1)\big)-\dim V\big((k+1)\lambda_1\big)-\dim V\big((k-1)\lambda_1\big)\\
&\quad=n\binom{k+n-3}{n-2}+n\binom{k+n-2}{n-2}-\binom{k+1+n-3}{n-2}\\
&\qquad-\binom{k+1+n-2}{n-2}-\binom{k-1+n-3}{n-2}-\binom{k-1+n-2}{n-2}\\
&\quad=\frac{k(k+n-2)(2k+n-2)(k+n-4)!}{(n-3)!(k+1)!}\\
&\quad=\dim V\big((k-1)\lambda_1+\lambda_2\big).
\end{aligned}\tag{2.51}$$

Hence all the three modules occur in $\mathcal{H}'_k\otimes_{\mathbb{C}}V_{\mathbb{C}}$.

Next we consider the case that $n=2m+1$ is odd with $m\geqslant 3$. The irreducible modules that may occur in $V(k\lambda_1)\otimes V(\lambda_1)$ are $V((k+1)\lambda_1)$, $V(k\lambda_1)$, $V((k-1)\lambda_1+\lambda_2)$ and $V((k-1)\lambda_1)$. Their multiplicities are all less than or equal to 1 by Lemma 2.2, (2.12) and (2.40). Note that $V((k+1)\lambda_1)$ must occur. Since

$$\begin{aligned}
&\dim V\big((k-1)\lambda_1+\lambda_2\big)\\
&\quad=\frac{\prod_{\alpha\succ 0}((k-1)\lambda_1+\lambda_2+\delta,\alpha)}{\prod_{\alpha\succ 0}(\delta,\alpha)}\\
&\quad=\frac{k(k+2m-1)(k+m-\frac{1}{2})\prod_{s=3}^{m}(k+2m-s)(k+s-1)\prod_{j=3}^{m}(2m-j)(j-1)}{(2m-2)(m-\frac{3}{2})\prod_{s=3}^{m}(2m-s)(s-1)\prod_{j=3}^{m}(2m-1-j)(j-2)}\\
&\quad=\frac{2k(k+2m-1)(k+m-\frac{1}{2})(k+2m-3)!}{(2m-2)!(k+1)!}\\
&\quad=\frac{k(k+n-2)(2k+n-2)(k+n-4)!}{(n-3)!(k+1)!}\\
&\quad=\dim\big(V(k\lambda_1)\otimes V(\lambda_1)\big)-\dim V\big((k+1)\lambda_1\big)-\dim V\big((k-1)\lambda_1\big),
\end{aligned}\tag{2.52}$$

at most two of $\{V(k\lambda_1), V((k-1)\lambda_1+\lambda_2), V((k-1)\lambda_1)\}$ occur. So (2.45) with $n=2m+1$ follows from the fact

$$\dim V\big((k-1)\lambda_1+\lambda_2\big)>\dim V(k\lambda_1)>\dim V\big((k-1)\lambda_1\big).\tag{2.53}$$

(ii) It follows by a similar argument as that of $n=2m$ in (i).

(iii) It is obtained by a similar argument as that of $n=2m+1$ in (i).

(iv) The irreducible modules may occur in $\mathcal{H}'_k\otimes V_{\mathbb{C}}$ are the four ones in the right side of (2.48) by Lemma 2.2. The conclusion follows from the fact

$$\dim V\big((k+1)(\lambda_1+\lambda_2)\big)=(k+2)^2,\qquad \dim V\big((k-1)(\lambda_1+\lambda_2)\big)=k^2,\tag{2.54}$$

$$\dim V\big((k+1)\lambda_1+(k-1)\lambda_2\big)=\dim V\big((k-1)\lambda_1+(k+1)\lambda_2\big)=k(k+2)\tag{2.55}$$

and

$$\dim V\big(k(\lambda_1+\lambda_2)\big)\otimes V(\lambda_1+\lambda_2)=4(k+1)^2.\tag{2.56}$$

(v) The set of weights occurring in $V(2\lambda_1)$ of $o'(3,\mathbb{C})$ is $\Pi(\lambda_1)=\{2\lambda_1,0,-2\lambda_1\}$. The statement holds because $\dim V(2k\lambda_1)=2k+1$ for any $k\geqslant 1$. $\square$

For convenience, we treat κ_r and ς_r as variables. With these notations, we may write

$$(E_{r,s}-E_{s,r})\left(\sum_{j=1}^{n} f_j\varsigma_j\right)=(x_r\partial_{x_s}-x_s\partial_{x_r}+\varsigma_r\partial_{\varsigma_s}-\varsigma_s\partial_{\varsigma_r})\left(\sum_{j=1}^{n} f_j\varsigma_j\right) \tag{2.57}$$

for $E_{r,s}-E_{s,r}\in o(n,\mathbb{R})$.

We define the bar operation on $\hat{\mathcal{A}}_{\mathbb{C}}$ by

$$\overline{\sum_{r_1,\dots,r_n}^{\infty}\sum_{j=1}^{n} a_{r_1,\dots,r_n,j}x_1^1\cdots x_n^n\varsigma_j}=\sum_{r_1,\dots,r_n}^{\infty}\sum_{j=1}^{n}\overline{a_{r_1,\dots,r_n,j}}x_1^1\cdots x_n^n\varsigma_j,\quad a_{r_1,\dots,r_n,j}\in\mathbb{C}. \tag{2.58}$$

For a transformation T on $\hat{\mathcal{A}}_{\mathbb{C}}$, we define its conjugate operator by

$$\overline{T}(\vec{f})=\overline{T(\vec{f})}\quad \text{for } \vec{f}\in\hat{\mathcal{A}}_{\mathbb{C}}. \tag{2.59}$$

For expository convenience, we shift the subscripts by -1 when $n=2m+1$ is odd; for instance,

$$x_r\to x_{r-1},\quad y_r\to y_{r-1},\quad \varsigma_r\to\varsigma_{r-1},\quad \kappa_r\to\kappa_{r-1},\quad E_{r,s}\to E_{r-1,s-1} \tag{2.60}$$

for $1\leqslant r,s\leqslant 2m+1$.

In this way, we always have

$$y_r=\frac{x_r+ix_{m+r}}{2},\qquad y_{m+r}=\frac{ix_r+x_{m+r}}{2} \tag{2.61}$$

and

$$\kappa_r=\frac{\varsigma_r+i\varsigma_{m+r}}{2},\qquad \kappa_{m+r}=\frac{i\varsigma_r+\varsigma_{m+r}}{2} \tag{2.62}$$

for $1\leqslant r\leqslant m$ by (2.25) and (2.30) in the both cases of $n=2m$ and $n=2m+1$. Moreover,

$$y_0=\frac{1+i}{2}x_0\quad\text{and}\quad \kappa_0=\frac{1+i}{2}\varsigma_0\quad \text{if } n=2m+1. \tag{2.63}$$

Note that

$$y_{m+r}=i\overline{y_r}\quad\text{and}\quad \kappa_{m+r}=i\overline{\kappa_r}\quad \text{for } 1\leqslant r\leqslant m, \tag{2.64}$$

also

$$\overline{y_0}=-iy_0\quad\text{and}\quad \overline{\kappa_0}=-i\kappa_0\quad \text{if } n=2m+1. \tag{2.65}$$

Furthermore, (2.61) yields

$$\partial_{x_r}=\frac{\partial_{y_r}+i\partial_{y_{m+r}}}{2},\qquad \partial_{x_{m+r}}=\frac{i\partial_{y_r}+\partial_{y_{m+r}}}{2} \tag{2.66}$$

and (2.63) says

$$\partial_{x_0}=\frac{1+i}{2}\partial_{y_0}. \tag{2.67}$$

Thus the Laplacian operator

$$\Delta=\sum_{r=1}^{n}\partial_{x_r}^2=\begin{cases} i\sum_{r=1}^{m}\partial_{y_r}\partial_{y_{m+r}} & \text{if } n=2m;\\ i(\frac{1}{2}\partial_{y_0}^2+\sum_{r=1}^{m}\partial_{y_r}\partial_{y_{m+r}}) & \text{if } n=2m+1. \end{cases} \tag{2.68}$$

In particular, (2.38) holds. Expressions (2.64)–(2.67) also imply

$$\overline{\partial_{y_s}}=\partial_{\overline{y_s}},\qquad \overline{\partial_{\kappa_s}}=\partial_{\overline{\kappa_s}}\quad \text{for } 1\leqslant s\leqslant n. \tag{2.69}$$

A complex module W of $o(n,\mathbb{R})$ is of *real type* if there exists a real module W_0 of $o(n,\mathbb{R})$ satisfying $W=W_0\otimes_{\mathbb{R}}\mathbb{C}$. In this case, we call W_0 a *real form* of W.

Because we have to deal with the real module finally, we give a lemma to describe how to find a real form of a module in $\mathcal{H}'_kV_{\mathbb{C}}$. Recall that we have chosen a Cartan subalgebra and positive (negative) root vectors for $o'(n,\mathbb{C})$ in (2.6)–(2.9) and (2.14)–(2.16) with sub-indices shifted by -1. A *highest* (*lowest*) *weight vector* of an $o'(n,\mathbb{C})$-module is a weight vector nullified by positive (negative) root vectors.

Lemma 2.4. *Suppose that $W\subset\mathcal{H}'_kV_{\mathbb{C}}$ is an irreducible submodule of $o'(n,\mathbb{C})$. Then $W_0=\{\vec{h}\in W\mid \overline{\vec{h}}=\vec{h}\}$, which is an irreducible real $o(n,\mathbb{R})$-module, is a real form of W.*

Proof. Recall the action of $o'(n,\mathbb{C})$ on $\hat{\mathcal{A}}_{\mathbb{C}}$ by $\hat{\rho}_1$ given in (2.32). Observe that

$$\overline{y_s\partial_{y_r}-y_{m+r}\partial_{y_{m+s}}+\kappa_s\partial_{\kappa_r}-\kappa_{m+r}\partial_{\kappa_{m+s}}}=-(y_r\partial_{y_s}-y_{m+s}\partial_{y_{m+r}}+\kappa_r\partial_{\kappa_s}-\kappa_{m+s}\partial_{\kappa_{m+r}}), \tag{2.70}$$

$$\overline{y_{m+s}\partial_{y_r}-y_{m+r}\partial_{y_s}+\kappa_{m+s}\partial_{\kappa_r}-\kappa_{m+r}\partial_{\kappa_s}}=-(y_r\partial_{y_{m+s}}-y_s\partial_{y_{m+r}}+\kappa_r\partial_{\kappa_{m+s}}-\kappa_s\partial_{\kappa_{m+r}}). \tag{2.71}$$

Moreover,

$$\overline{y_r\partial_{y_0}-y_0\partial_{y_{m+r}}+\kappa_r\partial_{\kappa_0}-\kappa_0\partial_{\kappa_{m+r}}}=-(y_0\partial_{y_r}-y_{m+r}\partial_{y_0}+\kappa_0\partial_{\kappa_r}-\kappa_{m+r}\partial_{\kappa_0}) \tag{2.72}$$

if $n=2m+1$. Note for a positive root vector $A\in o'(n,\mathbb{C})$, its transpose A^T is a negative root vector. The above expressions says that

$$A^T(\overline{\vec{f}})=-\overline{A(\vec{f})} \tag{2.73}$$

for $\vec{f}\in\hat{\mathcal{A}}_{\mathbb{C}}$.

Now let $\vec{f}$ be a highest weight vector of W with weight λ. Then $\overline{\vec{f}}$ is a lowest weight vector of some submodule of $\mathcal{H}'_kV_{\mathbb{C}}$ by (2.70)–(2.72). By (2.36), (2.37) and (2.68), $\overline{\mathcal{H}'_k}=\mathcal{H}'_k$. Moreover, (2.64) and (2.65) imply the weight subspaces

$$\overline{(\mathcal{H}'_k)_{\mu_1}}=(\mathcal{H}'_k)_{-\mu_1},\qquad \overline{(V_{\mathbb{C}})_{\mu_2}}=(V_{\mathbb{C}})_{-\mu_2}. \tag{2.74}$$

Thus

$$(\mathcal{H}'_kV_{\mathbb{C}})_{\mu}=\bigoplus_{\mu_1+\mu_2=\mu}(\mathcal{H}'_k)_{\mu_1}(V_{\mathbb{C}})_{\mu_2}\quad\Rightarrow\quad \overline{(\mathcal{H}'_kV_{\mathbb{C}})_{\mu}}=(\mathcal{H}'_kV_{\mathbb{C}})_{-\mu}. \tag{2.75}$$

So $\overline{\vec{f}}$ is of weight $-\lambda$. Since the lowest weight of W is also $-\lambda$ by the highest weights listed in Lemma 2.3, the $o'(n,\mathbb{C})$-submodule generated by $\overline{\vec{f}}$ must be isomorphic to W. But the multiplicity of any $o'(n,\mathbb{C})$-irreducible submodule in $\mathcal{H}'_k V_{\mathbb{C}}$ is 1 by Lemma 2.3. Therefore, $\overline{\vec{f}} \in W$.

Recall that W is spanned by the elements of the form

$$\vec{g} = f_{r_1}^{a_1} f_{r_2}^{a_2} \cdots f_{r_m}^{a_m} \vec{f}, \tag{2.76}$$

where f_{r_j} is the operator of the r_jth negative simple root vector in $o'(n,\mathbb{C})$, $a_r \in \mathbb{N}$ and $1 \leqslant r_j \leqslant m$ for $n = 2m$ or $n = 2m+1$. Then

$$\begin{aligned}(-1)^{\sum_{r=1}^m a_r} \overline{\vec{g}} &= (-1)^{\sum_{r=1}^m a_r} \overline{f_{r_1}^{a_1} f_{r_2}^{a_2} \cdots f_{r_m}^{a_m} \vec{f}} \\ &= (-1)^{\sum_{r=1}^m a_r} \overline{f_{r_1}^{a_1}}\, \overline{f_{r_2}^{a_2}} \cdots \overline{f_{r_m}^{a_m}}\, \overline{\vec{f}} \\ &= e_{r_1}^{a_1} e_{r_2}^{a_2} \cdots e_{r_m}^{a_m} \overline{\vec{f}},\end{aligned} \tag{2.77}$$

where e_{r_j} is the operator of the r_jth positive simple root vector in $o'(n,\mathbb{C})$. It follows that the real and imaginary parts of $\vec{g}$ are in W whenever $\vec{g} \in W$. This shows $W = W_0 + iW_0$. By Lemma 2.1, W_0 must be a real irreducible $o(n,\mathbb{R})$-submodule. □

Now we want to find the highest weight vectors of the direct summands of $\mathcal{H}'_k V_{\mathbb{C}}$ in Lemma 2.3. We denote $\mathcal{H}'_k V_{\mathbb{C}} = W_1 \oplus W_2 \oplus W'_3$, where W_j (or W'_j) is correspondence to the jth direct summand in every equalities in Lemma 2.3 expect the case $n=4$. Let $W_2 = W_{2,-} \oplus W_{2,+}$ be the sum of the middle two submodules if $n=4$. According to Lemma 2.4, we have real irreducible $o(n,\mathbb{R})$-submodules $\hat{\mathcal{H}}_{k,1}$, $\hat{\mathcal{H}}_{k,2}$, $\hat{\mathcal{H}}'_{k,3}$ such that $W_j = \hat{\mathcal{H}}_{k,j} + i\hat{\mathcal{H}}_{k,j}$ for $j = 1,2$ and $W'_3 = \hat{\mathcal{H}}'_{k,3} + i\hat{\mathcal{H}}'_{k,3}$. In particular, $\mathcal{H}_k V = \hat{\mathcal{H}}_{k,1} \oplus \hat{\mathcal{H}}_{k,2} \oplus \hat{\mathcal{H}}'_{k,3}$ by (2.35) and (2.38). in the case of $n=4$, $\hat{\mathcal{H}}_{k,2\mp}$ are the similar real forms of $W_{2,\mp}$.

Lemma 2.5. (i) *The following vectors v_j (respectively v'_3) of W_j (respectively W'_3) and $v_{2,\mp}$ of $W_{2,\mp}$ are highest weight vectors*:

$$v_1 = y_1^k \kappa_1 = \frac{1}{2^{k+1}} (x_1 + i x_{m+1})^k (\varsigma_1 + i\varsigma_{m+1}), \tag{2.78}$$

$$v_2 = -y_2 y_1^{k-1} \kappa_1 + y_1^k \kappa_2 \quad \textit{for } n > 4, \tag{2.79}$$

$$v_{2,-} = -y_2 y_1^{k-1} \kappa_1 + y_1^k \kappa_2, \qquad v_{2,+} = -y_4 y_1^{k-1} \kappa_1 + y_1^k \kappa_4 \quad \textit{for } n = 4, \tag{2.80}$$

$$v_2 = -y_0 y_1^{k-1} \kappa_1 + y_1^k \kappa_0 \quad \textit{for } n = 3, \tag{2.81}$$

$$v'_3 = 2(k-1) \sum_{j=1}^m y_j y_{m+j} y_1^{k-2} \kappa_1 - (2k+n-4) \sum_{j=1}^m y_1^{k-1} (y_{m+j}\kappa_j + y_j \kappa_{m+j}) \tag{2.82}$$

for $n = 2m$,

$$\begin{aligned} v'_3 = {}& -(2k+n-4) y_0 y_1^{k-1} \kappa_0 + 2(k-1) \left(\sum_{j=1}^m y_j y_{m+j} + \frac{1}{2} y_0^2 \right) y_1^{k-2} \kappa_1 \\ & - (2k+n-4) \sum_{j=1}^m y_1^{k-1} (y_{m+j}\kappa_j + y_j \kappa_{m+j}) \end{aligned} \tag{2.83}$$

for $n = 2m+1$.

