JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 21, 479-484 (1968)

Integrated Lagrange Expansions for a Monge-Ampere Equation

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1. INTRODUCTION

Nonlinear partial differential equations of order higher than the first present complications which make their systematic classification and solution difficult. Some classical methods for solving a given second-order equation seek a solvable first-order equation which is such that the original equation can be obtained from it and the two relations derived by partial differentiation (Jones and Ames [1] use a related development in examining a large class of hyperbolic equations). Such a first-order equation, called an intermediate integral, is often obtainable for the Monge-Ampere equation

$$Rr + Ss + Tt + U(rt - s^2) = V, \tag{1}$$

where $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$, and R, S, T, U and V are functions of x, y, z, p, and q, by application of such methods as those of Monge or Boole. Details of these methods are found in Forsyth [2, pp. 200-220]. However, the solutions so developed are not always in a form directly applicable to a particular problem.

In this note we shall describe several physical problems that lead to equa-

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[2] Research supported by the National Science Foundation under Grant GK-136.

tions of Monge-Ampere type (Eq. (1)) and then shall develop a generalized Lagrange series solution for the equation

$$rt - s^2 = 0. \tag{2}$$

2. Examples

(a) *Anisentropic Flow of Gas*

The unsteady, one dimensional, anisentropic flow of a polytropic gas, neglecting viscosity, conduction and radiation is represented by the equations

$$\begin{aligned} \rho_t + (\rho u)_x = 0, \qquad (\rho u)_t + (p + \rho u^2)_x = 0 \\ s_t + us_x = 0, \qquad s = f(p, \rho), \end{aligned} \tag{3}$$

where ρ, p, u and s are density, pressure, velocity, and entropy respectively. Martin [3, 4] (see also Ames [5, pp. 94-100, pp. 404] and Giese [6]) obtains the Monge-Ampere equation

$$\xi_{\psi\psi}\xi_{pp} - \xi_{\psi p}^2 = \tau_p \tag{4}$$

for the flow in terms of the independent variables p and particle trajectory function ψ. The function $\tau(\psi, p) = 1/\rho$ is determined once the entropy function f of Eq. (3) is specified.

(b) *Longitudinal Wave Propagation in a Moving Threadline*

Recent studies by the first author and his students on the transverse and longitudinal wave propagation in a moving threadline have disclosed a central role for Eq. (1). Herein we describe the situation for the longitudinal wave propagation when the transverse amplitude is small. In that case the dimensionless equations become

$$u_t + uu_x = \frac{T_x}{M} \tag{5a}$$

$$M_t + (uM)_x = 0 \tag{5b}$$

$$M(T + N) = BN, \tag{5c}$$

where u, T, M, t, x, N and B are velocity, tension, mass per unit length, time, distance and two physical constant groups.

Upon multiplying Eq. (5a) by M and Eq. (5b) by u and adding, we obtain

$$(uM)_t + (Mu^2 - T)_x = 0; \tag{6}$$

which form will be used in place of Eq. (5a). If θ is defined by means of the relations

$$\theta_{xx} = M, \qquad \theta_{xt} = -uM, \tag{7}$$

then Eq. (5b) is identically satisfied; and by choosing

$$\theta_{tt} = Mu^2 - T \tag{8}$$

Eq. (6) is also satisfied. From Eqs. (7) and (8) we obtain the defining relations

$$u = -\frac{\theta_{xt}}{\theta_{xx}}, \qquad M = \theta_{xx}, \qquad T = \frac{\theta_{xt}^2 - \theta_{xx}\theta_{tt}}{\theta_{xx}}. \tag{9}$$

Upon substituting Eq. (9) into Eq. (5c) the Monge-Ampere equation

$$N\theta_{xx} - (\theta_{xx}\theta_{tt} - \theta_{xt}^2) = - BN \tag{10}$$

is obtained

An alternate to this formulation is offered by applying Martin's method. Let us define two auxiliary functions $\bar{\phi}(x, t)$ and $\psi(x, t)$ by means of the expressions

$$\bar{\phi}_x = uM, \qquad \bar{\phi}_t = -(Mu^2 - T) \tag{11}$$

$$\psi_x = M, \qquad \psi_t = -uM, \tag{12}$$

whereupon

$$d\bar{\phi} = uM\,dx + (T - Mu^2)\,dt \tag{13}$$

$$d\psi = M\,dx - uM\,dt. \tag{14}$$

Equation (13) may be simplified by noting that

$$d\bar{\phi} = u(M\,dx - uM\,dt) + T\,dt = u\,d\psi - t\,dT + d(tT), \tag{15}$$

whereupon the introduction of $\phi = \bar{\phi} - tT$ yields the new expression

$$d\phi = u\,d\psi - t\,dT. \tag{16}$$

From Eq. (16) we deduce that

$$\frac{\partial\phi}{\partial\psi} = u, \qquad \frac{\partial\phi}{\partial T} = -t. \tag{17}$$

The introduction of T and ψ as independent variables is further facilitated by developing the equation for dx. From Eq. (14)

$$\begin{aligned} dx &= \gamma\,d\psi + u\,dt, \qquad \gamma = M^{-1} \\ &= \gamma\,d\psi + \phi_\psi[-\phi_{T\psi}\,d\psi - \phi_{TT}\,dT] \\ &= [\gamma - \phi_\psi\phi_{T\psi}]\,d\psi - \phi_\psi\phi_{TT}\,dT, \end{aligned} \tag{18}$$

whereupon

$$x_\psi = \gamma - \phi_\psi \phi_{T\psi}, \qquad x_T = -\phi_\psi \phi_{TT}. \tag{19}$$

From Eq. (19) $x_{\psi T}$ and $x_{T\psi}$ are developed and equated thereby obtaining the Monge-Ampere equation

$$\phi_{TT}\phi_{\psi\psi} - \phi_{T\psi}^2 = -\frac{1}{BN} = \text{constant}. \tag{20}$$

For the problem at hand the quantity $(BN)^{-1}$ is sufficiently small that a regular perturbation solution is feasible. The zero order equation in such a development would be

$$\phi_{TT}\phi_{\psi\psi} - \phi_{T\psi}^2 = 0. \tag{21}$$

An integrated Lagrange series solution of Eq. (21) will be constructed in the sequel.

