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OSCILLATION OF A NEUTRAL DIFFERENCE EQUATION

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Abstract—This paper is concerned with the oscillation of the bounded solutions of neutral difference equation

$$\Delta[a_n\Delta^{m-1}(x_n - p_n x_{n-k})] + \delta q_n f(x_{\sigma_n}) = 0,$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$.

1. INTRODUCTION

Let N denote the natural numbers and let $N(a) = \{a, a+1, \dots,\}$. Consider

$$\Delta[a_n\Delta^{m-1}(x_n - p_n x_{n-k})] + \delta q_n f(x_{\sigma_n}) = 0, \tag{1}$$

where $\delta = \pm 1$, k is a positive integer, $\{a_n\}$, $\{p_n\}$, $\{q_n\}$, and $\{\sigma_n\}$ are sequences of real numbers on $N(n_0)$ for $n_0 \geq 0$, $a_n > 0$ with $\Delta a_n \geq 0$ and

$$\sum^{\infty} \frac{1}{a_n} = \infty, \tag{2}$$

$1 < a \leq p_n \leq b < \infty$ for some real numbers a and b, $q_n \geq 0$, $\lim_{n\to\infty} \sigma_n = \infty$, and $f : R \to R$ is continuous such that $xf(x) > 0$ for $x \neq 0$.

By a solution of (1) we mean a sequence $\{x_n\}$ which is defined for $n \geq \min_{m\geq 0}\{m - k, \sigma_m\}$ and satisfies (1) for n sufficiently large. A nontrivial solution $\{x_n\}$ of (1) is said to be oscillatory if the terms x_n are not eventually positive or eventually negative.

The oscillation theorem we prove here is the discrete analogue of a theorem we have recently obtained. The proof proceeds in a similar manner but quite different due to discrete nature of the equation (1). Furthermore, it requires discrete anologues of Kiguradze's lemmas [1] that are not available to us. We have established these analogues, but in the following, we can only state them due to limited space.

2. LEMMAS

LEMMA 1. *Let $\{y_n\}$ be a sequence of real numbers on $N = \{0, 1, 2, 3, \dots\}$. Let y_n and $\Delta^m y_n$ be of constant sign with $\Delta^m y_n$ being not identically zero in any subset of the form $\{n_1, n_1 + 1, \dots\}$ of N. If*

$$y_n \Delta^m y_n \leq 0,$$

then

(i) *there is a natural number $n_2 \geq n_1$ such that the sequences $\{\Delta^j y_n\}$, $j = 1, 2, \dots, n-1$ are of constant sign on $\{n_2, n_2 + 1, \dots\}$;*
(ii) *there exists a number $l \in \{0, 1, 2, \dots, m-1\}$ with $(-1)^{m-l-1} = 1$ such that*

$$\begin{aligned} y_n\Delta^j y_n &> 0 && \text{for } j = 0, 1, \dots, l, && n \geq n_2 \\ (-1)^{j-l} y_n \Delta^j y_n &> 0 && \text{for } j = l+1, \dots, m-1, && n \geq n_2. \end{aligned}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

LEMMA 2. *Assume that y_n together with $\Delta^j y_n$, $j = 1,2,\dots,m-1$, are of constant sign on $N(n_1)$. Moreover*

$$y_n\Delta^m y_n \geq 0.$$

Then either

$$y_n\Delta^j y_n \geq 0, \qquad j = 1,2,\dots,m$$

or one can find a number l, $l \in \{0,1,\dots,m-2\}$, $(-1)^{m-l} = 1$, such that

$$y_n\Delta^j y_n > 0, \qquad j = 1,2,\dots,l,$$
$$(-1)^{j-l}y_n\Delta^j y_n > 0, \qquad j = l+1,\dots,m-2.$$

3. THE MAIN RESULT

THEOREM. *In addition to above conditions, suppose that*

$$\sum^{\infty} n^{m-1}\frac{q_n}{a_n} = \infty. \tag{3}$$

Then

(i) *every bounded solution $\{x_n\}$ of (1) is oscillatory when $(-1)^m\delta = -1$, and*
(ii) *every bounded solution $\{x_n\}$ of (1) is either oscillatory or satisfies*

$$\liminf_{n\to\infty} x_n = 0$$

when $(-1)^m\delta = 1$.

PROOF. Let x_n be an eventually positive solution of (1). Set $z_n = x_n - p_n x_{n-k}$. If z_n is eventually positive, then we have

$$x_n > p_n x_{n-k} \geq a x_{n-k},$$

and therefore by induction,

$$x_n > a^j x_{n-jk}$$

or

$$x_{n+jk} > a^j x_n$$

for every positive integer j. Letting $j \to \infty$ in the last inequality we see that

$$\lim_{j\to\infty} x_j = \infty.$$

Since this is a contradiction with x_n being bounded, we conclude that z_n is eventually negative. It follows from (1) that $\delta\Delta[a_n\Delta^{m-1}z_n]$ is also eventually negative. Thus, it can be claimed that eventually $\delta\Delta^{m-1}z_n$ is either positive or negative. Suppose that it is eventually negative, then there is an $N_1 \geq n_0$ such that for $n \geq N_1$,

$$\delta a_n\Delta^{m-1}z_n \leq \delta a_{N_1}\Delta^{m-1}z_{N_1} < 0.$$

Dividing both sides of this inequality by a_n and summing from N_1 to n, we obtain

$$\delta\Delta^{m-1}z_{n+1} \leq \delta a_{N_1}\Delta^{m-1}z_{N_1}\sum_{i=N_1}^{n}\frac{1}{a_n}.$$