(ii) *The vectors* v_1, v_2 *and* $v_{2,\mp}$ *are solutions of Navier equations* (1.1). *Moreover,* $\hat{\mathcal{H}}_{k,1}$ *and* $\hat{\mathcal{H}}_{k,2}$ *are subspaces of the solution space.*

Proof. (a) We want to find the highest weight vectors of W_1. The action of positive root vectors of $o'(2m,\mathbb{C})$ on $\hat{\mathcal{A}}_{\mathbb{C}}$ are operators:

$$y_r\partial_{y_s} - y_{m+s}\partial_{y_{m+r}} + \kappa_r\partial_{\kappa_s} - \kappa_{m+s}\partial_{\kappa_{m+r}} \tag{2.84}$$

and

$$y_r\partial_{y_{m+s}} - y_s\partial_{y_{m+r}} + \kappa_r\partial_{\kappa_{m+s}} - \kappa_s\partial_{\kappa_{m+r}} \tag{2.85}$$

for $1 \leqslant r < s \leqslant m$ (cf. (2.14) and (2.57)). They annihilate v_1. In the case of $n = 2m+1$, we have additional operators of positive roots:

$$y_r\partial_{y_0} - y_0\partial y_{m+r} + \kappa_r\partial_{\kappa_0} - \kappa_0\partial\kappa_{m+r}. \tag{2.86}$$

Then (2.84)–(2.86) annihilate v_1.

Since y_1^k is harmonic, v_1 is a singular vector of $\mathcal{H}'_k V_{\mathbb{C}}$. The weight of y_1^k is $k\lambda_1$ if $n > 4$, is $k(\lambda_1 + \lambda_2)$ if $n = 4$ and is $2k\lambda_1$ if $n = 3$. The weight of κ_1 is λ_1 if $n > 4$, is $\lambda_1 + \lambda_2$ if $n = 4$ and is $2\lambda_1$ if $n = 3$. Then the weight of v_1 is $(k+1)\lambda_1$ if $n > 4$, is $(k+1)(\lambda_1 + \lambda_2)$ if $n = 4$ and is $2(k+1)\lambda_1$ if $n = 3$, which implies that v_1 is the highest weight vector of W_1. Furthermore, $\Delta(v_1) + b(\nabla^T\nabla)(v_1) = b(\nabla^T\nabla)(v_1)$. But

$$\begin{aligned} \nabla(v_1) &= \frac{1}{2}\partial_{x_1}\big(y_1^k\big) + \frac{i}{2}\partial_{x_{m+1}}\big(y_1^k\big) \\ &= \frac{k}{2^{k+1}}(x_1 + ix_{m+1})^{k-1} + i^2\frac{k}{2^{k+1}}(x_1 + ix_{m+1})^{k-1} = 0, \end{aligned} \tag{2.87}$$

which implies that v_1 is a solution of Navier equations. By the invariant property of Navier equations under $o(n,\mathbb{R})$, $\hat{\mathcal{H}}_{k,1}$ is a solution subspace.

(b) Now we approach the highest weight vectors in W_2. It is obviously that $v_2 \in \mathcal{H}'_k V_{\mathbb{C}}$.

If $m > 1$, then

$$\begin{aligned} &(E_{r,s} - E_{m+s,m+r})(v_2) \\ &\quad = (y_r\partial_{y_s} - y_{m+s}\partial_{y_{m+r}} + \kappa_r\partial_{\kappa_s} - \kappa_{m+s}\partial_{\kappa_{m+r}})\big({-y_2y_1^{k-1}\kappa_1} + y_1^k\kappa_2\big) = 0 \end{aligned} \tag{2.88}$$

for $2 \leqslant r < s \leqslant m$,

$$\begin{aligned} (E_{1,2} - E_{m+2,m+1})(v_2) &= (y_1\partial_{y_2} - y_{m+2}\partial_{y_{m+1}} + \kappa_1\partial_{\kappa_2} - \kappa_{m+2}\partial_{\kappa_{m+1}})\big({-y_2y_1^{k-1}\kappa_1} + y_1^k\kappa_2\big) \\ &= -y_1^k\kappa_1 + y_1^k\kappa_1 = 0 \end{aligned} \tag{2.89}$$

and

$$\begin{aligned} &(E_{r,m+s} - E_{s,m+r})(v_2) \\ &\quad = (y_r\partial_{y_{m+s}} - y_s\partial_{y_{m+r}} + \kappa_r\partial_{\kappa_{m+s}} - \kappa_s\partial_{\kappa_{m+r}})\big({-y_2y_1^{k-1}\kappa_1} + y_1^k\kappa_2\big) = 0. \end{aligned} \tag{2.90}$$

Moreover, if $n = 2m+1$ is odd,

$$\begin{aligned}&(E_{r,0}-E_{0,m+r})(v_2)\\&\quad=(y_r\partial_{y_0}-y_0\partial y_{m+r}+\kappa_r\partial_{\kappa_0}-\kappa_0\partial\kappa_{m+r})\big(-y_2y_1^{k-1}\kappa_1+y_1^k\kappa_2\big)=0.\end{aligned}\tag{2.91}$$

For $n=3$, $v_2=-y_0y_1^{k-1}\kappa_1+y_1^k\kappa_0$. The operator of positive root vector of $o'(3,\mathbb{C})$ is

$$y_1\partial_{y_0}-y_0\partial_{y_2}+\kappa_1\partial_{\kappa_0}-\kappa_0\partial_{\kappa_2}\tag{2.92}$$

which also annihilates v_2. In the case of $n=4$, we have that

$$\begin{aligned}&(y_1\partial_{y_4}-y_2\partial_{y_3}+\kappa_1\partial_{\kappa_4}-\kappa_2\partial_{\kappa_3})(v_{2,+})\\&\quad=(y_1\partial_{y_4}+\kappa_1\partial_{\kappa_4})\big(-y_4y_1^{k-1}\kappa_1+y_1^k\kappa_4\big)=-y_1^k\kappa_1+y_1^k\kappa_1=0\end{aligned}\tag{2.93}$$

and similarly for $v_{2,-}$.

Note that the weight of v_2 is $(k+1)\lambda_1-\alpha_1$ ($2(k+1)\lambda_1-\alpha_1$ if $n=3$). So it is the highest weight of W_2. Moreover, the weight of vector $v_{2,-}$ (respectively $v_{2,+}$) is $(k+1)(\lambda_1+\lambda_2)-\alpha_1$ (respectively $(k+1)(\lambda_1+\lambda_2)-\alpha_2$). Hence it is the highest weight of $W_{2,-}$ (respectively $W_{2,+}$).

In the case of $n\neq 3$,

$$\begin{aligned}2^{k+1}\nabla(v_2)&=\partial_{x_1}\big(-(x_2+ix_{m+2})(x_1+ix_{m+1})^{k-1}\big)+\partial_{x_2}(x_1+ix_{m+1})^k\\&\quad+\partial_{x_{m+1}}\big(-i(x_2+ix_{m+2})(x_1+ix_{m+1})^{k-1}\big)+\partial_{x_{m+2}}\big(i(x_1+ix_{m+1})^k\big)\\&=-(k-1)(x_2+ix_{m+2})(x_1+ix_{m+1})^{k-2}\\&\quad-i^2(k-1)(x_2+ix_{m+2})(x_1+ix_{m+1})^{k-2}=0\end{aligned}\tag{2.94}$$

and it is easy to verify $\nabla(v_{2,-})=0$. Moreover,

$$\begin{aligned}2^{k+1}\nabla(v_{2,+})&=\partial_{x_1}\big(-(ix_2+x_4)(x_1+ix_3)^{k-1}\big)+\partial_{x_2}\big(i(x_1+ix_3)^k\big)\\&\quad+\partial_{x_3}\big(-i(ix_2+x_4)(x_1+ix_3)^{k-1}\big)+\partial_{x_4}\big((x_1+ix_3)^k\big)=0,\end{aligned}\tag{2.95}$$

in the case of $n=3$,

$$\begin{aligned}2^{k+1}\nabla(v_2)&=\partial_{x_0}\big((x_1+ix_2)^{k-1}(1+i)\big)+\partial_{x_1}\big(-(1+i)x_0(x_1+ix_2)^{k-1}\big)\\&\quad+\partial_{x_2}\big(-(1+i)ix_0(x_1+ix_2)^{k-1}\big)=0.\end{aligned}\tag{2.96}$$

Since y_1^k, $y_2y_1^{k-1}$ and $y_0y_1^{k-1}$ are harmonic, $\Delta(v_2)+b(\nabla^T\nabla)(v_2)=b(\nabla^T\nabla)(v_2)=0$. Similarly, the same equation holds for $v_{2,\mp}$. That is, v_2 and $v_{2,\mp}$ are solutions of Navier equations. Hence $\hat{\mathcal{H}}_{k,2}$ is a solution subspace.

(c) We now deal with the highest weight vectors of W_3'.

In the case of $n=2m$, observe that $E_{m-1,2m}-E_{m,2m-1}$ annihilate v_3' and

$$\begin{aligned}&(y_l\partial_{y_{l+1}}-y_{m+l+1}\partial_{y_{m+l}}+\kappa_l\partial_{\kappa_{l+1}}-\kappa_{m+l+1}\partial_{\kappa_{m+l}})\big(v_3'\big)\\&\quad=2(k-1)y_ly_{m+l+1}y_1^{k-2}\kappa_1-(2k+n-4)y_ly_1^{k-1}\kappa_{m+l+1}-2(k-1)y_{m+l+1}y_ly_1^{k-2}\kappa_1\\&\qquad+(2k+n-4)y_{m+l+1}y_1^{k-1}\kappa_l-(2k+n-4)y_{m+l+1}y_1^{k-1}\kappa_l+(2k+n-4)y_ly_1^{k-1}\kappa_{m+l+1}\\&\quad=0\end{aligned}\tag{2.97}$$

for $l=1,\dots,m-1$. On the other hand,

$$\Delta\left(2(k-1)\sum_{j=1}^{m} y_j y_{m+j} y_1^{k-2} - (2k+n-4) y_{m+1} y_1 y_1^{k-2}\right)$$
$$= i\big(2(k-1)(k-1) - (2k+n-4)(k-1) + 2(k-1)(m-1)\big) y_1^{k-2} = 0, \tag{2.98}$$

and $\Delta y_j y_1^{k-2} = 0$ for $j \neq m+1$ by (2.68). That is, $v_3' \in \mathcal{H}_k' V_{\mathbb{C}}$ is harmonic, and so it is a singular vector of $\mathcal{H}_k' V_{\mathbb{C}}$ by (a), (b) and Lemma 2.3.

In the case of $n=2m+1$, we similarly have that

$$(y_l \partial_{y_{l+1}} - y_{m+l+1}\partial_{y_{m+l}} + \kappa_l \partial_{\kappa_{l+1}} - \kappa_{m+l+1}\partial_{\kappa_{m+l}})\big(v_3'\big) = 0 \tag{2.99}$$

and

$$\begin{aligned}
&(y_r \partial_{y_0} - y_0 \partial_{y_{m+r}} + \kappa_r \partial_{\kappa_0} - \kappa_0 \partial_{\kappa_{m+r}})\big(v_3'\big)\\
&= -(2k+n-4) y_r y_1^{k-1}\kappa_0 + 2(k-1) y_r y_0 y_1^{k-2}\kappa_1 - 2(k-1) y_r y_0 y_1^{k-2}\kappa_1\\
&\quad + (2k+n-4) y_0 y_1^{k-1}\kappa_r - (2k+n-4) y_0 y_1^{k-1}\kappa_r + (2k+n-4) y_r y_1^{k-1}\kappa_0\\
&= 0.
\end{aligned} \tag{2.100}$$

In addition,

$$\Delta\left(2(k-1)\left(\sum_{j=1}^{m} y_j y_{m+j} + \frac{1}{2} y_0^2\right) y_1^{k-2} - (2k+n-4) y_{m+1} y_1 y_1^{k-2}\right)$$
$$= \frac{i}{2}(k-1)\big(2n+4(k-2)\big) y_1^{k-2} - i(2k+n-4)(k-1) y_1^{k-2} = 0 \tag{2.101}$$

and $\Delta y_j y_1^{k-2} = 0$ for $j \neq m+1$ by (2.68). That is, $v_3' \in \mathcal{H}_k' V_{\mathbb{C}}$ is harmonic, and so it is a singular vector of $\mathcal{H}_k' V_{\mathbb{C}}$ by (a), (b) and Lemma 2.3. Finally, one gets $v_3' \in W_3'$ by checking the weight of v_3'. □

Set

$$v_3'' = 2\sum_{j=1}^{m} y_j y_{m+j} y_1^{k-2}\kappa_1 \quad \text{if } n = 2m \tag{2.102}$$

and

$$v_3'' = \left(2\sum_{j=1}^{m} y_j y_{m+j} + y_0^2\right) y_1^{k-2}\kappa_1 \quad \text{if } n = 2m+1. \tag{2.103}$$

Since $[x_r\partial_{x_s} - x_s\partial_{x_r}, \sum_{l=1}^{n} x_l^2] = 0$, the irreducible module with v_3'' as its highest weight vector is isomorphic to $V((k-1)\lambda_1)$ for $n > 4$, $V((k-1)(\lambda_1+\lambda_2))$ for $n=4$ and $V(2(k-1)\lambda_1)$ for $n=3$. Let

$$c = \frac{(2k+n-2)(k+n-3)(k-1)}{2(b^{-1}(2k+n-4)+k-1)}. \tag{2.104}$$

Lemma 2.6. *The vector $v_3 = v_3' + cv_3''$ is a complex solution of Navier equations. Then the complex irreducible $o(n,\mathbb{C})$-module W_3 generated by v_3 is a complex solution subspace of Navier equations.*

Proof. Suppose $n = 2m$. We have

$$v_3' = 2(k-1)\sum_{j=1}^{m} y_j y_{m+j} y_1^{k-2}\kappa_1 - (2k+n-4)\sum_{j=1}^{m} y_1^{k-1}(y_{m+j}\kappa_j + y_j\kappa_{m+j})$$
$$= \frac{i}{2^k}(k-1)\sum_{j=1}^{n} x_j^2(x_1+ix_{m+1})^{k-2}\varsigma_1 - \frac{1}{2^k}(k-1)\sum_{j=1}^{n} x_j^2(x_1+ix_{m+1})^{k-2}\varsigma_{m+1}$$
$$- i\frac{2k+n-4}{2^k}\sum_{s=1}^{n} x_s(x_1+ix_{m+1})^{k-1}\varsigma_s, \tag{2.105}$$

$$v_3'' = \sum_{j=1}^{m} y_j y_{m+j} y_1^{k-2}\varsigma_1 + i\sum_{j=1}^{m} y_j y_{m+j} y_1^{k-2}\varsigma_{m+1}$$
$$= \frac{i}{2^k}\sum_{j=1}^{n} x_j^2(x_1+ix_{m+1})^{k-2}\varsigma_1 - \frac{1}{2^k}\sum_{j=1}^{n} x_j^2(x_1+ix_{m+1})^{k-2}\varsigma_{m+1}. \tag{2.106}$$

Hence

$$\Delta(v_3') + b(\nabla^T\nabla)(v_3') = -\frac{i(k-1)}{2^k}b(k+n-3)(2k+n-2)(x_1+ix_{m+1})^{k-2}\varsigma_1$$
$$+ \frac{(k-1)}{2^k}b(k+n-3)(2k+n-2)(x_1+ix_{m+1})^{k-2}\varsigma_{m+1}, \tag{2.107}$$

$$\Delta(v_3'') + b(\nabla^T\nabla)(v_3'') = \frac{i}{2^k}\big(2(2k+n-4)+2b(k-1)\big)(x_1+ix_{m+1})^{k-2}\varsigma_1$$
$$- \frac{1}{2^k}\big(2(2k+n-4)+2b(k-1)\big)(x_1+ix_{m+1})^{k-2}\varsigma_{m+1}. \tag{2.108}$$

Assume $n = 2m+1$. Then

$$v_3' = \frac{i}{2^k}(k-1)\sum_{j=0}^{2m} x_j^2(x_1+ix_{m+1})^{k-2}\varsigma_1 - \frac{1}{2^k}(k-1)\sum_{j=0}^{2m} x_j^2(x_1+ix_{m+1})^{k-2}\varsigma_{m+1}$$
$$- i\frac{2k+n-4}{2^k}\sum_{s=0}^{2m} x_s(x_1+ix_{m+1})^{k-1}\varsigma_s, \tag{2.109}$$

$$v_3'' = \frac{i}{2^k}\sum_{j=0}^{2m} x_j^2(x_1+ix_{m+1})^{k-2}\varsigma_1 - \frac{1}{2^k}\sum_{j=0}^{2m} x_j^2(x_1+ix_{m+1})^{k-2}\varsigma_{m+1}. \tag{2.110}$$

Thus

$$\Delta(v_3') + b(\nabla^T\nabla)(v_3') = -\frac{i(k-1)}{2^k} b(k+n-3)(2k+n-2)(x_1+ix_{m+1})^{k-2}\varsigma_1 + \frac{(k-1)}{2^k} b(k+n-3)(2k+n-2)(x_1+ix_{m+1})^{k-2}\varsigma_{m+1}, \tag{2.111}$$

$$\Delta(v_3'') + b(\nabla^T\nabla)(v_3'') = \frac{i}{2^k}\big(2(2k+n-4)+2b(k-1)\big)(x_1+ix_{m+1})^{k-2}\varsigma_1 - \frac{1}{2^k}\big(2(2k+n-4)+2b(k-1)\big)(x_1+ix_{m+1})^{k-2}\varsigma_{m+1}. \tag{2.112}$$

The conclusion follows from (2.107), (2.108) and (2.111), (2.112). □

Now we can describe the irreducible modules of $o(n,\mathbb{R})$ which are solution spaces of Navier equations.

