Advances in Mathematics **146**, 167–174 (1999)
Article ID aima.1999.1829, available online at http://www.idealibrary.com on IDEAL®

On Disjoint Borel Uniformizations

Howard Becker[1]

Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208
E-mail: becker@cs.scarolina.edu

and

Randall Dougherty[1]

Department of Mathematics, Ohio State University, Columbus, Ohio 43210
E-mail: rld@math.ohio-state.edu

Larman showed (1973, *Mathematika* **20**, 233–246) that any closed subset of the plane with uncountable vertical cross-sections has $\aleph_1$ disjoint Borel uniformizing sets. Here we show that Larman's result is best possible: there exist closed sets with uncountable cross sections which do not have more than $\aleph_1$ disjoint Borel uniformizations, even if the continuum is much larger than $\aleph_1$. This negatively answers some questions of Mauldin (1990, "Open Problems in Topology" (J. van Mill and G. M. Reed, Eds.), 617–629). The proof is based on a result of Stern, stating that certain Borel sets cannot be written as a small union of low-level Borel sets (1978, *C. R. Acad. Sci. Pan's Ser. A–B* **286**, A855–A857). The proof of the latter result uses Steel's method of forcing with tagged trees (1978, *Ann. Math. Logic* **15**, 55–74); a full presentation of this method, written in terms of Baire category rather than forcing, is given here. © 1999 Academic Press

Let I be the unit interval $[0, 1]$. It is well known that there exist Borel sets $B \subseteq I \times I$ such that all cross sections $B_x = \{y: (x, y) \in B\}$ are nonempty but there does not exist a Borel uniformization of B (a Borel set $U \subseteq B$ such that for every x there is a unique y such that $(x, y) \in U$; this can also be viewed as a Borel function from I to I which selects a point from each cross section). On the other hand, in some cases (e.g., if the cross sections B_x are all σ-compact or all nonmeager), one can prove that a Borel uniformization exists. See Moschovakis [9] for these results.

[1] The authors were supported by NSF Grant DMS-9158092; the second author was also supported by a fellowship from the Sloan Foundation.

0001-8708/99 $30.00

If all of the cross sections B_x are uncountable, then it is natural to ask whether one can find a large number of disjoint Borel uniformizations of B. Larman [5] has shown that, if the sets B_x are all uncountable and closed (or just $\boldsymbol{\Delta}^0_2$), then one can always find $\aleph_1$ disjoint Borel uniformizations of B. The main purpose of the present paper is to show that the $\aleph_1$ in Larman's result is best possible.

THEOREM 1. *There is a closed set $B \subseteq I \times I$ such that all cross sections $B_x = \{y: (x, y) \in B\}$ are uncountable but there do not exist uncountably many disjoint Borel uniformizations of B whose ranks (as Borel functions from I to I) are bounded below ω_1.*

Hence, there cannot exist $\aleph_2$ disjoint Borel uniformizations of this set B (since one would be able to choose $\aleph_2$ of them with the same Borel function rank). So, unless the Continuum Hypothesis is true, there do not exist continuum many disjoint Borel uniformizations of B. This answers a question raised by Mauldin [8].

It also follows from Theorem 1 that there do not exist uncountably many Borel measurable selector functions of bounded Borel rank for the space $K(I)$ of nonempty compact subsets of I which select distinct points within any uncountable compact set (since, as noted in Mauldin [8], such selector functions could be applied to the cross sections of B to get disjoint Borel uniformizations of B). Hence, one cannot find $\aleph_2$ (or $2^{\aleph_0}$ if CH fails) Borel measurable selector functions for this space which select distinct points within any uncountable compact set. (Mauldin [8] had shown that one can find $\aleph_1$ such functions.) This settles Problem 5.1 from Mauldin [7].

The main step in the proof of Theorem 1 is the following result. Let $\mathrm{cov}(\mathbf{K})$ be the least cardinal κ such that a perfect Polish space can be expressed as a union of κ meager sets. (It does not matter which perfect Polish space is used to define $\mathrm{cov}(\mathbf{K})$, because any such space has a comeager subset homeomorphic to the Baire space.) Clearly $\aleph_1 \leqslant \mathrm{cov}(\mathbf{K}) \leqslant 2^{\aleph_0}$.