In view of (2), we see that $\delta\Delta^{m-1}z_n \to -\infty$ as $n \to \infty$, which is of course a contradiction with z_n being bounded. Thus we see that $\delta\Delta^{m-1}z_n$ is eventually positive. Now from (1), it follows that

$$\delta a_n\Delta^m z_n = -(\Delta a_n)(\delta\Delta^{m-1}z_{n+1}) - q_n f(x_{\sigma_n}). \tag{4}$$

Since Δa_n and q_n are nonnegative, (4) implies that $\delta\Delta^m z_n$ is eventually negative. In view of the fact that z_n is bounded, applying Lemma 1 and Lemma 2, one can easily see that there are numbers $n_1 \geq N_1$ and $l \in \{0,1\}$, $(-1)^{m-l}\delta = 1$, such that for $n \geq n_1$

$$\begin{aligned} \Delta^j z_n < 0, &\qquad j = 0,1,2,\ldots,l, \\ (-1)^{j-l}\Delta^j z_n < 0, &\qquad j = l+1,\ldots,m. \end{aligned} \tag{5}$$

It is clear from (4) that

$$\delta\Delta^m z_n + \frac{q_n}{a_n} f(x_{\sigma_n}) \leq 0. \tag{6}$$

Multiplying (6) by n^{m-1} and summing from n_1 to n and then applying the summation by parts formula to the first term in the resulting inequality, we obtain

$$\sum_{i=0}^{m-2}(-1)^{i+1}\delta(\Delta^i n_1^{m-1})(\Delta^{m-i-1}z_{n_1+i}) + (-1)^{m-1}\delta(m-1)![z_{n+m} - z_{n_1+m-1}] + \sum_{j=n_1}^{n} j^{m-1}\frac{q_j}{a_j}f(x_{\sigma_j}) \leq 0. \tag{7}$$

Since $\{z_n\}$ is bounded, if we let $n \to \infty$ in (7) then we must have

$$\sum_{j=n_1}^{\infty} j^{m-1}\frac{q_j}{a_j}f(x_{\sigma_j}) < \infty. \tag{8}$$

From (3) and (8), it follows that

$$\liminf_{n\to\infty} x_n = 0.$$

Now we shall show that $\lim_{n\to\infty} z_n = 0$. Clearly,

$$z_{n+k} - z_n = x_{n+k} - (p_{n+k}+1)x_n + p_n x_{n-k}. \tag{9}$$

Let $\{n_j\}$ be such that $n_j \to \infty$ as $j \to \infty$, and $x_{n_j} \to 0$ as $j \to \infty$. Then from (9) we get

$$0 = \lim_{k\to\infty}[x_{n_j+k} + p_{n_j}x_{n_j-k}].$$

As $x_{n_j+k} > 0$ and $p_{n_j}x_{n_j-k} > 0$, we see that $p_{n_j}x_{n_j-k} \to 0$ as $j \to \infty$. If we now use the fact that p_n is bounded and $z_{n_j} = x_{n_j} - p_{n_j}x_{n_j-k}$, we see that

$$\lim_{n\to\infty} z_n = 0.$$

Note that if $(-1)^m\delta = -1$, then it follows from (5) that $l = 1$ and consequently z_n is negative and decreasing. In this case, $\lim_{n\to\infty} z_n = 0$ is not possible, and therefore x_n must be oscillatory.

Suppose that $(-1)^m\delta = 1$. Then $l = 0$ and so z_n increases to 0 as n grows to infinity. That is, given $\epsilon > 0$, there exists an $n_2 \geq n_1$ such that

$$z_n > -\epsilon, \qquad \text{for all } n \geq n_2.$$

Thus,

$$x_n - p_n x_{n-k} > -\epsilon, \qquad \text{for } n \geq n_1$$

or

$$x_n > -\epsilon + a x_{n-k}, \qquad \text{for } n \geq n_1$$

or

$$a x_n < \epsilon + x_{n+k}, \qquad \text{for } n \geq n_1.$$

By induction,

$$a^j x_n < \epsilon a\epsilon + \cdots + a^{j-1}\epsilon + x_{n+jk}, \qquad \text{for } n \geq n_1.$$

Let M be a bound for x_n, then it follows from the last inequality that

$$x_n < \frac{a^{-j}-1}{1-a}\epsilon + Ma^{-j}. \tag{10}$$

Since $\lim_{j\to\infty} a^{-j} = 0$ and $\epsilon > 0$ is arbitrary, (10) implies that

$$\lim_{n\to\infty} x_n = 0.$$

This completes the proof.

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