Lemma 2.7. *The linear map determined by*

$$\psi(x_{i_1}\cdots x_{i_{k+1}}) = \sum_{j=1}^{k+1} x_{i_1}\cdots x_{i_{j-1}}x_{i_{j+1}}\cdots x_{i_{k+1}}\varsigma_{i_j} \tag{2.113}$$

is an $o(n,\mathbb{R})$-module isomorphism from $\mathcal{H}_{k+1}$ to $\hat{\mathcal{H}}_{k,1}$.

Proof. Set $f = x_1^{l_1}\cdots x_n^{l_n}$, and then $x_1\partial_{x_2}(f) = l_2x_1^{l_1+1}x_2^{l_2-1}x_3^{l_3}\cdots x_n^{l_n}$. It follows that

$$\psi\big(x_1\partial_{x_2}(f)\big) = \psi\big(l_2x_1^{l_1+1}x_2^{l_2-1}x_3^{l_3}\cdots x_n^{l_n}\big) = l_2(l_1+1)x_1^{l_1}x_2^{l_2-1}x_3^{l_3}\cdots x_n^{l_n}\varsigma_1 + l_2(l_2-1)x_1^{l_1+1}x_2^{l_2-2}x_3^{l_3}\cdots x_n^{l_n}\varsigma_2 + l_2\sum_{j=3}^{n} l_jx_1^{l_1+1}x_2^{l_2-1}x_3^{l_3}\cdots x_{j-1}^{l_{j-1}}x_j^{l_j-1}x_{j+1}^{l_{j+1}}\cdots x_n^{l_n}\varsigma_j. \tag{2.114}$$

On the other hand, $\psi(f) = \sum_{j=1}^n l_jx_1^{l_1}\cdots x_{j-1}^{l_{j-1}}x_j^{l_j-1}x_{j+1}^{l_{j+1}}\cdots x_n^{l_n}\varsigma_j$. Then

$$(x_1\partial_{x_2}+\varsigma_1\partial_{\varsigma_2})(f) = (x_1\partial_{x_2}+\varsigma_1\partial_{\varsigma_2})\left(\sum_{j=1}^n l_jx_1^{l_1}\cdots x_{j-1}^{l_{j-1}}x_j^{l_j-1}x_{j+1}^{l_{j+1}}\cdots x_n^{l_n}\varsigma_j\right) = l_1l_2x_1^{l_1}x_2^{l_2-1}x_3^{l_3}\cdots x_n^{l_n}\varsigma_1 + l_2(l_2-1)x_1^{l_1+1}x_2^{l_2-2}x_3^{l_3}\cdots x_n^{l_n}\varsigma_2 + l_2\sum_{j=3}^n l_jx_1^{l_1+1}x_2^{l_2-1}x_3^{l_3}\cdots x_{j-1}^{l_{j-1}}x_j^{l_j-1}x_{j+1}^{l_{j+1}}\cdots x_n^{l_n}\varsigma_j + l_2x_1^{l_1}x_2^{l_2-1}x_3^{l_3}\cdots x_n^{l_n}\varsigma_1 = \psi\big(x_1\partial_{x_2}(f)\big). \tag{2.115}$$

By symmetry, $\psi(A(f)) = A(\psi(f))$ for any $A \in gl(n,\mathbb{R})$. Moreover, f is harmonic and $\psi(f) = \sum_{j=1}^n g_j\varsigma_j$, then all g_j are harmonic. Indeed, we write $f = \sum_{l=0}^{k+1} f_l(x_2,\dots,x_n)\frac{x_1^l}{l!}$, and then f is harmonic if and only if $f_{l+2} = -\Delta f_l$ for $0 \leqslant l \leqslant k-1$. It follows that $g_1 = \sum_{l=0}^k f_{l+1}(x_2,\dots,x_n)\frac{x_1^l}{l!}$ is harmonic. By symmetry, all g_j are harmonic. Hence $\psi(\mathcal{H}_{k+1}) \subset \mathcal{H}_kV$ is an irreducible module of $o(n,\mathbb{R})$. Thus $\psi(\mathcal{H}_{k+1}) = \hat{\mathcal{H}}_{k,1}$ by Lemmas 2.3 and 2.4. □

Lemma 2.8. (i) *Denote* $\tilde{x}_j=(k-1)\sum_{r=1}^n x_r^2\varsigma_j-(2k+n-4)x_j\sum_{r=1}^n x_r\varsigma_r$ *for* $1\leqslant j\leqslant n$. *Then the linear map* φ_1 *determined by*

$$x_{i_1}\cdots x_{i_{k-1}}\mapsto\sum_{s=1}^{k-1}x_{i_1}\cdots x_{i_{s-1}}x_{i_{s+1}}\cdots x_{i_{k-1}}\tilde{x}_{i_s} \tag{2.116}$$

is an $o(n,\mathbb{R})$*-module isomorphism from* $\mathcal{H}_{k-1}$ *to* $\hat{\mathcal{H}}'_{k,3}$. *Moreover,* $\varphi_2=(\sum_{r=1}^n x_r^2)\psi$ *is an* $o(n,\mathbb{R})$*-module isomorphism from* $\mathcal{H}_{k-1}$ *to the real form* $\hat{\mathcal{H}}''_{k,3}$ *of the complex* $o'(n,\mathbb{R})$*-submodule generated by* v''_3 (*cf.* (2.102) *and* (2.103)).

(ii) *The module* W_3 *is of real type,* $\varphi=\varphi_1+c\varphi_2$ (*cf.* (2.104)) *is an* $o(n,\mathbb{R})$*-module isomorphism from* $\mathcal{H}_{k-1}$ *to a real form* (*denoted by* $\hat{\mathcal{H}}_{k,3}$) *of* W_3 (*cf. Lemma* 2.6).

Proof. Firstly, it is clear that

$$(x_1\partial_{x_2}-x_2\partial_{x_1}+\varsigma_1\partial_{\varsigma_2}-\varsigma_2\partial_{\varsigma_1})(\tilde{x}_1)=-\tilde{x}_2, \tag{2.117}$$

$$(x_1\partial_{x_2}-x_2\partial_{x_1}+\varsigma_1\partial_{\varsigma_2}-\varsigma_2\partial_{\varsigma_1})(\tilde{x}_2)=\tilde{x}_1, \tag{2.118}$$

$$(x_1\partial_{x_2}-x_2\partial_{x_1}+\varsigma_1\partial_{\varsigma_2}-\varsigma_2\partial_{\varsigma_1})(\tilde{x}_j)=0 \tag{2.119}$$

for $j>2$. Moreover,

$$\begin{aligned}\varphi_1\big((E_{1,2}-E_{2,1})(\zeta)\big)&=l_2(l_1+1)\frac{\zeta}{x_2}\tilde{x}_1+l_2(l_2-1)\frac{x_1\zeta}{x_2^2}\tilde{x}_2+\sum_{j=3}^n l_2l_j\frac{x_1\zeta}{x_2x_j}\tilde{x}_j\\&\quad-l_1(l_1-1)\frac{x_2\zeta}{x_1^2}\tilde{x}_1-l_1(l_2+1)\frac{\zeta}{x_1}\tilde{x}_2-\sum_{j=3}^n l_1l_j\frac{x_2\zeta}{x_1x_j}\tilde{x}_j,\end{aligned} \tag{2.120}$$

where $\zeta=x_1^{l_1}\cdots x_n^{l_n}$. On the other hand,

$$\begin{aligned}&(E_{1,2}-E_{2,1})\big(\varphi_1(\zeta)\big)\\&\quad=l_1l_2\frac{\zeta}{x_2}\tilde{x}_1-l_1(l_1-1)\frac{x_2\zeta}{x_1^2}\tilde{x}_1+l_1\frac{\zeta}{x_1}\big((x_1\partial_{x_2}-x_2\partial_{x_1}+\varsigma_1\partial_{\varsigma_2}-\varsigma_2\partial_{\varsigma_1})(\tilde{x}_1)\big)\\&\qquad+l_2(l_2-1)\frac{x_1\zeta}{x_2^2}\tilde{x}_2-l_1l_2\frac{\zeta}{x_1}\tilde{x}_2+l_2\frac{\zeta}{x_2}\big((x_1\partial_{x_2}-x_2\partial_{x_1}+\varsigma_1\partial_{\varsigma_2}-\varsigma_2\partial_{\varsigma_1})(\tilde{x}_2)\big)\\&\qquad+\sum_{j=3}^n\left(l_2l_j\frac{x_1\zeta}{x_2x_j}\tilde{x}_j-l_1l_j\frac{x_2\zeta}{x_1x_j}\tilde{x}_j\right)+\sum_{j=3}^n\frac{\zeta}{x_j}\big((x_1\partial_{x_2}-x_2\partial_{x_1}+\varsigma_1\partial_{\varsigma_2}-\varsigma_2\partial_{\varsigma_1})(\tilde{x}_j)\big)\end{aligned} \tag{2.121}$$

by (2.117)–(2.119). Thus $\varphi_1((E_{1,2}-E_{2,1})(\zeta))=(E_{1,2}-E_{2,1})(\varphi_1(\zeta))$. By the symmetry on subscripts, φ_1 is an $o(n,\mathbb{R})$-module monomorphism from $\mathcal{H}_{k-1}$ to $\hat{\mathcal{A}}_k$. If $n=2m$,

$$\begin{aligned}\varphi_1\big(y_1^{k-1}\big)&=\varphi_1\left(\frac{1}{2^{k-1}}(x_1+ix_{m+1})^{k-1}\right)\\&=\frac{1}{2^{k-1}}\sum_{r=0}^{k-1}\binom{k-1}{r}\big((k-1-r)x_1^{k-2-r}(ix_{m+1})^r\tilde{x}_1+rix_1^{k-1-r}(ix_{m+1})^{r-1}\tilde{x}_{m+1}\big)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{k-1}} \sum_{r=0}^{k-2} \binom{k-2}{r} (k-1) x_1^{k-2-r} (i x_{m+1})^r (\tilde{x}_1 + i\tilde{x}_{m+1}) \\
&= \frac{k-1}{2} y_1^{k-2} (\tilde{x}_1 + i\tilde{x}_{m+1}) \\
&= -2i(k-1) v_3' \qquad (2.122)
\end{aligned}
$$

(cf. (2.82)). So φ_1 is an $o(n,\mathbb{R})$-module isomorphism from $\mathcal{H}_{k-1}$ to $\hat{\mathcal{H}}'_{k,3}$. It holds similarly when $n = 2m+1$.

Lemma 2.8 implies that φ_2 is an $o(n,\mathbb{R})$-module monomorphism from $\mathcal{H}_{k-1}$ to $\hat{\mathcal{A}}_k$. If $n = 2m$, we denote $\xi = \sum_{j=1}^n x_j^2$, and then

$$
\begin{aligned}
\varphi_2\big(y_1^{k-1}\big) &= \frac{1}{2^{k-1}} \sum_{r=0}^{k-1} \binom{k-1}{r} \big((k-1-r)\xi x_1^{k-2-r} (i x_{m+1})^r \varsigma_1 + r i \xi x_1^{k-1-r} (i x_{m+1})^{r-1} \varsigma_{m+1}\big) \\
&= -4i(k-1) \sum_{j=1}^m y_j y_{m+j} y_1^{k-2} \kappa_1 = -2i(k-1) v_3'' \qquad (2.123)
\end{aligned}
$$

(cf. (2.102)). So $\varphi_2 = (\sum_{r=1}^n x_r^2)\psi$ is an $o(n,\mathbb{R})$-module isomorphism from $\mathcal{H}_{k-1}$ to the real form $\hat{\mathcal{H}}''_{k,3}$. It holds similarly when $n = 2m+1$. Since $\varphi(\mathcal{H}'_{k-1}) = W_3$ and $\varphi(\overline{f}) = \overline{\varphi(f)}$ for $f \in \mathcal{H}'_{k-1}$, we know that W_3 is of real type by Lemma 2.4. Moreover, $\varphi(\mathcal{H}_{k-1})$ is a real form of W_3. That is $\varphi(\mathcal{H}_{k-1}) = \hat{\mathcal{H}}_{k,3}$. $\square$

Lemma 2.9. *The subspace*

$$
\hat{\mathcal{H}}_{k,2} = \left\{ \sum_{r=1}^n f_r \varsigma_r \;\middle|\; f_r \in \mathcal{H}_k,\ \sum_{r=1}^n x_r f_r = 0 \right\}. \qquad (2.124)
$$

Proof. Denote by $\tilde{V}$ the right side of the above equality. Note that the linear map $\nu : \sum_{r=1}^n f_r \varsigma_r \mapsto \sum_{r=1}^n x_r f_r$ is an $o'(n,\mathbb{C})$-module homomorphism from $\hat{\mathcal{A}}_{\mathbb{C}}$ to $\mathcal{A}_{\mathbb{C}}$ by Lemma 2.1. In particular,

$$
\nu(\kappa_r) = y_r \quad \text{for } 1 \leqslant r \leqslant n. \qquad (2.125)
$$

Moreover, $\nu(\mathrm{Re}(v_1)) = \mathrm{Re}(\nu(v_1)) = \mathrm{Re}(y_1^{k+1}) \in \mathcal{H}_{k+1}$ by (2.78) and the classical harmonic analysis. Lemma 2.1 also tells us that $\nu_{\hat{\mathcal{A}}}$ is an $o(n,\mathbb{R})$-module homomorphism from $\hat{\mathcal{A}}$ to $\mathcal{A}$. Recall that $\hat{\mathcal{H}}_{k,1}$ is an irreducible $o(n,\mathbb{R})$-submodule generated by $\mathrm{Re}(v_1)$ and $\mathcal{H}_{k+1}$ is an irreducible $o(n,\mathbb{R})$-submodule. So $\nu|_{\hat{\mathcal{H}}_{k,1}}$ is an $o(n,\mathbb{R})$-module isomorphism from $\hat{\mathcal{H}}_{k,1}$ to $\mathcal{H}_{k+1}$.

According to (2.61) and (2.63),

$$
4\sum_{r=1}^m y_r y_{m+r} = i \sum_{s=1}^n x_s^2, \qquad 2y_0^2 = i x_0^2. \qquad (2.126)
$$

Thus

$$
\nu\big(v_3'\big) = \frac{3-n-k}{2^k} \big(x_1^2 + \cdots + x_n^2\big) i (x_1 + i x_{m+1})^{k-1} \qquad (2.127)
$$

by (2.82) if $n=2m$. When $n=2m+1$, we shift index $r\mapsto r+1$ and have

$$\nu\big(v_3'\big)=\frac{3-n-k}{2^k}\big(x_1^2+\cdots+x_n^2\big)i(x_2+ix_{m+2})^{k-1} \tag{2.128}$$

by (2.83). Thus $0\neq\nu(\mathrm{Re}(v_3'))=\mathrm{Re}(\nu(v_3'))\in(x_1^2+\cdots+x_n^2)\mathcal{H}_{k-1}$. By the $o(n,\mathbb{R})$-irreducibility of $\hat{\mathcal{H}}'_{k,3}$ and $(x_1^2+\cdots+x_n^2)\mathcal{H}_{k-1}$, $\nu|_{\hat{\mathcal{H}}'_{k,3}}$ is an $o(n,\mathbb{R})$-module isomorphism from $\hat{\mathcal{H}}'_{k,3}$ to $(x_1^2+\cdots+x_n^2)\mathcal{H}_{k-1}$.