THEOREM 2 (Stern). *For any $\alpha < \omega_1$, there is a Borel subset of the Baire space ${}^{\omega}\omega$ which cannot be expressed as the union of fewer than* $\mathrm{cov}(\mathbf{K})$ $\boldsymbol{\Pi}^0_\alpha$ *sets.*

The proof of this, described in Stern [12] (although the result is not stated as generally there), uses Steel's method of forcing with tagged trees. Actually, Stern combines this method with an analysis of the Borel ranks of collections of well-founded trees to produce a stronger result: for any $\alpha < \omega_1$, any Borel set which is a union of fewer than $\mathrm{cov}(\mathbf{K})$ $\boldsymbol{\Sigma}^0_\alpha$ sets must itself be $\boldsymbol{\Sigma}^0_\alpha$. The weaker version above (which was rediscovered independently by the authors) suffices for the application here.

The method of Steel forcing is presented in Harrington [2] and Steel [11]; we will give another presentation here, in terms of Baire category rather than forcing.

Solecki [10] has recently given a different proof of Stern's results, using effective descriptive set theory.

Corollary 3. *A complete analytic or coanalytic set in an uncountable Polish space cannot be written as a union of fewer than* cov($\mathbf{K}$) *Borel sets with ranks bounded below* ω_1.

(It is well known that any analytic or coanalytic set can be written as a union of $\aleph_1$ Borel sets [9].)

Proof. Let X be an analytic (or coanalytic) subset of ${}^{\omega}\omega$ which is complete for analytic (coanalytic) subsets of ${}^{\omega}\omega$ using continuous maps. It will suffice to show that X cannot be written as a union of fewer than cov($\mathbf{K}$) Borel sets with ranks bounded below ω_1, because if Y were a complete analytic (coanalytic) set which could be written as such a union, then one could fix a Borel map reducing X to Y and take preimages of the Borel sets of bounded rank with union Y to get Borel sets of bounded rank with union X. Now, for any $\alpha < \omega_1$, we can find a Borel set $W \subseteq {}^{\omega}\omega$ as in Theorem 2. Let $g: {}^{\omega}\omega \to {}^{\omega}\omega$ be a continuous map reducing W to X. Then X cannot be a union of fewer than cov($\mathbf{K}$) $\mathbf{\Pi}^0_\alpha$ sets, because if it were one could take preimages under g to get fewer than cov($\mathbf{K}$) $\mathbf{\Pi}^0_\alpha$ sets with union W, which is impossible. Since α was arbitrary, we are done. ∎

Actually, one can get a slightly stronger result.

Corollary 4. *In any uncountable Polish space, there exist two disjoint coanalytic sets which cannot be separated by a set which is a union of fewer than* cov($\mathbf{K}$) *Borel sets of bounded rank.*

Proof. Since all uncountable Polish spaces are Borel isomorphic, it will suffice to work in the space $({}^{\omega}\omega)^3$. We follow the usual construction of a universal pair of disjoint coanalytic sets: Let U be a universal coanalytic set in $({}^{\omega}\omega)^2$ (i.e., all coanalytic subsets of ${}^{\omega}\omega$ occur as cross sections U_x); let $C = \{(x, y, z): (x, z) \in U\}$ and $D = \{(x, y, z): (y, z) \in U\}$; and apply the reduction principle for coanalytic sets to get disjoint coanalytic sets $C' \subseteq C$ and $D' \subseteq D$ such that $C' \cup D' = C \cup D$. Now, for any $\alpha < \omega_1$, let B be the Borel set obtained from Theorem 2 and find x and y such that $U_x = B$ and $U_y = {}^{\omega}\omega \backslash B$. Then $C_{x,y} = B$ and $D_{x,y} = {}^{\omega}\omega \backslash B$, so $C'_{x,y} = B$ and $D'_{x,y} = {}^{\omega}\omega \backslash B$. Hence, $C'_{x,y}$ and $D'_{x,y}$ cannot be separated by a union of fewer than cov($\mathbf{K}$) $\mathbf{\Pi}^0_\alpha$ sets, so C' and D' cannot either. ∎

Of course, the preceding results say little if $\operatorname{cov}(\mathbf{K}) = \aleph_1$ (e.g., if CH holds). However, under Martin's Axiom, the union of fewer than $2^{\aleph_0}$ meager sets is meager, so $\operatorname{cov}(\mathbf{K}) = 2^{\aleph_0}$ and these results are more interesting.