3. Series Solution

Consider the Monge-Ampere equation

$$z_{xx}z_{tt} - z_{xt}^2 = 0. \tag{22}$$

Upon setting $p = z_x$, $q = z_t$ Eq. (22) can be written as the system

$$p_x q_t - p_t q_x = 0, \qquad p_t - q_x = 0. \tag{23}$$

The first of these equations expresses the relation

$$\frac{\partial(p, q)}{\partial(x, t)} = 0, \tag{24}$$

where the left-hand side is a notation for the Jacobian of p and q. Equation (24) implies that p and q are functionally dependent so that

$$q = F(p) \tag{25}$$

for all differentiable functions F. This is the well-known intermediate integral for Eq. (22).

Upon substituting Eq. (25) into the second of Eq. (23), we have the first-order partial differential equation,

$$p_t - F'(p)\, p_x = 0, \tag{26}$$

for the determination of p. By an elementary application of the Lagrange-Charpit theory (e.g., see Ames [5, pp. 50-58]) the general solution of Eq. (26) is found to be

$$G[p, x + F'(p)\,t] = G[z_x\,, x + F'(z_x)\,t] = 0 \tag{27}$$

for arbitrary but differentiable G.

Equation (27) is not the most tractable form for the determination of z. Alternatively, we can obtain a series solution to Eq. (26). Here we are motivated by a remark from Bellman [7, p. 7]—"The Lagrange expansion is derivable from the observation that if $u = g(x - ut)$ then $u_t + uu_x = 0$." Lagrange expansions have been utilized by Banta [8] in the development of solutions for finite amplitude sound waves.

An integrated Lagrange series solution is now constructed for Eq. (26). Suppose p has a Taylor series expansion about $t = 0$,

$$p(x, t) = \Gamma_0(x) + \sum_{n=1}^{\infty} \Gamma_n(x) \frac{t^n}{n!}$$

$$\Gamma_n(x) = \left.\frac{\partial^n p(x, t)}{\partial t^n}\right|_{t=0}. \tag{28}$$

This form is inconvenient since the derivatives are with respect to t and not x. Replacement of these time derivatives is accomplished by using the differential Eq. (26). By an inductive proof, similar to that in Goursat [9, p. 405], Banta [8] shows that if

$$u_t + f(u)\, u_x = 0 \tag{29}$$

then

$$\frac{\partial^n u}{\partial t^n} = (-1)^n \frac{\partial^{n-1}}{\partial x^{n-1}} \left[\frac{\partial u}{\partial x} f^n\right]. \tag{30}$$

Upon applying this theorem to Eq. (28) using Eq. (26), we find for $n \geqslant 1$

$$\Gamma_n(x) = \left.\frac{\partial^{n-1}}{\partial x^{n-1}} \left\{\frac{\partial p}{\partial x} [F'(p)]^n\right\}\right|_{t=0}, \tag{31}$$

a form which involves only space derivatives of $p(x, t)$. The expression for $z(x, t)$ is then obtained by integration as

$$z(x, t) = \int^x \Gamma_0(s)\, ds + \sum_{n=1}^{\infty} \frac{t^n}{n!} \int^x \left.\frac{\partial^{n-1}}{\partial s^{n-1}} \left\{\frac{\partial p}{\partial s} [F'(p)]^n\right\}\right|_{t=0} ds, \tag{32}$$

where Γ_0 and F are arbitrary and we have discarded the function of t arising from the integration. If the process of x differentiation and evaluation (at $t = 0$) are interchangeable Eq. (32) takes the form

$$z(x, t) = \int^{x} \Gamma_0(s)\, ds + \sum_{n=1}^{\infty} \frac{t^n}{n!} \int^{x} \frac{d^{n-1}}{ds^{n-1}} \left\{\frac{d\Gamma_0}{ds} [F'(\Gamma_0(s))]^n\right\} ds. \tag{33}$$

Upon setting $\bar{G}(x) = F'(\Gamma_0(x))$ and integrating we obtain

$$z(x, t) = \int^{x} \Gamma_0(s)\, ds + t \int^{x} \Gamma_0'(s)\, \bar{G}(s)\, ds + \sum_{n=2}^{\infty} \frac{t^n}{n!} \frac{d^{n-2}}{dx^{n-2}} \{\Gamma_0'(x)\, [\bar{G}(x)]^n\}, \tag{34}$$

which is the integrated Lagrange expansion (Goursat [9, p. 404] or Whittaker and Watson [10, p. 133]). That Eq. (34) satisfies Eq. (22) is easily demonstrated.

Uniqueness and convergence questions for Eq. (34) are now immediately obtainable from the Lagrange theorem. Local uniqueness is guaranteed by appeal to the implicit function theorem if F' has a convergent power series. The region of convergence is clearly dependent upon x. When it is examined by any of the standard tests information concerning Γ_0, F and their derivatives is required. Since systems such as Eq. (5) can develop shocks the series representation must be used with caution.

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