By (2.79)–(2.81) and (2.125), $\nu(v_2)=0$ and $\nu(v_{2,\pm})=0$. Similarly, we have $\nu|_{\hat{\mathcal{H}}_{k,2}}=0$. Since $\mathcal{H}_kV=\hat{\mathcal{H}}_{k,1}\oplus\hat{\mathcal{H}}_{k,2}\oplus\hat{\mathcal{H}}_{k,3}$, $\ker\nu|_{\mathcal{H}_kV}=\hat{\mathcal{H}}_{k,2}$, that is, (2.124) holds. □

Corollary 2.10. *The subspace*

$$\hat{\mathcal{H}}_{k,1}+\hat{\mathcal{H}}_{k,2}=\left\{\sum_{r=1}^n f_r\varsigma_r\;\Big|\; f_r\in\mathcal{H}_k,\ \sum_{r=1}^n\partial_{x_r}(f_r)=0\right\}. \tag{2.129}$$

Proof. Denote by $\tilde{V}$ the set in right side in (2.129). Note $(\partial_{x_1}+i\partial_{x_{m+1}})((x_1+ix_{m+1})^k)=0$. By (2.78), $\mathrm{Re}(v_1)\in\tilde{V}\cap\hat{\mathcal{H}}_{k,1}$. Moreover, the map $\sigma:\sum_{r=1}^n f_r\varsigma_r\to\sum_{r=1}^n\partial_{x_r}(f_r)$ is an $o(n,\mathbb{R})$-module homomorphism from $\hat{\mathcal{A}}$ to $\mathcal{A}$. The irreducibility of $\hat{\mathcal{H}}_{k,1}$ as an $o(n,\mathbb{R})$-submodule implies $\hat{\mathcal{H}}_{k,1}\subset\tilde{V}$. For $\vec{f}\in\hat{\mathcal{H}}_{k,2}$, $0=\Delta(\sum_{r=1}^n x_rf_r)=2\sum_{r=1}^n\partial_{x_r}(f_r)$ by straightforward calculation. Thus $\hat{\mathcal{H}}_{k,2}\subset\tilde{V}$. Now $\sigma(\hat{\mathcal{H}}_{k,3})=\sigma(\mathcal{H}_kV)\neq\{0\}$. Since $\hat{\mathcal{H}}_{k,3}$ is an irreducible $o(n,\mathbb{R})$-submodule, $\ker\sigma|_{\hat{\mathcal{H}}_{3,k}}=\{0\}$. Hence (2.129) holds. □

Denote $d_{r,s}=x_s\partial_{x_r}-x_r\partial_{x_s}$ and

$$\mathcal{D}=\begin{pmatrix}0 & d_{3,4} & d_{4,2} & d_{2,3}\\ d_{4,3} & 0 & d_{1,4} & d_{3,1}\\ d_{2,4} & d_{4,1} & 0 & d_{1,2}\\ d_{3,2} & d_{1,3} & d_{2,1} & 0\end{pmatrix}. \tag{2.130}$$

Then if $n=4$, we have that

Lemma 2.11. *The subspaces*

$$\hat{\mathcal{H}}_{k,2\pm}=\big\{\vec{f}\in\hat{\mathcal{H}}_{k,2}\;\big|\;\mathcal{D}\vec{f}=\pm(k+1)\vec{f}\big\}. \tag{2.131}$$

Proof. We can verify that $\mathcal{D}$ commutes with $o(4,\mathbb{R})$. Moreover, $\mathcal{D}(v_{2,\pm})=\pm(k+1)v_{2,\pm}$ by (2.80). Expression (2.131) holds because $v_{2,\pm}$ are generators of $\hat{\mathcal{H}}_{k,2\pm}$ as $o(4,\mathbb{R})$-modules. □

Now we can get the main theorem of this section.

Theorem 2.12. *Assume that integer $n\geqslant 3$. Let $\hat{\mathcal{H}}_{k,1}=\psi(\mathcal{H}_{k+1})$ (cf. (2.113)) and $\hat{\mathcal{H}}_{k,3}=\varphi(\mathcal{H}_{k-1})$ (cf. Lemma 2.8), which are irreducible $o(n,\mathbb{R})$-submodules. Take $\hat{\mathcal{H}}_{k,2}$ in (2.124), which is an irreducible $o(n,\mathbb{R})$-submodule if $n\neq 4$. When $n=4$, $\hat{\mathcal{H}}_{k,2}=\hat{\mathcal{H}}_{k,2+}\oplus\hat{\mathcal{H}}_{k,2-}$ and $\hat{\mathcal{H}}_{k,2\pm}$ are irreducible $o(4,\mathbb{R})$-submodules characterized by (2.131). Then the subspace of homogeneous polynomial solutions with degree k of Navier equation is $\hat{\mathcal{H}}_k=\hat{\mathcal{H}}_{k,1}\oplus\hat{\mathcal{H}}_{k,2}\oplus\hat{\mathcal{H}}_{k,3}$, and*

$$\hat{\mathcal{A}}_k=\hat{\mathcal{H}}_k\oplus\big(x_1^2+\cdots+x_n^2\big)\hat{\mathcal{A}}_{k-2}. \tag{2.132}$$

Proof. We have $\hat{\mathcal{H}}_{k,j} \subset \hat{\mathcal{H}}_k$ for $j = 1, 2, 3$ by Lemmas 2.5 and 2.6. By Xu's method, we can calculate $\dim \hat{\mathcal{H}}_k = n \dim \mathcal{H}_k = \dim \hat{\mathcal{H}}_{k,1} + \dim \hat{\mathcal{H}}_{k,2} + \dim \hat{\mathcal{H}}_{k,3}$. (The details will be given later for technical convenience (see Theorem 3.8).) Since

$$\hat{\mathcal{H}}_k \equiv \mathcal{H}_k V \mod \big(x_1^2 + \cdots + x_n^2\big)\mathcal{H}_{k-2} V \tag{2.133}$$

by the way of our taking subspaces $\hat{\mathcal{H}}_{k,1}, \hat{\mathcal{H}}_{k,2}$ and $\hat{\mathcal{H}}_{k,3}$ in the theorem, (2.132) follows from the facts $\mathcal{A}_k = \mathcal{H}_k + (x_1^2 + \cdots + x_n^2)\mathcal{H}_{k-2}$, $\hat{\mathcal{A}}_k = \mathcal{A}_k V$ and induction on k. □

3. Bases

In this section, we will construct some bases of the subspaces $\hat{\mathcal{H}}_{k,1}$, $\hat{\mathcal{H}}_{k,2}$, $\hat{\mathcal{H}}_{k,3}$ and $\hat{\mathcal{H}}_{k,2\mp}$ defined in Section 2.

Since $\mathcal{H}_{k+1} \overset{\psi}{\cong} \hat{\mathcal{H}}_{k,1}$ (cf. Lemma 2.7) and $\mathcal{H}_{k-1} \overset{\varphi}{\cong} \hat{\mathcal{H}}_{k,3}$ (cf. Lemma 2.8), we can obtain the bases of $\hat{\mathcal{H}}_{k,1}$ and $\hat{\mathcal{H}}_{k,3}$ by a basis of $\mathcal{H}_k$ introduced in [10]:

$$\left\{ w(\epsilon, l_2, \ldots, l_n) \;\middle|\; \epsilon \in \{0,1\};\ l_2, \ldots, l_n \in \mathbb{N},\ \epsilon + \sum_{j=2}^{n} l_j = k \right\} \tag{3.1}$$

with

$$w(\epsilon, l_2, \ldots, l_n) = \sum_{r_2,\ldots,r_n=0}^{\infty} \frac{(-1)^{\sum_{j=2}^n r_j} \binom{r_2+\cdots+r_n}{r_2,\ldots,r_n} \prod_{s=2}^n \binom{l_s}{2r_s}}{(1 + 2\epsilon \sum_{j=2}^n r_j)\binom{2(r_2+\cdots+r_n)}{2r_2,\ldots,2r_n}} x_1^{\epsilon + 2\sum_{j=2}^n r_j} \prod_{j=2}^{n} x_j^{l_j - 2r_j}. \tag{3.2}$$

Take $\epsilon \in \{0,1\}$, $l_2, \ldots, l_n \in \mathbb{N}$ such that $\epsilon + \sum_{j=2}^n l_j = k+1$, and define

$$\vec{f}(\epsilon, l_2, \ldots, l_n) = \psi\big(w(\epsilon, l_2, \ldots, l_n)\big) = \sum_{j=1}^{n} f_j(\epsilon, l_2, \ldots, l_n)\varsigma_j. \tag{3.3}$$

Then straightforward calculation shows that

$$\begin{aligned} & f_1(\epsilon, l_2, \ldots, l_n) \\ &= \sum_{r_2,\ldots,r_n=0}^{\infty} \left(\epsilon + 2\sum_{j=2}^{n} r_j\right) \frac{(-1)^{\sum_{j=2}^n r_j} \binom{r_2+\cdots+r_n}{r_2,\ldots,r_n} \prod_{s=2}^n \binom{l_s}{2r_s}}{(1 + 2\epsilon \sum_{j=2}^n r_j)\binom{2(r_2+\cdots+r_n)}{2r_2,\ldots,2r_n}} x_1^{\epsilon + 2\sum_{j=2}^n r_j - 1} \prod_{s=2}^{n} x_s^{l_s - 2r_s} \end{aligned} \tag{3.4}$$

and for $j = 2, \ldots, n$,

$$\begin{aligned} & f_j(\epsilon, l_2, \ldots, l_n) \\ &= \sum_{r_2,\ldots,r_n=0}^{\infty} (l_j - 2r_j) \frac{(-1)^{\sum_{j=2}^n r_j} \binom{r_2+\cdots+r_n}{r_2,\ldots,r_n} \prod_{s=2}^n \binom{l_s}{2r_s}}{(1 + 2\epsilon \sum_{j=2}^n r_j)\binom{2(r_2+\cdots+r_n)}{2r_2,\ldots,2r_n}} x_1^{\epsilon + 2\sum_{j=2}^n r_j} x_j^{-1} \prod_{s=2}^{n} x_s^{l_s - 2r_s}. \end{aligned} \tag{3.5}$$

Proposition 3.1. *The set*

$$\left\{\vec{f}(\epsilon, l_2, \ldots, l_n) \;\Big|\; \epsilon = 0 \text{ or } 1;\ l_j \in \mathbb{N};\ \epsilon + \sum_{j=2}^{n} l_j = k+1\right\} \tag{3.6}$$

forms a basis of $\hat{\mathcal{H}}_{k,1}$*, where the components of* $\vec{f}(\epsilon, l_2, \ldots, l_n)$ *are given by* (3.4) *and* (3.5).

Similarly, we define

$$\tilde{w}(\epsilon, l_2, \ldots, l_n) = \sum_{r_2,\ldots,r_n=0}^{\infty} \frac{(-1)^{\sum_{j=2}^{n} r_j} \binom{r_2+\cdots+r_n}{r_2,\ldots,r_n} \prod_{s=2}^{n} \binom{l_s}{2r_s}}{(1+2\epsilon\sum_{j=2}^{n} r_j)\binom{2(r_2+\cdots+r_n)}{2r_2,\ldots,2r_n}} x_1^{\epsilon+2\sum_{j=2}^{n} r_j} \prod_{j=2}^{n} x_j^{l_j-2r_j}, \tag{3.7}$$

where $\epsilon \in \{0, 1\}$, $l_2, \ldots, l_n \in \mathbb{N}$ and $\epsilon + \sum_{j=2}^{n} l_j = k-1$. Thus $\tilde{w} \in \mathcal{H}_{k-1}$. Denote

$$\vec{g}(\epsilon, l_2, \ldots, l_n) = \varphi\big(\tilde{w}(\epsilon, l_2, \ldots, l_n)\big) = \sum_{j=1}^{n} g_j(\epsilon, l_2, \ldots, l_n)\varsigma_j, \tag{3.8}$$

then we get that

$$\begin{aligned} &g_1(\epsilon, l_2, \ldots, l_n) \\ &\quad= \sum_{r_2,\ldots,r_n=0}^{\infty} \frac{(-1)^{\sum_{j=2}^{n} r_j} \binom{r_2+\cdots+r_n}{r_2,\ldots,r_n} \prod_{s=2}^{n} \binom{l_s}{2r_s}}{(1+2\epsilon\sum_{j=2}^{n} r_j)\binom{2(r_2+\cdots+r_n)}{2r_2,\ldots,2r_n}} \Bigg(\bigg(\epsilon + 2\sum_{j=2}^{n} r_j\bigg) \\ &\qquad\times \bigg((k-1) + \frac{(2k+n-2)(k+n-3)(k-1)}{2(b^{-1}(2k+n-4)+k-1)}\bigg)\bigg(\sum_{p=1}^{n} x_p^2\bigg) x_1^{\epsilon+2\sum_{j=2}^{n} r_j - 1} \prod_{s=2}^{n} x_s^{l_s-2r_s} \\ &\qquad- (k-1)(2k+n-4) x_1^{\epsilon+2\sum_{j=2}^{n} r_j + 1} \prod_{s=2}^{n} x_s^{l_s-2r_s}\Bigg) \end{aligned} \tag{3.9}$$

and for $j = 2, \ldots, n$,

$$\begin{aligned} &g_j(\epsilon, l_2, \ldots, l_n) \\ &\quad= \sum_{r_2,\ldots,r_n=0}^{\infty} \frac{(-1)^{\sum_{j=2}^{n} r_j} \binom{r_2+\cdots+r_n}{r_2,\ldots,r_n} \prod_{s=2}^{n} \binom{l_s}{2r_s}}{(1+2\epsilon\sum_{j=2}^{n} r_j)\binom{2(r_2+\cdots+r_n)}{2r_2,\ldots,2r_n}} \Bigg((l_j - 2r_j) \\ &\qquad\times \bigg((k-1) + \frac{(2k+n-2)(k+n-3)(k-1)}{2(b^{-1}(2k+n-4)+k-1)}\bigg)\bigg(\sum_{p=1}^{n} x_p^2\bigg) x_1^{\epsilon+2\sum_{j=2}^{n} r_j} x_j^{-1} \prod_{s=2}^{n} x_s^{l_s-2r_s} \\ &\qquad- (k-1)(2k+n-4) x_1^{\epsilon+2\sum_{j=2}^{n} r_j} x_j \prod_{s=2}^{n} x_s^{l_s-2r_s}\Bigg). \end{aligned} \tag{3.10}$$

Proposition 3.2. *The set*

$$\left\{\vec{g}(\epsilon, l_2, \ldots, l_n) \;\middle|\; \epsilon = 0 \text{ or } 1,\ l_j \in \mathbb{N},\ \epsilon + \sum_{j=2}^{n} l_j = k-1\right\} \tag{3.11}$$

forms a basis of $\hat{\mathcal{H}}_{k,3}$, where the components of $\vec{g}(\epsilon, l_2, \ldots, l_n)$ are given by (3.9) *and* (3.10).

Now we will find a basis of $\hat{\mathcal{H}}_{k,2}$ by solving the equation in (2.124). Let $\vec{f} = \sum_{j=1}^{n} f_j \varsigma_j \in \hat{\mathcal{H}}_{k,2}$. Then $f_j \in \mathcal{H}_k$ for $j = 1, \ldots, n$ and $f_n = -x_n^{-1} \sum_{j=1}^{n-1} x_j f_j$. We write

$$f_j = \sum_{l=0}^{k} f_{n-1,l}^{j}(x_1, \ldots, x_{n-1}) \frac{x_n^l}{l!} \quad \text{for } j = 1, \ldots, n, \tag{3.12}$$

where $f_{n-1,l}^{j}(x_1, \ldots, x_{n-1})$ are homogeneous polynomials with degree $k-l$. Denote

$$\Delta_s = \sum_{p=1}^{s} \partial_{x_p}^2 \quad \text{for } s = 1, \ldots, n-1. \tag{3.13}$$

Then f_j is determined by $f_{n-1,0}^{j}$ and $f_{n-1,1}^{j}$ via

$$f_{n-1,l+2}^{j} = -\Delta_{n-1}\big(f_{n-1,l}^{j}\big) \quad \text{for } 0 \leqslant l \leqslant k-2. \tag{3.14}$$

Note

$$\begin{aligned} f_n = -x_n^{-1} \sum_{j=1}^{n-1} x_j f_j &= -x_n^{-1} \sum_{j=1}^{n-1} \sum_{l=0}^{k} x_j f_{n-1,l}^{j} \frac{x_n^l}{l!} \\ &= -x_n^{-1} \sum_{j=1}^{n-1} x_j f_{n-1,0}^{j} - \sum_{j=1}^{n-1} \sum_{l=0}^{k-1} \frac{x_j f_{n-1,l+1}^{j}}{l+1} \frac{x_n^l}{l!}. \end{aligned} \tag{3.15}$$

Thus $f_n \in \mathcal{H}_k$ if and only if

$$\begin{cases} \text{(a)} \quad \displaystyle\sum_{j=1}^{n-1} x_j f_{n-1,0}^{j} = 0, \\ \text{(b)} \quad \displaystyle\sum_{j=1}^{n-1} \frac{x_j f_{n-1,l+3}^{j}}{l+3} = -\sum_{j=1}^{n-1} \Delta_{n-1}\left(\frac{x_j f_{n-1,l+1}^{j}}{l+1}\right) \quad \text{for } l \geqslant 0 \end{cases} \tag{3.16}$$

by (3.14) and (3.15). To write down a basis of $\hat{\mathcal{H}}_{k,2}$, it is sufficient to solve (3.16).