In order to prove Theorem 1, we will need to use Corollary 4 to rule out separating sets that are the union of $\aleph_1$ Borel sets of bounded rank. This can be done directly if $\operatorname{cov}(\mathbf{K}) > \aleph_1$; if $\operatorname{cov}(\mathbf{K}) = \aleph_1$, then we will need to do a forcing and absoluteness argument.

The closed set for Theorem 1 will be obtained from a construction given in Mauldin [6, Example 3.2]. The construction uses the following well-known fact, proved by methods probably due to Hurewicz [3].

LEMMA 5 (Hurewicz?). *For any analytic set $A \subseteq I$, there is a closed set $B \subseteq I \times I$ such that if $x \in A$, then B_x is uncountable, and if $x \notin A$, then $B_x \subseteq \mathbf{Q}$.*

Proof. Since A is analytic, there is a closed set $C \subseteq I \times {}^\omega\omega$ whose projection to I is A. Define $C' \subseteq I \times {}^\omega\omega \times {}^\omega\omega$ so that $(x, y, z) \in C'$ iff $(x, y) \in C$. Then, if $x \in A$, there are uncountably many $(y, z) \in {}^\omega\omega \times {}^\omega\omega$ such that $(x, y, z) \in C'$; if $x \notin A$, then there is no such (y, z). But ${}^\omega\omega \times {}^\omega\omega$ is homeomorphic to ${}^\omega\omega$, which is homeomorphic to the set of irrationals in I; let f be a homeomorphism from ${}^\omega\omega \times {}^\omega\omega$ to the irrationals in I. Let B be the closure in $I \times I$ of the set $\{(x, f(y, z)) : (x, y, z) \in C'\}$; then B has the desired properties. ∎

Proof of Theorem 1. Let D_1 and D_2 be disjoint inseparable coanalytic subsets of I as given by Corollary 4, and let $A_1 = I \setminus D_1$ and $A_2 = I \setminus D_2$; then $A_1 \cup A_2 = I$. Let B_1 and B_2 be closed sets in $I \times I$ obtained by applying Lemma 5 to A_1 and A_2. Apply linear mappings to the second coordinate to compress B_1 and B_2 to sets $\hat{B}_1 \subseteq I \times [0, 1/3]$ and $\hat{B}_2 \subseteq I \times [2/3, 1]$ with the same properties. Now let $B = \hat{B}_1 \cup \hat{B}_2$. Since $A_1 \cup A_2 = I$, all cross sections B_x are uncountable. It remains to show that B does not have $\aleph_1$ disjoint Borel uniformizations of bounded rank.

First, let us assume that $\operatorname{cov}(\mathbf{K}) > \aleph_1$. Suppose that we have a collection $\{u_\gamma : \gamma < \omega_1\}$ of pairwise disjoint functions from I to I each uniformizing B, whose Borel function ranks are bounded by some fixed $\alpha < \omega_1$. For each γ, let E_γ be the set of $x \in I$ such that $u_\gamma(x)$ is an irrational number greater than 1/2. Since the set of irrational numbers above 1/2 is $\mathbf{\Pi}^0_2$, each set E_γ is $\mathbf{\Pi}^0_{\alpha+1}$. Now, if $x \in D_2$, then $x \notin A_2$, so B_x contains no irrationals above 1/2, so $x \notin E_\gamma$ for all γ; if $x \in D_1$, then $x \notin A_1$, so B_x contains no irrationals below 1/2, and since the values $u_\gamma(x)$ for $\gamma < \omega_1$ are distinct, only countably many of them can be rational, so $x \in E_\gamma$ for all but countably many γ. Therefore, the set $\bigcup_{\gamma < \omega_1} E_\gamma$ is a union of $\aleph_1$ Borel sets of bounded rank which separates D_1 from D_2; since D_1 and D_2 were obtained from Corollary 4 and $\aleph_1 < \operatorname{cov}(\mathbf{K})$, we have a contradiction.