Lemma 3.3. *The following system is equivalent to* (3.16)*.*

$$\begin{cases} \text{(a)} \quad \sum_{j=1}^{n-1} x_j f_{n-1,0}^j = 0, \\ \text{(b)} \quad \sum_{j=1}^{n-1} \partial_{x_j}\big(f_{n-1,1}^j\big) = -\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j f_{n-1,1}^j\right). \end{cases} \tag{3.17}$$

Proof. Note that

$$\begin{aligned} &-(l+1)\sum_{j=1}^{n-1} x_j f_{n-1,l+3}^j = (l+3)\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j f_{n-1,l+1}^j\right) \\ &\Leftrightarrow \quad (l+1)\sum_{j=1}^{n-1} x_j \Delta_{n-1}\big(f_{n-1,l+1}^j\big) = (l+3)\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j f_{n-1,l+1}^j\right) \\ &\Leftrightarrow \quad (l+1)\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j f_{n-1,l+1}^j\right) - 2(l+1)\sum_{j=1}^{n-1}\partial_{x_j}\big(f_{n-1,l+1}^j\big) = (l+3)\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j f_{n-1,l+1}^j\right) \\ &\Leftrightarrow \quad (l+1)\sum_{j=1}^{n-1}\partial_{x_j}\big(f_{n-1,l+1}^j\big) = -\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j f_{n-1,l+1}^j\right). \end{aligned} \tag{3.18}$$

Set $l=0$ in the above equalities, we get that

$$\sum_{j=1}^{n-1}\frac{x_j f_{n-1,3}^j}{3} = -\sum_{j=1}^{n-1}\Delta_{n-1}\left(\frac{x_j f_{n-1,1}^j}{2}\right) \quad \Rightarrow \quad \sum_{j=1}^{n-1}\partial_{x_j}\big(f_{n-1,1}^j\big) = -\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j f_{n-1,1}^j\right). \tag{3.19}$$

Thus (3.16) implies (3.17). But

$$\begin{aligned} &(l+1)\sum_{j=1}^{n-1}\partial_{x_j}\big(f_{n-1,l+1}^j\big) = -\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j f_{n-1,l+1}^j\right) \\ &\Leftrightarrow \quad -(l+1)\Delta_{n-1}\left(\sum_{j=1}^{n-1}\partial_{x_j}\big(f_{n-1,l-1}^j\big)\right) = -\Delta_{n-1}\left(\sum_{j=1}^{n-1} x_j \Delta_{n-1}\big(f_{n-1,l-1}^j\big)\right) \\ &\Leftrightarrow \quad (l+1)\Delta_{n-1}\left(\sum_{j=1}^{n-1}\partial_{x_j}\big(f_{n-1,l-1}^j\big)\right) = -\Delta_{n-1}^2\left(\sum_{j=1}^{n-1} x_j f_{n-1,l-1}^j\right) + 2\Delta_{n-1}\left(\sum_{j=1}^{n-1}\partial_{x_j}\big(f_{n-1,l-1}^j\big)\right) \\ &\Leftrightarrow \quad (l-1)\Delta_{n-1}\left(\sum_{j=1}^{n-1}\partial_{x_j}\big(f_{n-1,l-1}^j\big)\right) = -\Delta_{n-1}^2\left(\sum_{j=1}^{n-1} x_j f_{n-1,l-1}^j\right). \end{aligned} \tag{3.20}$$

Then by induction, one gets that (3.17(a)) implies (3.16(b)) if l is odd, and (3.17(b)) implies (3.16(b)) when l is even. Hence (3.17) is equivalent to (3.16). □

Using (3.17) and the condition $f_j \in \mathcal{H}_k$ for $j = 1, \ldots, n-1$, we can obtain a basis of $\hat{\mathcal{H}}_{k,2}$. For convenience, we classify these base vectors into three disjoint subsets satisfying the following conditions, respectively: Let f_j be the first nonzero component of the base vector $\vec{f} = \sum_{l=1}^{n} f_l \varsigma_l \in \hat{\mathcal{H}}_{k,2}$.

Condition $(*)$: $1 \leqslant j \leqslant n-2$, and the powers of x_n in f_j are even.
Condition $(**)$: $1 \leqslant j \leqslant n-2$, and the powers of x_n in f_j are odd.
Condition $(***)$: $j = n-1$.

To find the base vectors satisfying Condition $(*)$, we set $f_{n-1,1}^{l} = 0$ in (3.17(b)) for $l = 1, \ldots, n-1$. Moreover, for convenience, we write

$$f_{n-1,0}^{j} = \sum_{l=0}^{k} f_{n-2,l}^{j}(x_1, \ldots, x_{n-2}) \frac{x_{n-1}^{l}}{l!} \quad \text{for } j = 1, \ldots, n-1. \tag{3.21}$$

Continue the process. In general, we write

$$f_{s,0}^{j} = \sum_{l=0}^{k} f_{s-1,l}^{j}(x_1, \ldots, x_{s-1}) \frac{x_s^{l}}{l!} \quad \text{for } s = 1, \ldots, n-1. \tag{3.22}$$

Then by induction, the following system is equivalent to (3.17(a))

$$\begin{cases} f_{1,0}^{1} = 0 \\ f_{j,0}^{j} = -x_j^{-1} \sum_{r=1}^{j-1} x_r f_{j,0}^{r} \quad \text{for } j \geqslant 2. \end{cases} \tag{3.23}$$

Now we can write down those base vectors satisfying Condition $(*)$ by (3.23).

From now on, the notations r_s are always nonnegative integers. Assume that f_j is the first nonzero component of the base vector $\vec{f} = \sum_{l=1}^{n} f_l \varsigma_l \in \hat{\mathcal{H}}_{k,2}$. Then $f_{j,0}^{j} = 0$ by (3.23). We take $f_{n-1,0}^{j}$ to be the following monomials:

$$f_{n-1,0}^{j} = x_1^{r_1} \cdots x_{n-1}^{r_{n-1}}, \tag{3.24}$$

where $r_1 + \cdots + r_{n-1} = k$ and $r_{j+1} + \cdots + r_{n-1} > 0$. Moreover, we set

$$f_{n-1,0}^{q} = f_{q,0}^{q} \quad \text{for } j < q \leqslant n-1. \tag{3.25}$$

Thus

$$f_{n-1,0}^{q} = -\delta_{r_1+\cdots+r_q,k} \left(\sum_{s=1}^{n-1} \delta_{r_q,s} \right) \frac{x_j}{x_q} \prod_{s=1}^{n-1} x_s^{r_s}, \tag{3.26}$$

and so

$$f_q = \sum_{l=0}^{[\![\frac{k}{2}]\!]} (-1)^{l+1} \delta_{r_1+\cdots+r_q,k} \left(\sum_{s=1}^{n-1} \delta_{r_q,s} \right) \Delta_{n-1}^{l} \left(\frac{x_j x_n^{2l}}{(2l)! x_q} \prod_{s=1}^{n-1} x_s^{r_s} \right) \tag{3.27}$$

for $j < q \leqslant n-1$. Then we obtain that

(I) The following vectors are the base vectors satisfying Condition $(*)$:

$$\vec{f}=\sum_{l=j}^{n-1} f_l\big(\varsigma_l - x_n^{-1}x_l\varsigma_n\big) \tag{3.28}$$

for some $j\in\{1,\dots,n-2\}$, where

$$f_j=\sum_{l=0}^{[\![\frac{k}{2}]\!]}(-1)^l\Delta_{n-1}^l\left(\frac{x_n^{2l}}{(2l)!}\prod_{s=1}^{n-1}x_s^{r_s}\right) \tag{3.29}$$

and $f_{j+1},\dots,f_{n-1}$ are given in (3.27) for any nonnegative integers $r_1,\dots,r_{n-1}$ with $r_1+\cdots+r_{n-1}=k$ and $r_{j+1}+\cdots+r_{n-1}>0$.

To get the base vectors satisfying Conditions $(**)$ and $(***)$, we set $f_{n-1,0}^l=0$ in (3.17(a)), and simplify (3.17(b)). Note that it can be written as

$$\partial_{x_{n-1}}\big(f_{n-1,1}^{n-1}\big)+\Delta_{n-1}\big(x_{n-1}f_{n-1,1}^{n-1}\big)=-\sum_{l=1}^{n-2}\partial_{x_l}\big(f_{n-1,1}^l\big)-\Delta_{n-1}\left(\sum_{l=1}^{n-2}x_lf_{n-1,1}^l\right). \tag{3.30}$$

We write

$$f_{n-1,1}^s=\sum_{l=0}^{k-1}g_l^s(x_1,\dots,x_{n-2})\frac{x_{n-1}^l}{l!}\quad\text{for } 1\leqslant s\leqslant n-1. \tag{3.31}$$

Substituting (3.31) to (3.30), we get

$$\begin{aligned}\text{left}&=\partial_{x_{n-1}}\left(\sum_{l=0}^{k-1}g_l^{n-1}\frac{x_{n-1}^l}{l!}\right)+\big(\Delta_{n-2}+\partial_{x_{n-1}}^2\big)\left(x_{n-1}\sum_{l=0}^{k-1}g_l^{n-1}\frac{x_{n-1}^l}{l!}\right)\\&=\sum_{l=0}^{k-2}(l+3)g_{l+1}^{n-1}\frac{x_{n-1}^l}{l!}+\sum_{l=0}^{k}l\Delta_{n-2}\left(g_{l-1}^{n-1}\frac{x_{n-1}^l}{l!}\right)\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\text{right}&=-\sum_{s=1}^{n-2}\partial_{x_s}\left(\sum_{l=0}^{k-1}g_l^s\frac{x_{n-1}^l}{l!}\right)-\big(\Delta_{n-2}+\partial_{x_{n-1}}^2\big)\left(\sum_{s=1}^{n-2}\sum_{l=0}^{k-1}x_sg_l^s\frac{x_{n-1}^l}{l!}\right)\\&=-\sum_{l=0}^{k-1}\sum_{s=1}^{n-2}(\Delta_{n-2}x_s+\partial_{x_s})\left(g_l^s\frac{x_{n-1}^l}{l!}\right)-\sum_{l=0}^{k-3}\sum_{s=1}^{n-2}x_sg_{l+2}^s\frac{x_{n-1}^l}{l!}.\end{aligned} \tag{3.33}$$

Thus we obtain

$$g_{l+1}^{n-1}=-\frac{l}{l+3}\Delta_{n-2}\big(g_{l-1}^{n-1}\big)-\frac{1}{l+3}\sum_{s=1}^{n-2}(\Delta_{n-2}x_s+\partial_{x_s})\big(g_l^s\big)-\frac{1}{l+3}\sum_{s=1}^{n-2}x_sg_{l+2}^s \tag{3.34}$$

for $0 \leqslant l \leqslant k-2$, where $g_{-1}^s = g_k^s = 0$. If we define $(-1)!! = 0$, then

$$g_{2l}^{n-1} = \sum_{r=1}^{n-2}\sum_{s=1}^{l}(-1)^{l-s+1}\frac{(2s-2)!!(2l-1)!!}{(2s-1)!!(2l+2)!!}\big(\Delta_{n-2}^{l-s+1}x_r + 2s\Delta_{n-2}^{l-s}\partial_{x_r}\big)\big(g_{2s-1}^r\big) + (-1)^l\frac{(2l-1)!!}{(2l+2)!!}\Delta_{n-2}^l\left(2g_0^{n-1} + \sum_{r=1}^{n-2}x_r g_1^r\right) - \frac{1}{2l+2}\sum_{r=1}^{n-2}x_r g_{2l+1}^r \tag{3.35}$$

and

$$g_{2l+1}^{n-1} = \sum_{r=1}^{n-2}\sum_{s=0}^{l}(-1)^{l-s}\frac{(2s-1)!!(2l)!!}{(2s)!!(2l+3)!!}\big(\Delta_{n-2}^{l-s+1}x_r + (2s+1)\Delta_{n-2}^{l-s}\partial_{x_r}\big)\big(g_{2s}^r\big) - \frac{1}{2l+3}\sum_{r=1}^{n-2}x_r g_{2l+2}^r \tag{3.36}$$

for $l \geqslant 0$. The above two equalities tell us that f_{n-1} is determined by f_r whenever $r \leqslant n-2$ and g_0^{n-1} under the condition $f_{n-1,0}^l = 0$ for $l = 1, \dots, n-1$. Moreover, f_r is determined by $g_s^r(x_1, \dots, x_{n-2})$ which can be any homogeneous polynomial with degree $k-1-s$. These help us to write down those base vectors satisfying Conditions $(**)$ and $(***)$.

Setting $g_0^{n-1} = 0$, similarly as (I), we get that

(II) The following vectors are the base vectors satisfying Condition $(**)$:

$$\vec{f} = f_j\varsigma_j + f_{n-1}\varsigma_{n-1} - x_n^{-1}(x_j f_j + x_{n-1}f_{n-1})\varsigma_n \tag{3.37}$$

for some $j \in \{1, \dots, n-2\}$, where

$$f_j = \sum_{l=0}^{[\![\frac{k}{2}]\!]}(-1)^l\Delta_{n-1}^l\left(\frac{x_n^{2l+1}}{(2l+1)!}\prod_{s=1}^{n-1}x_s^{r_s}\right), \tag{3.38}$$

$$\begin{aligned} f_{n-1} = \sum_{l=0}^{[\![\frac{k}{2}]\!]}(-1)^{l+1}\Bigg\{&\sum_{p=1}^{k-1}\frac{\delta_{r_{n-1},p}\,p!}{(p+1)(p-1)!(2l+1)!}\Delta_{n-1}^l\left(x_j x_{n-1}^{p-1}x_n^{2l+1}\prod_{s=1}^{n-2}x_s^{r_s}\right) \\ &+ \sum_{q=0}^{[\![\frac{k}{2}]\!]}\Bigg[\frac{\delta_{r_{n-1},1}(-1)^{q+1}(2q-1)!!}{(2q+2)!!(2q)!(2l+1)!}\Delta_{n-1}^l\Delta_{n-2}^q\left(x_j x_{n-1}^{2q+1}x_n^{2l+1}\prod_{s=1}^{n-2}x_s^{r_s}\right) \\ &+ \frac{(-1)^{q+1}(r_{n-1}-1)!!(r_{n-1}+2q)!!r_{n-1}!}{r_{n-1}!!(r_{n-1}+2q+3)!!(r_{n-1}+2q+1)!(2l+1)!}\Bigg(\Delta_{n-1}^l\Delta_{n-2}^{q+1}\left(x_j x_{n-1}^{2q+1}x_n^{2l+1}\prod_{s=1}^{n-1}x_s^{r_s}\right) \\ &+ (r_{n-1}+1)r_j\Delta_{n-1}^l\Delta_{n-2}^q\left(x_j^{-1}x_{n-1}^{2q+1}x_n^{2l+1}\prod_{s=1}^{n-1}x_s^{r_s}\right)\Bigg)\Bigg]\Bigg\} \end{aligned} \tag{3.39}$$

for any nonnegative integers $r_1, \dots, r_{n-1}$ with $r_1 + \cdots + r_{n-1} = k-1$.

We set $f_r = 0$ for $r \leqslant n-2$ and

$$g_0^{n-1} = x_1^{r_1} \cdots x_{n-2}^{r_{n-2}}, \tag{3.40}$$

where $r_1 + \cdots + r_{n-2} = k-1$. Then we get

(III) The following vectors are the base vectors satisfying Condition (∗∗∗):

$$\vec{f} = f_{n-1}\big(\varsigma_{n-1} - x_n^{-1} x_{n-1} \varsigma_{n-1}\big), \tag{3.41}$$

where

$$f_{n-1} = \sum_{l=0}^{[\![\frac{k}{2}]\!]} \sum_{q=0}^{[\![\frac{k}{2}]\!]} \frac{(-1)^{l+q} 2(2q-1)!!}{(2q+2)!!(2q)!(2l+1)!} \Delta_{n-1}^{l} \Delta_{n-2}^{q} \left(x_{n-1}^{2q} x_n^{2l+1} \prod_{s=1}^{n-2} x_s^{r_s} \right) \tag{3.42}$$

for any nonnegative integers $r_1, \ldots, r_{n-2}$ with $r_1 + \cdots + r_{n-2} = k-1$.

Proposition 3.4. *The set of the vectors $\vec{f}$ given in* (3.27)–(3.29), (3.37)–(3.39) *and* (3.41)–(3.42) *forms a basis of $\hat{\mathcal{H}}_{k,2}$.*

Now we give bases of $\hat{\mathcal{H}}_{k,2\mp}$ in the case of $n = 4$. Recall that the representation operators of negative simple root vectors of $o'(4, \mathbb{C})$ are

$$f_{\alpha_1} = y_2 \partial_{y_1} - y_3 \partial_{y_4} + \kappa_2 \partial_{\kappa_1} - \kappa_3 \partial_{\kappa_4} \tag{3.43}$$

and

$$f_{\alpha_2} = y_4 \partial_{y_1} - y_3 \partial_{y_2} + \kappa_4 \partial_{\kappa_1} - \kappa_3 \partial_{\kappa_2}. \tag{3.44}$$