Now let us drop the assumption that $\operatorname{cov}(\mathbf{K}) > \aleph_1$. Suppose that we have disjoint functions u_γ for $\gamma < \omega_1$ as above. Fix Borel codes of the appropriate ranks for the functions u_γ and the set B. By going through the details of the construction of B, one can check that one obtains the same Borel code for B no matter what transitive model of set theory one is working in. Now, using the current universe as the ground model, construct a generic extension with the same ω_1 in which Martin's Axiom plus $\neg$CH holds. (Collapse some cardinals above $\aleph_1$ in order to make $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, and then do the standard c.c.c. forcing iteration to get $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$.) All of the properties we assumed about the functions u_γ, including the property of being a function with domain I, are easily seen to be $\mathbf{\Pi}^1_2$ assertions about the Borel codes (which are only used one or two at a time), so by the Shoenfield absoluteness theorem these codes define functions in the generic extension which satisfy the same assertions. But $\operatorname{cov}(\mathbf{K}) > \aleph_1$ holds in the extension, so we get a contradiction, as in the preceding paragraph. ▮

It now remains to give the proof of Theorem 2.

Proof of Theorem 2. Let $\mathscr{T}$ be the space of trees of ω, viewed as a (closed) subspace of the space of subsets of ${}^{<\omega}\omega$ with the usual Cantor topology (which in turn is homeomorphic to a closed subspace of the Baire space). For any tree $T \in \mathscr{T}$, define the rank function $\mathrm{rk}_T\colon T \to \omega_1 \cup \{\infty\}$ by the following condition: for any $s \in T$, $\mathrm{rk}_T(s)$ is an ordinal iff $\mathrm{rk}_T(sn)$ is an ordinal for all immediate successors sn of s which are in T, and in this case $\mathrm{rk}_T(s)$ is the least ordinal greater than all of the ordinals $\mathrm{rk}_T(sn)$. So $\mathrm{rk}_T(s) = 0$ iff s is a leaf of T, and $\mathrm{rk}_T(s) = \infty$ iff T is not well-founded below s.

We will work with a slightly restricted set of trees: let $\mathscr{T}'$ be the set of $T \in \mathscr{T}$ such that the null sequence $\langle\ \rangle$ is in T, and for any sequence s either all immediate successors sn ($n \in \omega$) of s are in T or none of them are. Clearly $\mathscr{T}'$ is closed in $\mathscr{T}$.

For any $\beta < \omega_1$, let R_β be the set of trees $T \in \mathscr{T}'$ such that, for any $s \in T$, $\mathrm{rk}_T(s)$ is either ∞ or less than β. Equivalently, R_β is the set of trees which have no nodes of rank exactly β. Since an easy induction on γ shows that $\{T \in \mathscr{T}\colon \mathrm{rk}_T(s) = \gamma\}$ is Borel for any s and any $\gamma < \omega_1$, the sets R_β are all Borel. We will prove Theorem 2 by showing that if $\beta \geqslant \omega \cdot \alpha$, then R_β is not a union of fewer than $\operatorname{cov}(\mathbf{K})$ $\mathbf{\Pi}^0_\alpha$ sets.

Define a *tagged tree* to be a pair (T, H) where $T \in \mathscr{T}$ and H is a function from T to $\omega_1 \cup \infty$ such that, for any $s \in T$ and $s' \subset s$, we have $H(s') > H(s)$ (where ∞ is defined to be greater than any ordinal and greater than itself). For example, if $T \in \mathscr{T}$, then (T, rk_T) is a tagged tree, and so is $(T', \mathrm{rk}_T \upharpoonright T')$ for any subtree T' of T. We write $(T, H) \subseteq (T', H')$ when $T \subseteq T'$ and $H \subseteq H'$. For $\beta < \omega_1$, a *β-tagged tree* is a tagged tree (T, H) such that $H\colon T \to \beta \cup \{\infty\}$. A β-tagged tree can be viewed as a subset of ${}^{<\omega}\omega \times (\beta \cup \{\infty\})$; the set $\mathscr{T}_\beta$ of all β-tagged trees is a $\mathbf{\Pi}^0_2$ subset of the space of subsets of ${}^{<\omega}\omega \times (\beta \cup \{\infty\})$, so, with the inherited topology, it is itself a Polish space by Alexandrov's Theorem [4].