Then

$$\hat{\mathcal{H}}_{k,2-} = \operatorname{span}\big\{\operatorname{Re}\big(f_{\alpha_2}^s f_{\alpha_1}^r (v_{2,-})\big), \operatorname{Im}\big(f_{\alpha_2}^s f_{\alpha_1}^r (v_{2,-})\big) \bigm| r, s \in \mathbb{N}\big\} \tag{3.45}$$

by Lemma 2.4. Observe that

$$\begin{aligned} f_{\alpha_1}^r (v_{2,-}) &= (y_2 \partial_{y_1} + \kappa_2 \partial_{\kappa_1})^r \big(-y_2 y_1^{k-1} \kappa_1 + y_1^k \kappa_2\big) \\ &= \big(y_2^r \partial_{y_1}^r + r y_2^{r-1} \kappa_2 \partial_{y_1}^{r-1} \partial_{\kappa_1}\big)\big(-y_2 y_1^{k-1} \kappa_1 + y_1^k \kappa_2\big) \\ &= \left(\prod_{j=0}^{r-1} k-1-j\right)\big(-y_2^{r+1} y_1^{k-1-r} \kappa_1 + y_2^r y_1^{k-r} \kappa_2\big) \end{aligned} \tag{3.46}$$

for $r \leqslant k-1$, and

$$\begin{aligned}
&f_{\alpha_2}^{s}\big(-y_2^{r+1}y_1^{k-1-r}\kappa_1+y_2^{r}y_1^{k-r}\kappa_2\big)\\
&\quad=(y_4\partial_{y_1}-y_3\partial_{y_2}+\kappa_4\partial_{\kappa_1}-\kappa_3\partial_{\kappa_2})^s\big(-y_2^{r+1}y_1^{k-1-r}\kappa_1+y_2^{r}y_1^{k-r}\kappa_2\big)\\
&\quad=\big((y_4\partial_{y_1}-y_3\partial_{y_2})^s+s(y_4\partial_{y_1}-y_3\partial_{y_2})^{s-1}(\kappa_4\partial_{\kappa_1}-\kappa_3\partial_{\kappa_2})\big)\big(-y_2^{r+1}y_1^{k-1-r}\kappa_1+y_2^{r}y_1^{k-r}\kappa_2\big)\\
&\quad=\frac{1}{2}\begin{pmatrix}-(y_4\partial_{y_1}-y_3\partial_{y_2})^s y_2^{r+1}y_1^{k-1-r}-is(y_4\partial_{y_1}-y_3\partial_{y_2})^{s-1}y_2^{r}y_1^{k-r}\\ (y_4\partial_{y_1}-y_3\partial_{y_2})^s y_2^{r}y_1^{k-r}-is(y_4\partial_{y_1}-y_3\partial_{y_2})^{s-1}y_2^{r+1}y_1^{k-1-r}\\ -i(y_4\partial_{y_1}-y_3\partial_{y_2})^s y_2^{r+1}y_1^{k-1-r}-s(y_4\partial_{y_1}-y_3\partial_{y_2})^{s-1}y_2^{r}y_1^{k-r}\\ i(y_4\partial_{y_1}-y_3\partial_{y_2})^s y_2^{r}y_1^{k-r}-s(y_4\partial_{y_1}-y_3\partial_{y_2})^{s-1}y_2^{r+1}y_1^{k-1-r}\end{pmatrix}
\end{aligned}\tag{3.47}$$

for $0\leqslant s\leqslant k+1$. Denote

$$g_+(r,s)=2^k\,\mathrm{Re}\big((y_4\partial_{y_1}-y_3\partial_{y_2})^s y_2^r y_1^{k-r}\big)\tag{3.48}$$

and

$$g_-(r,s)=2^k\,\mathrm{Im}\big((y_4\partial_{y_1}-y_3\partial_{y_2})^s y_2^r y_1^{k-r}\big).\tag{3.49}$$

Then the real part of (3.47) is

$$\vec{v}(r,s)=\frac{1}{2^{k+1}}\begin{pmatrix}-g_+(r+1,s)+sg_-(r,s-1)\\ g_+(r,s)+sg_-(r+1,s-1)\\ g_-(r+1,s)-sg_+(r,s-1)\\ -g_-(r,s)-sg_+(r+1,s-1)\end{pmatrix},\tag{3.50}$$

and the imaginary part of (3.47) is

$$\vec{w}(r,s)=\frac{1}{2^{k+1}}\begin{pmatrix}-g_-(r+1,s)-sg_+(r,s-1)\\ g_-(r,s)-sg_+(r+1,s-1)\\ -g_+(r+1,s)-sg_-(r,s-1)\\ g_+(r,s)-sg_-(r+1,s-1)\end{pmatrix}.\tag{3.51}$$

Moreover, a straightforward calculation shows that

(i) If $0\leqslant r+s\leqslant k$ and $0\leqslant r<s$, then

$$\begin{aligned}
g_\pm(r,s)={}&\sum_{l=0}^{s}\sum_{p=0}^{k-r-s}\sum_{q=0}^{s-r}\delta_{(-1)^{r+p+q},\pm1}(-1)^{l+[\![\frac{r+p+q}{2}]\!]}s!\binom{k-r}{s-l}\binom{r}{l}\binom{k-r-s}{p}\binom{s-r}{q}\\
&\times\big(x_1^2+x_3^2\big)^l\big(x_2^2+x_4^2\big)^{r-l}x_1^{k-r-s-p}x_2^q x_3^p x_4^{s-r-q}.
\end{aligned}\tag{3.52}$$

(ii) If $0\leqslant r+s\leqslant k$ and $0\leqslant s\leqslant r$, then

$$\begin{aligned}
g_\pm(r,s)={}&\sum_{l=0}^{s}\sum_{p=0}^{k-r-s}\sum_{q=0}^{r-s}\delta_{(-1)^{r+p+q},\pm1}(-1)^{l+[\![\frac{r+p+q}{2}]\!]}s!\binom{k-r}{s-l}\binom{r}{l}\binom{k-r-s}{p}\binom{r-s}{q}\\
&\times\big(x_1^2+x_3^2\big)^l\big(x_2^2+x_4^2\big)^{s-l}x_1^{k-r-s-p}x_2^{r-s-q}x_3^p x_4^q.
\end{aligned}\tag{3.53}$$

(iii) If $r+s>k$ and $0\leqslant r<s$, then

$$g_{\pm}(r,s)=\sum_{l=0}^{s}\sum_{p=0}^{r+s-k}\sum_{q=0}^{s-r}\delta_{(-1)^{r-p+q},\pm 1}(-1)^{l+[\![\frac{r-p+q}{2}]\!]}s!\binom{k-r}{s-l}\binom{r}{l}\binom{r+s-k}{p}\binom{s-r}{q} \times \left(x_1^2+x_3^2\right)^{k-r-s+l}\left(x_2^2+x_4^2\right)^{r-l}x_1^{r+s-k-p}x_2^q x_3^p x_4^{s-r-q}. \tag{3.54}$$

(iv) If $r+s>k$ and $0\leqslant s\leqslant r$, then

$$g_{\pm}(r,s)=\sum_{l=0}^{s}\sum_{p=0}^{r+s-k}\sum_{q=0}^{r-s}\delta_{(-1)^{r-p-q},\pm 1}(-1)^{l+[\![\frac{r-p-q}{2}]\!]}s!\binom{k-r}{s-l}\binom{r}{l}\binom{r+s-k}{p}\binom{r-s}{q} \times \left(x_1^2+x_3^2\right)^{k-r-s+l}\left(x_2^2+x_4^2\right)^{s-l}x_1^{r+s-k-p}x_2^q x_3^p x_4^{r-s-q}. \tag{3.55}$$

Now we have the following base vectors of $\hat{\mathcal{H}}_{k,2-}$:

(i) $$\vec{f}=\vec{v}(r,s)\quad\text{if}\begin{cases}0\leqslant r\leqslant [\![\frac{k}{2}]\!],\\ 0\leqslant s\leqslant [\![\frac{k+1}{2}]\!],\end{cases}\quad\text{or}\quad\begin{cases}[\![\frac{k}{2}]\!]+1\leqslant r\leqslant k-1,\\ 0\leqslant s\leqslant [\![\frac{k}{2}]\!],\end{cases} \tag{3.56}$$

(ii) $$\vec{f}=\vec{w}(r,s)\quad\text{if}\begin{cases}0\leqslant r\leqslant [\![\frac{k}{2}]\!],\\ 0\leqslant s\leqslant [\![\frac{k+1}{2}]\!],\end{cases}\quad\text{or}\quad\begin{cases}[\![\frac{k}{2}]\!]+1\leqslant r\leqslant k-1,\\ 0\leqslant s\leqslant [\![\frac{k}{2}]\!],\end{cases} \tag{3.57}$$

(iii) $$\vec{f}=\begin{cases}\vec{v}(r,s) & \text{if } \vec{v}(r,s)\neq 0,\\ \vec{w}(r,s) & \text{if } \vec{v}(r,s)=0\end{cases}\quad\text{for } r=\frac{k-1}{2},\ s=\frac{k+1}{2}\text{ and } k \text{ is odd.} \tag{3.58}$$

Then by (3.45), we get that

Proposition 3.5. *The set of vectors $\vec{f}$ given in* (3.56)–(3.58) *forms a basis of $\hat{\mathcal{H}}_{k,2-}$.*

Remark 3.6. We observe that there exists a linear isomorphism between vector spaces $\hat{\mathcal{H}}_{k,2-}$ and $\hat{\mathcal{H}}_{k,2+}$

$$\sigma:\hat{\mathcal{H}}_{k,2-}\to\hat{\mathcal{H}}_{k,2+},\quad\text{where } \sigma(x_1)=x_1,\ \sigma(x_2)=x_4,\ \sigma(x_3)=x_3 \text{ and } \sigma(x_4)=x_2.$$

These give a basis of $\hat{\mathcal{H}}_{k,2+}$ by easily interchanging x_2 and x_4 in a basis of $\hat{\mathcal{H}}_{k,2-}$.

At the end of this section, we will use Xu's method in [10] to construct a uniform basis of the polynomial solution space of Navier equations, whose the cardinality was pre-used in the proof of Theorem 2.12. It is different from those bases given above. It is not listed in accordance to the irreducible summands of the polynomial solution space. Xu's method is also critical to solve the initial value problems of Navier equations and Lamé equations in the next section.

Lemma 3.7. *(See [10].) Suppose that $\mathscr{A}$ is a free module of a subalgebra $\mathscr{B}$ generated by a filtrated subspace $V=\bigcup_{r=0}^{\infty}V_r$ (i.e. $V_r\subset V_{r+1}$). Let T_0 be a linear operator on $\mathscr{A}$ with right inverse T_0^- such that $T_0(\mathscr{B})\subset\mathscr{B}$, $T_0^-(\mathscr{B})\subset\mathscr{B}$ and $T_0(\eta_1\eta_2)=T_0(\eta_1)\eta_2$ for $\eta_1\in\mathscr{B}$, $\eta_2\in V$. Let $T_1,\dots,T_m$ be linear operators on $\mathscr{A}$ such that $T_j(V)\subset V$, $T_j(f\zeta)=fT_j(\zeta)$ for $j=1,\dots,m$, $f\in\mathscr{B}$, $\zeta\in\mathscr{A}$. If $T_0^m(h)=0$ with $h\in\mathscr{B}$ and $g\in V$, then*

$$u=\sum_{j=0}^{\infty}\left(\sum_{s=1}^{m}\left(T_0^-\right)^s T_s\right)^j(hg) \tag{3.59}$$

is a solution of the equation

$$\left(T_0^m-\sum_{j=1}^{m}T_0^{m-j}T_j\right)(u)=0. \tag{3.60}$$

Suppose $T_j(V_r)\subset V_{r-1}$ for $j=1,\dots,m$, $r\in\mathbb{N}$, where $V_{-1}=0$. Then any polynomial solution of (3.60) *is a linear combination of the solutions of the form* (3.59)*. In particular, if $T_rT_s=T_sT_r$, $T_0T_j=T_jT_0$ and $T_0^-T_j=T_jT_0^-$ for any $j,r,s\in\{1,\dots,m\}$, then u can be written as follows*:

$$u=\sum_{i_1,\dots,i_m=0}^{\infty}\binom{i_1+\cdots+i_m}{i_1,\dots,i_m}\big(T_0^-\big)^{\sum_{s=1}^m si_s}(h)\left(\prod_{r=1}^{m}T_r^{i_r}\right)(g). \tag{3.61}$$

Note that Navier equations (1.1) can be written as the following form

$$\big(T_0^2-T_0T_1-T_2\big)(\vec{u})=0, \tag{3.62}$$

where

$$T_0=\partial_{x_1}I_n,\qquad T_1=-\begin{pmatrix}0&\frac{b}{b+1}\partial_{x_2}&\cdots&\frac{b}{b+1}\partial_{x_n}\\ b\partial_{x_2}&0&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ b\partial_{x_n}&0&\cdots&0\end{pmatrix}, \tag{3.63}$$

$$T_2=-\begin{pmatrix}\frac{1}{b+1}\sum_{j=2}^n\partial_{x_j}^2&0&0&\cdots&0\\ 0&b\partial_{x_2}^2+\sum_{j=2}^n\partial_{x_j}^2&b\partial_{x_2}\partial_{x_3}&\cdots&b\partial_{x_2}\partial_{x_n}\\ 0&b\partial_{x_3}\partial_{x_2}&b\partial_{x_3}^2+\sum_{j=2}^n\partial_{x_j}^2&\cdots&b\partial_{x_3}\partial_{x_n}\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&b\partial_{x_n}\partial_{x_2}&b\partial_{x_n}\partial_{x_3}&\cdots&b\partial_{x_n}^2+\sum_{j=2}^n\partial_{x_j}^2\end{pmatrix}, \tag{3.64}$$

where $b=(\iota_1+\iota_2)/\iota_1$. Set $\mathscr{B}=\mathbb{R}[x_1]I_n$ and

$$V=\sum_{r=1}^{n}\mathbb{R}[x_2,\dots,x_n]\varsigma_r. \tag{3.65}$$

By Lemma 3.7, any polynomial solution of (3.62) is a linear combination of

$$\vec{u}=\sum_{m=0}^{\infty}\big(\big(T_0^-\big)^m\big(T_1+T_0^-T_2\big)^m\big)(hg), \tag{3.66}$$

where $h=x_1^{\epsilon}I_n$ and $g\in V$. In order to write down these solutions explicitly, we need to calculate $(T_0^-)^m(T_1+T_0^-T_2)^m$. Recall that $E_{r,s}$ is the $n\times n$ matrix whose (r,s)th entry is 1 and the others are 0. We define the linear operator $\int_{(x_1)}$ on $\mathbb{R}[x_1,\dots,x_n]$ by

$$\int_{(x_1)}\big(x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}\big)=\frac{1}{r_1+1}x_1^{r_1+1}x_2^{r_2}\cdots x_n^{r_n}. \tag{3.67}$$

We take

$$T_0^- = \int_{(x_1)} I_n. \tag{3.68}$$

Then

$$-\left(T_1 + T_0^- T_2\right) = \left((b+1)^{-1}\sum_{j=2}^{n}\partial_{x_j}^2 \int_{(x_1)}\right)E_{1,1} + \sum_{r=2}^{n}\left(\frac{b\partial_{x_r}}{b+1}E_{1,r} + b\partial_{x_r}E_{r,1}\right)$$
$$+ \sum_{r=2}^{n}\left(\sum_{j=2}^{n}\partial_{x_j}^2 \int_{(x_1)}\right)E_{r,r} + \sum_{r,s=2}^{n}\left(b\partial_{x_r}\partial_{x_s}\int_{(x_1)}\right)E_{r,s}. \tag{3.69}$$

Observe that $\int_{(x_1)}, \partial_{x_2}, \ldots, \partial_{x_n}$ commute pairwise, and so the entries of $(T_0^-)^m(T_1 + T_0^- T_2)^m$ are polynomials in $\mathbb{R}[\int_{(x_1)}, \partial_{x_2}, \ldots, \partial_{x_n}]$. In order to use linear algebra, we replace these operators by real numbers as Xu did in [11]. Indeed, if we let

$$B(\hat a_1, \hat a_2, \ldots, \hat a_n) = \begin{pmatrix} \frac{\xi}{b+1} & \frac{b\hat a_2}{b+1} & \frac{b\hat a_3}{b+1} & \cdots & \frac{b\hat a_n}{b+1} \\ b\hat a_2 & b\hat a_1\hat a_2^2 + \xi & b\hat a_1\hat a_2\hat a_3 & \cdots & b\hat a_1\hat a_2\hat a_n \\ b\hat a_3 & b\hat a_1\hat a_3\hat a_2 & b\hat a_1\hat a_3^2 + \xi & \cdots & b\hat a_1\hat a_3\hat a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b\hat a_n & b\hat a_1\hat a_n\hat a_2 & b\hat a_1\hat a_n\hat a_3 & \cdots & b\hat a_1\hat a_n^2 + \xi \end{pmatrix} \tag{3.70}$$

with $\hat a_s \in \mathbb{R}$,

$$\eta = \sum_{j=2}^{n}\hat a_j^2 \quad \text{and} \quad \xi = \hat a_1\eta, \tag{3.71}$$

then

$$B\left(\int_{(x_1)} \partial_2, \ldots, \partial_{x_n}\right) = -\left(T_1 + T_0^- T_2\right). \tag{3.72}$$

Observe that for $m \geqslant 1$,

$$B^m = \begin{pmatrix} \frac{1}{\sqrt{b+1}} & 0 \\ 0 & I_{n-1} \end{pmatrix} A^m \begin{pmatrix} \sqrt{b+1} & 0 \\ 0 & I_{n-1} \end{pmatrix} \tag{3.73}$$

with

$$A = \begin{pmatrix} \frac{\xi}{b+1} & \frac{b\hat a_2}{\sqrt{b+1}} & \frac{b\hat a_3}{\sqrt{b+1}} & \cdots & \frac{b\hat a_n}{\sqrt{b+1}} \\ \frac{b\hat a_2}{\sqrt{b+1}} & b\hat a_1\hat a_2^2 + \xi & b\hat a_1\hat a_2\hat a_3 & \cdots & b\hat a_1\hat a_2\hat a_n \\ \frac{b\hat a_3}{\sqrt{b+1}} & b\hat a_1\hat a_3\hat a_2 & b\hat a_1\hat a_3^2 + \xi & \cdots & b\hat a_1\hat a_3\hat a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b\hat a_n}{\sqrt{b+1}} & b\hat a_1\hat a_n\hat a_2 & b\hat a_1\hat a_n\hat a_3 & \cdots & b\hat a_1\hat a_n^2 + \xi \end{pmatrix} \tag{3.74}$$

is symmetrical (note that $b+1=(2\iota_1+\iota_2)/\iota_1>0$). Moreover, we denote

$$a=(b+1)^2+1 \quad \text{and} \quad \varpi=(b+2)^2\xi^2+4(b+1)\eta. \tag{3.75}$$

It can be proved that the eigenvalues of A are

$$\xi, \quad \theta_1=\frac{a\xi+b\sqrt{\varpi}}{2(b+1)} \quad \text{and} \quad \theta_2=\frac{a\xi-b\sqrt{\varpi}}{2(b+1)}, \tag{3.76}$$

where the multiplicity of ξ is $n-2$. Recall that $\varsigma_r=(0,\dots,0,\overset{r}{1},0,\dots,0)^T$. We can take orthonormal eigenvectors $\vec{v}_j=(0,p_{j,2},\dots,p_{j,n})^T$ for $j=1,\dots,n-2$ corresponding to the eigenvalue ξ, a unit eigenvector

$$\vec{v}_{n-1}=\frac{\sqrt{2(b+1)\eta}}{\sqrt{\varpi+(b+2)\xi\sqrt{\varpi}}}\varsigma_1+\sum_{r=2}^{n}\frac{\hat{a}_r((b+2)\xi+\sqrt{\varpi})}{\sqrt{2\eta(\varpi+(b+2)\xi\sqrt{\varpi})}}\varsigma_r \tag{3.77}$$

corresponding to the eigenvalue θ_1, and a unit eigenvector

$$\vec{v}_{n}=\frac{\sqrt{2(b+1)\eta}}{\sqrt{\varpi-(b+2)\xi\sqrt{\varpi}}}\varsigma_1+\sum_{r=2}^{n}\frac{\hat{a}_r((b+2)\xi-\sqrt{\varpi})}{\sqrt{2\eta(\varpi-(b+2)\xi\sqrt{\varpi})}}\varsigma_r \tag{3.78}$$

corresponding to the eigenvalue θ_2. Setting $P=(\vec{v}_1,\dots,\vec{v}_n)$ and

$$J=\begin{pmatrix} \xi I_{n-2} & 0 & 0 \\ 0 & \theta_1 & 0 \\ 0 & 0 & \theta_2 \end{pmatrix}, \tag{3.79}$$

we have that

$$A^m=PJ^mP^t=P\cdot\operatorname{diag}\bigl(\xi^m,\dots,\xi^m,\theta_1^m,\theta_2^m\bigr)\cdot P^T=(c_{r,s})_{n\times n}, \tag{3.80}$$

where

$$c_{1,j}=c_{j,1}=p_{1,n-1}p_{j,n-1}\theta_1^m+p_{1,n}p_{j,n}\theta_2^m \tag{3.81}$$

for $1\leqslant j\leqslant n$, and

$$\begin{aligned} c_{r,s}&=\sum_{j=1}^{n-2}p_{r,j}p_{s,j}\xi^m+p_{r,n-1}p_{s,n-1}\theta_1^m+p_{r,n}p_{s,n}\theta_2^m \\ &=\delta_{r,s}\xi^m+p_{r,n-1}p_{s,n-1}\bigl(\theta_1^m-\xi^m\bigr)+p_{r,n}p_{s,n}\bigl(\theta_2^m-\xi^m\bigr) \end{aligned} \tag{3.82}$$

for $r,s\in\{2,\dots,n\}$ by the fact $PP^T=I_n$. Substituting (3.76) into the above two equations, we have that

$$c_{11} = \frac{2(b+1)\eta}{\varpi + (b+2)\xi\sqrt{\varpi}} \cdot \frac{(a\xi + b\sqrt{\varpi})^m}{2^m(b+1)^m} + \frac{2(b+1)\eta}{\varpi - (b+2)\xi\sqrt{\varpi}} \cdot \frac{(a\xi - b\sqrt{\varpi})^m}{2^m(b+1)^m}$$
$$= \frac{1}{2^m(b+1)^m}\left(\sum_{2|j}\binom{m}{j}a^{m-j}b^j\xi^{m-j}\varpi^{\frac{j}{2}} - (b+2)\sum_{2\nmid j}\binom{m}{j}a^{m-j}b^j\xi^{m-j+1}\varpi^{\frac{j-1}{2}}\right), \quad (3.83)$$

$$c_{r,1} = \frac{\sqrt{2(b+1)\eta}}{\sqrt{\varpi + (b+2)\xi\sqrt{\varpi}}} \cdot \frac{\hat{a}_r((b+2)\xi + \sqrt{\varpi})}{\sqrt{2\eta(\varpi + (b+2)\xi\sqrt{\varpi})}} \cdot \frac{(a\xi + b\sqrt{\varpi})^m}{2^m(b+1)^m}$$
$$+ \frac{\sqrt{2(b+1)\eta}}{\sqrt{\varpi - (b+2)\xi\sqrt{\varpi}}} \cdot \frac{\hat{a}_r((b+2)\xi - \sqrt{\varpi})}{\sqrt{2\eta(\varpi - (b+2)\xi\sqrt{\varpi})}} \cdot \frac{(a\xi - b\sqrt{\varpi})^m}{2^m(b+1)^m}$$
$$= \frac{\hat{a}_r}{2^{m-1}(b+1)^{m-\frac{1}{2}}}\sum_{2\nmid j}\binom{m}{j}a^{m-j}b^j\xi^{m-j}\varpi^{\frac{j-1}{2}} \quad (3.84)$$

for $2 \leqslant r \leqslant n$,

$$c_{r,s} = -\left(\frac{\hat{a}_r((b+2)\xi + \sqrt{\varpi})}{\sqrt{2\eta(\varpi + (b+2)\xi\sqrt{\varpi})}} \cdot \frac{\hat{a}_s((b+2)\xi + \sqrt{\varpi})}{\sqrt{2\eta(\varpi + (b+2)\xi\sqrt{\varpi})}}\right.$$
$$\left. + \frac{\hat{a}_r((b+2)\xi - \sqrt{\varpi})}{\sqrt{2\eta(\varpi - (b+2)\xi\sqrt{\varpi})}} \cdot \frac{\hat{a}_s((b+2)\xi - \sqrt{\varpi})}{\sqrt{2\eta(\varpi - (b+2)\xi\sqrt{\varpi})}}\right)\xi^m$$
$$+ \frac{\hat{a}_r((b+2)\xi + \sqrt{\varpi})}{\sqrt{2\eta(\varpi + (b+2)\xi\sqrt{\varpi})}} \cdot \frac{\hat{a}_s((b+2)\xi + \sqrt{\varpi})}{\sqrt{2\eta(\varpi + (b+2)\xi\sqrt{\varpi})}} \cdot \frac{(a\xi + b\sqrt{\varpi})^m}{2^m(b+1)^m}$$
$$+ \frac{\hat{a}_r((b+2)\xi - \sqrt{\varpi})}{\sqrt{2\eta(\varpi - (b+2)\xi\sqrt{\varpi})}} \cdot \frac{\hat{a}_s((b+2)\xi - \sqrt{\varpi})}{\sqrt{2\eta(\varpi - (b+2)\xi\sqrt{\varpi})}} \cdot \frac{(a\xi - b\sqrt{\varpi})^m}{2^m(b+1)^m}$$
$$= -\hat{a}_1\hat{a}_r\hat{a}_s\xi^{m-1} + \frac{\hat{a}_1\hat{a}_r\hat{a}_s}{2^m(b+1)^m}\left(\sum_{2|j}\binom{m}{j}a^{m-j}b^j\xi^{m-j-1}\varpi^{\frac{j}{2}}\right.$$
$$\left. + (b+2)\sum_{2\nmid j}\binom{m}{j}a^{m-j}b^j\xi^{m-j}\varpi^{\frac{j-1}{2}}\right) \quad (3.85)$$

for $2 \leqslant r, s \leqslant n$ with $r \neq s$, and

$$c_{r,r} = \hat{a}_1\left(\eta - \hat{a}_r^2\right)\xi^{m-1} + \frac{\hat{a}_1\hat{a}_r^2}{2^m(b+1)^m}\left(\sum_{2|j}\binom{m}{j}a^{m-j}b^j\xi^{m-j-1}\varpi^{\frac{j}{2}}\right.$$
$$\left. + (b+2)\sum_{2\nmid j}\binom{m}{j}a^{m-j}b^j\xi^{m-j}\varpi^{\frac{j-1}{2}}\right) \quad (3.86)$$

for $2 \leqslant r \leqslant n$.

For convenience, we denote

$$f(m,s) = \sum_{r=s}^{[\![\frac{m}{2}]\!]} \frac{4^s(b+1)^s(b+2)^{2r-2s}a^{m-2r}b^{2r}}{2^m(b+1)^m}\binom{r}{s}\binom{m}{2r} \quad (3.87)$$

and

$$g(m,s)=\sum_{r=s}^{[\![\frac{m}{2}]\!]}\frac{4^s(b+1)^s(b+2)^{2r-2s}a^{m-2r-1}b^{2r+1}}{2^m(b+1)^m}\binom{r}{s}\binom{m}{2r+1}. \tag{3.88}$$

Thus we have

$$\left(\hat{a}_1^m B^m\right)(\hat{a}_1,\hat{a}_2,\dots,\hat{a}_n)=\begin{pmatrix}\hat{a}_1^m c_{1,1} & \frac{c_{1,2}\hat{a}_1^m}{\sqrt{b+1}} & \cdots & \frac{c_{1,n}\hat{a}_1^m}{\sqrt{b+1}}\\ \sqrt{b+1}\hat{a}_1^m c_{2,1} & \hat{a}_1^m c_{2,2} & \cdots & \hat{a}_1^m c_{2,n}\\ \vdots & \vdots & \ddots & \vdots\\ \sqrt{b+1}\hat{a}_1^m c_{n,1} & \hat{a}_1^m c_{n,2} & \cdots & \hat{a}_1^m c_{n,n}\end{pmatrix} \tag{3.89}$$

for $m \geqslant 1$, where

$$\hat{a}_1^m c_{1,1}=\sum_{s=0}^{[\![\frac{m}{2}]\!]}\big(f(m,s)-(b+2)g(m,s)\big)\hat{a}_1^{2m-2s}\eta^{m-s}, \tag{3.90}$$

$$\frac{\hat{a}_1^m c_{1,j}}{\sqrt{b+1}}=\sum_{s=0}^{[\![\frac{m-1}{2}]\!]}2g(m,s)\hat{a}_1^{2m-2s-1}\eta^{m-s-1}\hat{a}_j, \tag{3.91}$$

$$\sqrt{b+1}\hat{a}_1^m c_{j,1}=\sum_{s=0}^{[\![\frac{m-1}{2}]\!]}2(b+1)g(m,s)\hat{a}_1^{2m-2s-1}\eta^{m-s-1}\hat{a}_j \tag{3.92}$$

for $2 \leqslant j \leqslant n$ and

$$\hat{a}_1^m c_{j,l}=\delta_{j,l}\hat{a}_1^{2m}\eta^m+\sum_{s=0}^{[\![\frac{m}{2}]\!]}\big(-\delta_{s,0}+f(m,s)+(b+2)g(m,s)\big)\hat{a}_1^{2m-2s}\eta^{m-s-1}\hat{a}_j\hat{a}_l \tag{3.93}$$

for $2 \leqslant j, l \leqslant n$ by (3.73), (3.83)–(3.89). According to (3.72),

$$\left(T_0^-\right)^m\left(T_1+T_0^-T_2\right)^m=(-1)^m\left(\hat{a}_1^m B^m\right)\left(\int_{(x_1)},\partial_{x_2},\dots,\partial_{x_n}\right). \tag{3.94}$$

Set

$$\Delta_{2,n}=\sum_{r=2}^{n}\partial_{x_r}^2. \tag{3.95}$$

By (3.66) and (3.89)–(3.94), we have the following solutions

$$\vec{u}_j(\epsilon,l_2,\dots,l_n)=\begin{pmatrix}u_j^1(\epsilon,l_2,\dots,l_n)\\ \vdots\\ u_j^n(\epsilon,l_2,\dots,l_n)\end{pmatrix} \tag{3.96}$$

of Navier equations, where

$$u_1^1 = x_1^{\epsilon} \prod_{q=2}^{n} x_q^{l_q} + \sum_{m=1}^{\infty} \sum_{s=0}^{[\![\frac{m}{2}]\!]} (-1)^m \big(f(m,s) - (b+2)g(m,s)\big) \times \frac{x_1^{\epsilon+2m-2s}}{(\epsilon+2m-2s)!} \Delta_{2,n}^{m-s} \left(\prod_{q=2}^{n} x_q^{l_q}\right), \tag{3.97}$$

$$u_1^r = \sum_{m=1}^{\infty} \sum_{s=0}^{[\![\frac{m}{2}]\!]} (-1)^m 2(b+1) g(m,s) l_r \frac{x_1^{\epsilon+2m-2s-1}}{(\epsilon+2m-2s-1)!} \Delta_{2,n}^{m-s-1} \left(x_r^{-1} \prod_{q=2}^{n} x_q^{l_q}\right) \tag{3.98}$$

for $2 \leqslant r \leqslant n$,

$$u_j^1 = \sum_{m=1}^{\infty} \sum_{s=0}^{[\![\frac{m-1}{2}]\!]} (-1)^m 2g(m,s) l_j \frac{x_1^{\epsilon+2m-2s-1}}{(\epsilon+2m-2s-1)!} \Delta_{2,n}^{m-s-1} \left(x_j^{-1} \prod_{q=2}^{n} x_q^{l_q}\right) \tag{3.99}$$

for $2 \leqslant j \leqslant n$,

$$u_j^r = \sum_{m=1}^{\infty} \sum_{s=0}^{[\![\frac{m-1}{2}]\!]} (-1)^m \big(-\delta_{s,0} + f(m,s) + (b+2)g(m,s)\big) l_r l_j \times \frac{x_1^{\epsilon+2m-2s}}{(\epsilon+2m-2s)!} \Delta_{2,n}^{m-s-1} \left(x_r^{-1} x_j^{-1} \prod_{q=2}^{n} x_q^{l_q}\right) \tag{3.100}$$

for $2 \leqslant r,\ j \leqslant n$ with $r \neq j$, and

$$u_j^j = x_1^{\epsilon} \prod_{q=2}^{n} x_q^{l_q} + \sum_{m=1}^{\infty} (-1)^m \frac{x_1^{\epsilon+2m}}{(\epsilon+2m)!} \Delta_{2,n}^{m} \left(\prod_{q=2}^{n} x_q^{l_q}\right) + \sum_{m=1}^{\infty} \sum_{s=0}^{[\![\frac{m}{2}]\!]} (-1)^m \big(-\delta_{s,0} + f(m,s) + (b+2)g(m,s)\big) l_j (l_j - 1) \times \frac{x_1^{\epsilon+2m-2s}}{(\epsilon+2m-2s)!} \Delta_{2,n}^{m-s-1} \left(x_j^{-2} \prod_{q=2}^{n} x_q^{l_q}\right) \tag{3.101}$$

for $2 \leqslant j \leqslant n$.

Theorem 3.8. *The set* $\{\vec{u}_j(\epsilon, l_2, \dots, l_n) \mid \epsilon = 0 \text{ or } 1;\ l_r \in \mathbb{N};\ j = 1, \dots, n\}$ *forms a basis of the space of polynomial solutions for Navier equations. In particular,*

$$\dim \hat{\mathcal{H}}_k = n \cdot \dim \mathcal{H}_k. \tag{3.102}$$

4. Initial value problems

In this section we will deal with the initial value problems of Navier equations and Lamé equations using Lemma 3.7 and Fourier expansions.