Let $P_\beta \subseteq \mathscr{T}_\beta$ be the set of finite β-tagged trees. Now define a new topology on R_β, to be called the β-topology, with basis consisting of the sets

$$N_p^\beta = \{T \in R_\beta : p \subseteq (T, \mathrm{rk}_T)\}$$

for $p \in P_\beta$. Then the β-topology is a Polish topology on R_β. To see this, let S be the set of $(T, H) \in \mathscr{T}_\beta$ such that $T \in \mathscr{T}'$ and H satisfies the recursive definition of rk_T; then it is easy to check that S is $\mathbf{\Pi}_2^0$ in $\mathscr{T}_\beta$ and hence Polish by Alexandrov's Theorem. It is not hard to show that the projection $(T, H) \mapsto T$ is a homeomorphism from S to R_β with the β-topology. (One needs the fact that, for any $s \in {}^{<\omega}\omega$, the set $\{T \in R_\beta : s \notin T\}$ is open in the β-topology; this set can in fact be written as the union of N_p^β for those $p = (t, h)$ such that $h(s') = 0$ for some $s' \subset s$, because we have restricted ourselves to trees in $\mathscr{T}'$.) So the β-topology is Polish and includes the original topology on R_β as a subspace of $\mathscr{T}$.

For any set $A \subseteq \mathscr{T}$ and any $p \in P_\beta$, define $p \Vdash_\beta A$ to means that $A \cap N_p^\beta$ is comeager in N_p^β under the β-topology. Easily, if $p \subseteq q$, then $p \Vdash_\beta A$ implies $q \Vdash_\beta A$; if $A \subseteq B$, then $p \Vdash_\beta A$ implies $p \Vdash_\beta B$; and $p \Vdash_\beta \bigcap_{n=0}^\infty A_n$ if and only if $p \Vdash_\beta A_n$ for all n. Furthermore, if $A \cap R_\beta$ has the Baire property in the β-topology, then $p \nVdash_\beta A$ if and only if there is $q \supseteq p$ such that $q \Vdash_\beta -A$. In particular, this is true whenever A is a Borel subset of $\mathscr{T}$, since then $A \cap R_\beta$ is Borel in R_β under the inherited topology and hence under the β-topology as well.

For example, let p_0 be the tagged tree $(\{\langle\ \rangle\}, h)$ where $h(\langle\ \rangle) = \infty$; then $R_\beta \backslash R_{\beta'}$ is β-open dense in $N_{p_0}^\beta$ for any $\beta' < \beta$ (since any $p \supseteq p_0$ in P_β can be extended by adding a new sequence of length 1 to the tree with tag β'), so $p_0 \Vdash_\beta R_\beta \backslash \bigcup_{\beta' < \beta} R_{\beta'}$.

If (t, h) and (t', h') are finite tagged trees and α is an ordinal, define $(t, h) \sim_\alpha (t', h')$ to mean that $t = t'$ and, for any $s \in t$, if either $h(s)$ or $h'(s)$ is an ordinal less than α, then $h(s) = h'(s)$.

The following lemma is known as the Retagging Lemma.

Lemma 6 (Steel). *If $\alpha \geqslant 1$ is a countable ordinal, $\beta_1, \beta_2 \geqslant \omega \cdot \alpha$, $p_1 \in P_{\beta_1}$, $p_2 \in P_{\beta_2}$, and $p_1 \sim_{\omega \cdot \alpha} p_2$, then, for any $\mathbf{\Pi}_\alpha^0$ set $A \subseteq \mathscr{T}$, $p_1 \Vdash_{\beta_1} A$ if and only if $p_2 \Vdash_{\beta_2} A$.*

Proof. Say $p_1 = (t, h_1)$ and $p_2 = (t, h_2)$. We will proceed by induction on α.