Recall that the initial value problem of Navier equations is as follows:

$$\begin{cases} \iota_1 \Delta(\vec{u}) + (\iota_1 + \iota_2)\big(\nabla^T \nabla\big)(\vec{u}) = 0, \\ \vec{u}(0, x_2, \ldots, x_n) = \vec{g}_0(x_2, \ldots, x_n), \\ \vec{u}_{x_1}(0, x_2, \ldots, x_n) = \vec{g}_1(x_2, \ldots, x_n) \end{cases} \tag{4.1}$$

with $x_1 \in \mathbb{R}$ and $x_r \in [-a_r,\ a_r]$ for $r = 2, \ldots, n$, where $a_2, \ldots, a_n$ are positive numbers, and the jth component g_ϵ^j of $\vec{g}_\epsilon$ is a continuous function for $\epsilon = 0$ or 1 and $j = 1, \ldots, n$. For convenience, we denote

$$k_r^\dagger = \frac{k_r}{a_r}, \qquad \vec{k}^\dagger = \big(k_2^\dagger, \ldots, k_n^\dagger\big)^T \quad \text{for } \vec{k} = (k_2, \ldots, k_n)^T \in \mathbb{N}^{n-1}, \tag{4.2}$$

and

$$\vec{x} = (x_2, \ldots, x_n)^T, \qquad \vec{k}^\dagger \cdot \vec{x} = \sum_{r=2}^{n} a_r^{-1} k_r x_r. \tag{4.3}$$

By Lemma 3.7, we have that

$$\vec{\phi}_\epsilon(x_1, \ldots, x_n) = \sum_{m=0}^{\infty} \big(T_0^-\big)^m \big(T_1 + T_0^- T_2\big)^m \big(x_1^\epsilon \vec{g}_\epsilon\big) \tag{4.4}$$

and

$$\vec{\psi}_1(x_1, \ldots, x_n) = \sum_{m=0}^{\infty} \big(T_0^-\big)^m \big(T_1 + T_0^- T_2\big)^m \big(x_1 \partial_{x_1} \big(\vec{\phi}_0(0, x_2, \ldots, x_n)\big)\big) \tag{4.5}$$

are solutions of Navier equations. Denote

$$\vec{u}(x_1, \ldots, x_n) = \vec{\phi}_0(x_1, \ldots, x_n) + \vec{\phi}_1(x_1, \ldots, x_n) - \vec{\psi}_1(x_1, \ldots, x_n). \tag{4.6}$$

Then by (3.90)–(3.94) and superposition principle, the function $\vec{u}(x_1, \ldots, x_n)$ is the solution of (4.1). Now we give the explicit expression of $\vec{u}(x_1, \ldots, x_n)$. For convenience, we write that

$$\vec{\phi}_\epsilon = \sum_{j=1}^{n} \phi_\epsilon^j \varsigma_j, \qquad \vec{\psi}_1 = \sum_{j=1}^{n} \vec{\psi}_1^j \varsigma_j. \tag{4.7}$$

Take the Fourier expansions of $\vec{g}_\epsilon$:

$$\vec{g}_\epsilon = \sum_{j=1}^{n} \sum_{\vec{0} \preccurlyeq \vec{k} \in \mathbb{N}^{n-1}} \big(b_\epsilon^j(\vec{k}) \cos 2\pi \big(\vec{k}^\dagger \cdot \vec{x}\big) + c_\epsilon^j(\vec{k}) \sin 2\pi \big(\vec{k}^\dagger \cdot \vec{x}\big)\big) \varsigma_j, \tag{4.8}$$

where

$$b_\epsilon^j(\vec{k}) = \frac{1}{2^{n-2}a_2\cdots a_n}\int_{-a_2}^{a_2}\cdots\int_{-a_n}^{a_n} g_\epsilon^j(x_2,\ldots,x_n)\cos 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)dx_n\cdots dx_2, \tag{4.9}$$

$$c_\epsilon^j(\vec{k}) = \frac{1}{2^{n-2}a_2\cdots a_n}\int_{-a_2}^{a_2}\cdots\int_{-a_n}^{a_n} g_\epsilon^j(x_2,\ldots,x_n)\sin 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)dx_n\cdots dx_2. \tag{4.10}$$

Substituting (4.8) into (4.4), and using (3.90)–(3.94) again, we get that

$$\begin{aligned}
\phi_\epsilon^1 = \sum_{\vec{0}\preccurlyeq\vec{k}\in\mathbb{N}^{n-1}}\sum_{m=0}^{\infty}\Bigg(&\sum_{s=0}^{[\![\frac{m}{2}]\!]}(-1)^s4^{m-s}\big(f(m,s)-(b+2)g(m,s)\big)\frac{x_1^{\epsilon+2m-2s}}{(\epsilon+2m-2s)!}\\
&\times\left(b_\epsilon^1(\vec{k})\cos 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)+c_\epsilon^1(\vec{k})\sin 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)\right)\left(\sum_{r=2}^{n}\left(k_r^\dagger\pi\right)^2\right)^{m-s}\\
&+\sum_{l=2}^{n}\sum_{s=0}^{[\![\frac{m-1}{2}]\!]}(-1)^s4^{m-s}g(m,s)\pi k_l^\dagger\frac{x_1^{\epsilon+2m-2s-1}}{(\epsilon+2m-2s-1)!}\\
&\times\left(b_\epsilon^l(\vec{k})\sin 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)-c_\epsilon^l(\vec{k})\cos 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)\right)\left(\sum_{r=2}^{n}\left(k_r^\dagger\pi\right)^2\right)^{m-s-1}\Bigg),
\end{aligned}\tag{4.11}$$

$$\begin{aligned}
\phi_\epsilon^j = \sum_{\vec{0}\preccurlyeq\vec{k}\in\mathbb{N}^{n-1}}\sum_{m=0}^{\infty}\Bigg(&\sum_{s=0}^{[\![\frac{m-1}{2}]\!]}(-1)^s4^{m-s}(b+1)g(m,s)\pi k_j^\dagger\frac{x_1^{\epsilon+2m-2s-1}}{(\epsilon+2m-2s-1)!}\\
&\times\left(b_\epsilon^1(\vec{k})\sin 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)-c_\epsilon^1(\vec{k})\cos 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)\right)\left(\sum_{r=2}^{n}\left(k_r^\dagger\pi\right)^2\right)^{m-s-1}\\
&+4^m\frac{x_1^{\epsilon+2m}}{(\epsilon+2m)!}\left(b_\epsilon^j(\vec{k})\cos 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)+c_\epsilon^j(\vec{k})\sin 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)\right)\left(\sum_{r=2}^{n}\left(k_r^\dagger\pi\right)^2\right)^{m}\\
&+\sum_{l=2}^{n}\sum_{s=0}^{[\![\frac{m}{2}]\!]}(-1)^s4^{m-s}\pi^2k_j^\dagger k_l^\dagger\big(-\delta_{s,0}+f(m,s)+(b+2)g(m,s)\big)\frac{x_1^{\epsilon+2m-2s}}{(\epsilon+2m-2s)!}\\
&\times\left(b_\epsilon^l(\vec{k})\cos 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)+c_\epsilon^l(\vec{k})\sin 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)\right)\left(\sum_{r=2}^{n}\left(k_r^\dagger\pi\right)^2\right)^{m-s-1}\Bigg)
\end{aligned}\tag{4.12}$$

for $j=2,\ldots,n$,

$$\begin{aligned}
\psi_1^1 = \sum_{\vec{0}\preccurlyeq\vec{k}\in\mathbb{N}^{n-1}}\sum_{m=0}^{\infty}\Bigg(&\sum_{l=2}^{n}\sum_{s=0}^{[\![\frac{m}{2}]\!]}\frac{(-1)^s4^{m-s}\cdot 2b\pi k_l^\dagger}{b+1}\big(f(m,s)-(b+2)g(m,s)\big)\frac{x_1^{1+2m-2s}}{(1+2m-2s)!}\\
&\times\left(b_0^l(\vec{k})\sin 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)-c_0^l(\vec{k})\cos 2\pi\left(\vec{k}^\dagger\cdot\vec{x}\right)\right)\left(\sum_{r=2}^{n}\left(k_r^\dagger\pi\right)^2\right)^{m-s}
\end{aligned}$$

$$+\sum_{l=2}^{n}\sum_{s=0}^{[\![\frac{m-1}{2}]\!]}(-1)^{s+1}4^{m-s}\cdot 2b\big(\pi k_l^{\dagger}\big)^2 g(m,s)\frac{x_1^{2m-2s}}{(2m-2s)!}$$

$$\times\big(b_0^1(\vec{k})\cos 2\pi\big(\vec{k}^{\dagger}\cdot\vec{x}\big)+c_0^1(\vec{k})\sin 2\pi\big(\vec{k}^{\dagger}\cdot\vec{x}\big)\big)\Bigg(\sum_{r=2}^{n}\big(k_r^{\dagger}\pi\big)^2\Bigg)^{m-s-1}\Bigg), \tag{4.13}$$

and

$$\psi_1^j=\sum_{\vec{0}\preccurlyeq\vec{k}\in\mathbb{N}^{n-1}}\sum_{m=0}^{\infty}\Bigg(\sum_{l=2}^{n}\sum_{s=0}^{[\![\frac{m-1}{2}]\!]}(-1)^{s+1}4^{m-s}\cdot 2b\pi^2 k_l^{\dagger}k_j^{\dagger}g(m,s)\frac{x_1^{2m-2s}}{(2m-2s)!}$$

$$\times\big(b_0^l(\vec{k})\cos 2\pi\big(\vec{k}^{\dagger}\cdot\vec{x}\big)+c_0^l(\vec{k})\sin 2\pi\big(\vec{k}^{\dagger}\cdot\vec{x}\big)\big)\Bigg(\sum_{r=2}^{n}\big(k_r^{\dagger}\pi\big)^2\Bigg)^{m-s-1}$$

$$+4^m\cdot 2b\pi k_j^{\dagger}\frac{x_1^{1+2m}}{(1+2m)!}\big(b_0^1(\vec{k})\sin 2\pi\big(\vec{k}^{\dagger}\cdot\vec{x}\big)-c_0^1(\vec{k})\cos 2\pi\big(\vec{k}^{\dagger}\cdot\vec{x}\big)\big)\Bigg(\sum_{r=2}^{n}\big(k_r^{\dagger}\pi\big)^2\Bigg)^{m}$$

$$+\sum_{l=2}^{n}\sum_{s=0}^{[\![\frac{m}{2}]\!]}(-1)^{s}4^{m-s}\cdot 2\pi^3 k_j^{\dagger}\big(k_l^{\dagger}\big)^2\big(-\delta_{s,0}+f(m,s)+(b+2)g(m,s)\big)\frac{x_1^{1+2m-2s}}{(1+2m-2s)!}$$

$$\times\big(b_0^1(\vec{k})\sin 2\pi\big(\vec{k}^{\dagger}\cdot\vec{x}\big)-c_0^1(\vec{k})\cos 2\pi\big(\vec{k}^{\dagger}\cdot\vec{x}\big)\big)\Bigg(\sum_{r=2}^{n}\big(k_r^{\dagger}\pi\big)^2\Bigg)^{m-s-1}\Bigg) \tag{4.14}$$

for $j=2,\dots,n$. Thus, as we mentioned, by superposition principle and Fourier expansions, we get

Theorem 4.1. *The solution of* (4.1) *is*

$$\vec{u}(x_1,\dots,x_n)=\sum_{l=1}^{n}\Bigg(\sum_{\epsilon=0}^{1}\phi_\epsilon^l(x_1,\dots,x_n)-\psi_1^l(x_1,\dots,x_n)\Bigg)\varsigma_l, \tag{4.15}$$

where $\phi_\epsilon^l(x_1,\dots,x_n)$ and $\psi_1^l(x_1,\dots,x_n)$ are defined by (4.11)–(4.14)*. The convergence of the series* (4.15) *is guaranteed by Kovalevskaya Theorem on the existence and uniqueness of the solution of linear partial differential equations when the given functions in* (4.1) *are analytic.*

The initial value problem of Lamé equations is as follows:

$$\begin{cases}\vec{u}_{tt}-b^{-1}\Delta(\vec{u})-\big(\nabla^T\nabla\big)(\vec{u})=0,\\ \vec{u}(0,x_1,\dots,x_n)=\vec{h}_0(x_1,\dots,x_n),\\ \vec{u}_t(0,x_1,\dots,x_n)=\vec{h}_1(x_1,\dots,x_n)\end{cases} \tag{4.16}$$

with $x_r\in[-a_r,a_r]$ for $r=1,\dots,n$, where $a_1,\dots,a_n$ are positive numbers and the jth component h_ϵ^j of $\vec{h}_\epsilon$ is continuous function for $\epsilon=0$ or 1 and $j=1,\dots,n$.

Set

$$T_1=\partial_t^2 I_n,\qquad T_2=b^{-1}\Delta I_n+H,\quad \text{where } H=\nabla^T\nabla. \tag{4.17}$$

Then by Lemma 3.7, the set

$$\left\{ \sum_{m=0}^{\infty} (T_1^-)^m \big(t^\epsilon \big(T_2^m\big)(\vec{g})\big) \;\Big|\; \vec{g} \in \hat{\mathcal{A}};\ \epsilon = 0 \text{ or } 1 \right\} \tag{4.18}$$

spans the polynomial solution space of Lamé equations. Note that

$$T_2^m = b^{-m}\big(\Delta^m I_n + \big((b+1)^m - 1\big)\Delta^{m-1} H\big) \tag{4.19}$$

for $m \geqslant 1$ and $T_2^0 = I_n$.

For convenience, we denote

$$k_r^\dagger = \frac{k_r}{a_r}, \qquad \vec{k}^\dagger = \big(k_1^\dagger, \ldots, k_n^\dagger\big)^T \quad \text{for } \vec{k} = (k_1, \ldots, k_n)^T \in \mathbb{N}^n, \tag{4.20}$$

and

$$\vec{x} = (x_1, \ldots, x_n)^T, \qquad \vec{k}^\dagger \cdot \vec{x} = \sum_{r=1}^{n} a_r^{-1} k_r x_r. \tag{4.21}$$

Take the Fourier expansions of $\vec{h}_\epsilon$:

$$\vec{h}_\epsilon = \sum_{j=1}^{n} \sum_{\vec{0} \preccurlyeq \vec{k} \in \mathbb{N}^n} \big(b_\epsilon^j(\vec{k}) \cos 2\pi \big(\vec{k}^\dagger \cdot \vec{x}\big) + c_\epsilon^j(\vec{k}) \sin 2\pi \big(\vec{k}^\dagger \cdot \vec{x}\big)\big) \varsigma_j, \tag{4.22}$$

where

$$b_\epsilon^j(\vec{k}) = \frac{1}{2^{n-1} a_1 \cdots a_n} \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} h_\epsilon^j(x_1, \ldots, x_n) \cos 2\pi \big(\vec{k}^\dagger \cdot \vec{x}\big)\, dx_n \cdots dx_1, \tag{4.23}$$

$$c_\epsilon^j(\vec{k}) = \frac{1}{2^{n-1} a_1 \cdots a_n} \int_{-a_1}^{a_1} \cdots \int_{-a_n}^{a_n} h_\epsilon^j(x_1, \ldots, x_n) \sin 2\pi \big(\vec{k}^\dagger \cdot \vec{x}\big)\, dx_n \cdots dx_1. \tag{4.24}$$

Note that

$$\vec{\phi}_\epsilon = \sum_{m=0}^{\infty} (T_1^-)^m \big(t^\epsilon \big(T_2^m\big)(\vec{h}_\epsilon)\big) \tag{4.25}$$

are solutions of Lamé equations for $\epsilon = 0, 1$. Then by superposition principle, the vector

$$\vec{u}(t, x_1, \ldots, x_n) = \sum_{\epsilon=0}^{1} \vec{\phi}_\epsilon(t, x_1, \ldots, x_n) \tag{4.26}$$

is also a solution. Moreover, one finds easily that

$$\vec{u}(0, x_1, \ldots, x_n) = \vec{h}_0(x_1, \ldots, x_n) \tag{4.27}$$

and

$$\vec{u}_t(0, x_1, \ldots, x_n) = \vec{h}_1(x_1, \ldots, x_n). \tag{4.28}$$

Thus it is the solution of (4.16). Substituting (4.22) into (4.26), we get

Theorem 4.2. *The solution of* (4.16) *is*

$$\begin{aligned}\vec{u} = {} & \sum_{\vec{0}\preccurlyeq\vec{k}\in\mathbb{N}^n} \sum_{\epsilon=0}^{1}\sum_{j=1}^{n}\sum_{m=0}^{\infty} \frac{(-1)^m t^{\epsilon+2m}}{b^m(\epsilon+2m)!}\Bigg(\big(b_\epsilon^j(\vec{k})\cos 2\pi\big(\vec{k}^\dagger\cdot\vec{x}\big) + c_\epsilon^j(\vec{k})\sin 2\pi\big(\vec{k}^\dagger\cdot\vec{x}\big)\big) \\ & \times \left(\sum_{r=1}^{n}\big(2\pi k_r^\dagger\big)^2\right)^m + \sum_{l=1}^{n}\big((b+1)^m-1\big)4\pi k_j^\dagger k_l^\dagger \\ & \times \big(b_\epsilon^l(\vec{k})\cos 2\pi\big(\vec{k}^\dagger\cdot\vec{x}\big) + c_\epsilon^l(\vec{k})\sin 2\pi\big(\vec{k}^\dagger\cdot\vec{x}\big)\big)\left(\sum_{r=1}^{n}\big(2\pi k_r^\dagger\big)^2\right)^{m-1}\Bigg)\varsigma_j. \end{aligned} \tag{4.29}$$

The convergence of the series (4.29) *is guaranteed by Kovalevskaya Theorem on the existence and uniqueness of the solution of linear partial differential equations when the given functions in* (4.16) *are analytic.*

Set

$$\vec{g}_r = \big(0, \ldots, 0, \overset{r}{x_1^{l_1}\cdots x_n^{l_n}}, 0, \ldots, 0\big)^T \tag{4.30}$$

and

$$T_2^m(\vec{g}_r) = (\tilde{g}_1, \ldots, \tilde{g}_n)^T. \tag{4.31}$$

Then by (4.19), we have

$$\tilde{g}_j = \delta_{j,r} b^{-m}\left(\Delta^m\left(\prod_{s=1}^{n} x_s^{l_s}\right) + l_j l_r\big((b+1)^m-1\big)\Delta^{m-1}\left(x_r^{-1}x_j^{-1}\prod_{s=1}^{n} x_s^{l_s}\right)\right). \tag{4.32}$$

Thus by (4.18), we obtain

Proposition 4.3. *The set*

$$\begin{aligned}\Bigg\{ & t^\epsilon\left(\prod_{s=1}^{n} x_s^{l_s}\right)\varsigma_r + \sum_{j=1}^{n}\sum_{m=1}^{\infty}\delta_{j,r}b^{-m}\frac{t^{\epsilon+2m}}{(\epsilon+2m)!}\left(\Delta^m\left(\prod_{s=1}^{n} x_s^{l_s}\right) + l_j l_r\big((b+1)^m-1\big)\right. \\ & \left.\times\,\Delta^{m-1}\left(x_r^{-1}x_j^{-1}\prod_{s=1}^{n} x_s^{l_s}\right)\right)\varsigma_j \;\Bigg|\; r=1,\ldots,n,\ \epsilon=0 \text{ or } 1,\ l_s\in\mathbb{N}\Bigg\} \end{aligned} \tag{4.33}$$

forms a basis of polynomial solution space of Lamé equations.

Acknowledgments

I would like to thank Professor Xiaoping Xu for his advice and suggesting this research topic. I also thank the referee for helpful suggestions which improved the exposition of this paper.

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