For $\alpha = 1$, suppose that $p_1 \nVdash_{\beta_1} A$; we will show that $p_2 \nVdash_{\beta_2} A$. (Of course, the reverse implication is identical.) Let T be a tree in $N_{p_1}^{\beta_1}$ which is not in A. Since A is $\mathbf{\Pi}_1^0$, $-A$ is open, so there exist finitely many sequences $s_1, ..., s_m \in T$ and $s'_1, ..., s'_k \notin T$ such that any tree containing all of the sequences s_i and none of the sequences s'_i is in $-A$. For each $i \leqslant k$, let s''_i

be the longest initial segment of s_i' that is in T; then, since $T \in \mathscr{T}'$, each s_i'' is a leaf of T (i.e., $\mathrm{rk}_T(s_i'') = 0$). Now let $\tau \supseteq t$ be a finite subtree of T containing all of the sequences s_i and s_i'', and let $q = (\tau, h)$ where $h = \mathrm{rk}_T \restriction \tau$; then $q \supseteq p_1$ and we have not only $q \Vdash_{\beta_1} -A$, but also $\bar{q} \Vdash_{\bar{\beta}} -A$ whenever $\bar{q} \in P_{\bar{\beta}}$ and $\bar{q} \sim_1 q$ (since, in any tree in $N_{\bar{q}}^{\bar{\beta}}$, all sequences s_i would be nodes and all sequences s_i'' would be leaves). Let M be a natural number greater than all natural numbers occurring as tags in q or p_2, and let $\gamma_0 < \cdots < \gamma_{n-1}$ list the infinite ordinals occurring as tags in q. Let L be the largest of the lengths of the sequences in τ. Now define $\hat{h}\colon \tau \to \beta_2 \cup \{\infty\}$ as follows:

$$\hat{h}(s) = \begin{cases} h_2(s) & \text{if } \ s \in t, \\ h(s) & \text{if } \ s \notin t \text{ and } h(s) < \omega, \\ M + j & \text{if } \ s \notin t \text{ and } h(s) = \gamma_j, \\ M + n + L - \mathrm{len}(s) & \text{if } \ s \notin t \text{ and } h(s) = \infty. \end{cases}$$

Then, since $p_1 \sim_\omega p_2$, it is easy to check that $(\tau, \hat{h})$ is a valid β_2-tagged tree extending p_2 and $(\tau, \hat{h}) \sim_1 q$. Hence, $(\tau, \hat{h}) \Vdash_{\beta_2} -A$, so $p_2 \not\Vdash_{\beta_2} A$, as desired.

Now suppose $\alpha > 1$, and write A as a countable intersection of sets A_k, each of which is $\mathbf{\Sigma}^0_{\alpha'}$ for some $\alpha' < \alpha$ (which may vary with k). Suppose that $p_1 \not\Vdash_{\beta_1} A$; we will show that $p_2 \not\Vdash_{\beta_2} A$. There must be a k such that $p_1 \not\Vdash_{\beta_1} A_k$, and hence $q \Vdash_{\beta_1} -A_k$ for some $q = (\tau, h) \supseteq p_1$. Fix $\alpha' < \alpha$ such that A_k is $\mathbf{\Sigma}^0_{\alpha'}$ and hence $-A_k$ is $\mathbf{\Pi}^0_{\alpha'}$. Arguing as before, let M be a natural number such that $\omega \cdot \alpha' + M$ is greater than all ordinals below $\omega \cdot \alpha' + \omega$ occurring as tags in q or p_2, and let $\gamma_0 < \cdots < \gamma_{n-1}$ list the ordinals at or above $\omega \cdot \alpha' + \omega$ occurring as tags in q. Let L be the largest of the lengths of the sequences in τ. Now define $\hat{h}\colon \tau \to \beta_2 \cup \{\infty\}$ as follows:

$$\hat{h}(s) = \begin{cases} h_2(s) & \text{if } \ s \in t, \\ h(s) & \text{if } \ s \notin t \text{ and } h(s) < \omega \cdot \alpha' + \omega, \\ \omega \cdot \alpha' + M + j & \text{if } \ s \notin t \text{ and } h(s) = \gamma_j, \\ \omega \cdot \alpha' + M + n + L - \mathrm{len}(s) & \text{if } \ s \notin t \text{ and } h(s) = \infty. \end{cases}$$

Then, since $p_1 \sim_{\omega \cdot \alpha} p_2$ and $\omega \cdot \alpha \geqslant \omega \cdot \alpha' + \omega$, it is easy to check that $(\tau, \hat{h})$ is a valid β_2-tagged tree extending p_2 and $(\tau, \hat{h}) \sim_{\omega \cdot \alpha'} q$. Hence, by the induction hypothesis, $(\tau, \hat{h}) \Vdash_{\beta_2} -A_k$, so $p_2 \not\Vdash_{\beta_2} A$, as desired. ∎

We are now ready to show that if $\beta \geqslant \omega \cdot \alpha$ then R_β cannot be expressed as the union of fewer than $\mathrm{cov}(\mathbf{K})$ $\mathbf{\Pi}^0_\alpha$ subsets of $\mathscr{T}$. Suppose it can. Then these subsets cover $N_{p_0}^\beta$, which can be viewed as a Polish space under the β-topology, so, by the definition of $\mathrm{cov}(\mathbf{K})$, at least one of these $\mathbf{\Pi}^0_\alpha$ sets, say W, must be β-nonmeager in $N_{p_0}^\beta$. We now have $p_0 \not\Vdash_\beta -W$, so there exists a $q \supseteq p_0$ in P_β such that $q \Vdash_\beta W$. By the Retagging Lemma, we have $q \Vdash_\gamma W$ for any $\gamma > \beta$. As noted before, $p_0 \Vdash_\gamma R_\gamma \setminus R_\beta$, so $q \Vdash_\gamma W \cap (R_\gamma \setminus R_\beta)$. But $W \subseteq R_\beta$, so $q \Vdash_\gamma \varnothing$, which is impossible. This completes the proof. ∎

A question raised by these results is: Exactly what is the least cardinal λ such that any analytic set is the union of λ Borel sets of bounded rank? (The same number of Borel sets of bounded rank would also suffice to give any coanalytic set or even any $\mathbf{\Sigma}^1_2$ set, since a $\mathbf{\Sigma}^1_2$ set is a union of $\aleph_1$ analytic sets.) Corollary 3 gives a lower bound of $\operatorname{cov}(\mathbf{K})$ for λ, while Theorem 8.10(e) of van Douwen [1] implies that the dominating number $\mathfrak{d}$ is an upper bound for λ, since it states that any analytic set is a union of $\mathfrak{d}$ *compact* sets. Solecki [10] gives related results.

REFERENCES

1. E. K. van Douwen, The integers and topology, *in* "Handbook of Set-Theoretic Topology" (K. Kunen and J. E. Vaughan, Eds.), pp. 111–167, North-Holland, Amsterdam, 1984.
2. L. Harrington, Analytic determinacy and $0^{\#}$, *J. Symbolic Logic* **43** (1978), 685–693.
3. W. Hurewicz, Zur Theorie der analytischen Mengen, *Fund. Math.* **15** (1930), 4–17.
4. K. Kuratowski, "Topology," Vol. 1, Academic Press, New York, 1966.
5. D. G. Larman, Projecting and uniformizing Borel sets with K_σ sets, II, *Mathematika* **20** (1973), 233–246.
6. R. D. Mauldin, Borel parametrizations, *Trans. Amer. Math. Soc.* **250** (1979), 223–234.
7. R. D. Mauldin, Problems in topology arising from analysis, *in* "Open Problems in Topology" (J. van Mill and G. M. Reed, Eds.), pp. 617–629, North-Holland, Amsterdam, 1990.
8. R. D. Mauldin, Some selection theorems and problems, *in* "Measure Theory, Oberwolfach 1979" (D. Kölzow, Ed.), Lecture Notes in Mathematics, Vol. 794, pp. 160–165, Springer-Verlag, Berlin, 1979.
9. Y. N. Moschovakis, "Descriptive Set Theory," North-Holland, Amsterdam, 1980.
10. S. Solecki, Decomposing Borel sets and functions and the structure of Baire class 1 functions, *J. Amer. Math. Soc.* **11** (1998), 521–550.
11. J. R. Steel, Forcing with tagged trees, *Ann. Math. Logic* **15** (1978), 55–74.
12. J. Stern, Évaluation du rang de Borel de certains ensembles, *C. R. Acad. Sci. Paris Sér. A–B* **286** (1978), A855–A857.