3-trees with few vertices of degree 3 in circuit graphs

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Abstract

A circuit graph $(G, C)$ is a 2-connected plane graph $G$ with an outer cycle $C$ such that from each inner vertex $v$, there are three disjoint paths to $C$. In this paper, we shall show that a circuit graph with $n$ vertices has a $3$-tree (i.e., a spanning tree with maximum degree at most 3) with at most $\frac{n-7}{3}$ vertices of degree 3. Our estimation for the number of vertices of degree 3 is sharp. Using this result, we prove that a $3$-connected graph with $n$ vertices on a surface $F_\chi$ with Euler characteristic $\chi \geq 0$ has a $3$-tree with at most $\frac{n}{3} + c_\chi$ vertices of degree 3, where $c_\chi$ is a constant depending only on $F_\chi$.

Keywords: 3-connected graph; Circuit graph; 3-tree; Surface

1. Introduction

We consider only finite simple graphs embedded in the sphere, the projective plane, the torus and the Klein bottle. These surfaces have Euler characteristics at least 0 and at most 2. For a graph $G$, we denote the vertex set and the edge set of $G$ by $V(G)$ and $E(G)$, respectively. In particular, let $V_i(G)$ denote the set of vertices of $G$ whose degree are exactly $i$. Let $\Delta(G)$ denote the maximum degree of $G$. For an edge $e$ of $G$, let $G - e$ and $G/e$ denote the graphs obtained from $G$ by deleting and contracting $e$, respectively. (An edge-contraction of $e$ or contracting $e$ is to remove $e$, identify the endpoints of $e$ and replace all pairs of multiple edges by single edges, respectively. The inverse operation of an edge-contraction is called a vertex splitting or splitting a vertex.) For a plane graph $G$, let $\partial G$ denote the subgraph of $G$ induced by the vertices and the edges incident with the infinite region. (If $G$ is a 2-connected plane graph, then $\partial G$ is a cycle, and is called the outer cycle.) A vertex or an edge of $G$ is said to be outer (resp., inner) if it is (resp., is not) contained in $\partial G$.

A spanning tree of maximum degree at most $k$ is called a $k$-tree. Tutte [9] proved that every 4-connected planar graph has a Hamiltonian cycle, i.e., a cycle passing through all vertices exactly once (hence a 4-connected planar graph has a 2-tree), but every 3-connected planar graph is not necessarily Hamiltonian. On the other hand, it has been shown in [1] that a 3-connected planar graph has a 3-tree. Furthermore, every 3-connected graph embedded in a

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A circuit graph \((G, C)\) is a 2-connected plane graph \(G\) with an outer cycle \(C\) such that for each inner vertex \(v\) of \(G\), there exist three disjoint paths from \(v\) to \(C\). Such a condition of a 2-connected plane graph to be a circuit graph is called the three path condition. Observe that a 3-connected planar graph is a circuit graph, and moreover, a 3-connected planar graph with one vertex removed is also a circuit graph. (Such a 2-connected graph obviously has a planar embedding satisfying the three path condition.)

In this paper, we shall bound the number of vertices of degree 3 of 3-trees in circuit graphs, as follows:

**Theorem 1.** Let \(G\) be a circuit graph with \(n\) vertices. Then \(G\) has a 3-tree with at most \(\max\{0, \frac{n-7}{3}\}\) vertices of degree 3. Moreover, the estimation for the number of vertices of degree 3 is best possible.

Using Theorem 1, we shall prove the following theorems:

**Theorem 2.** Let \(G\) be a 3-connected graph with \(n\) vertices on the sphere or the projective plane. Then \(G\) has a 3-tree with at most \(\max\{0, \frac{n-7}{3}\}\) vertices of degree 3. The bound for the number of vertices of degree 3 is best possible when \(G\) is on the projective plane.

**Theorem 3.** Let \(G\) be a 3-connected graph with \(n\) vertices on the torus or the Klein bottle. Then \(G\) has a 3-tree with at most \(\frac{n-3}{3}\) vertices of degree 3. The bound for the number of vertices of degree 3 is best possible.

A \(k\)-walk in a graph \(G\) is a walk in \(G\) passing through every vertex of \(G\) at least once and at most \(k\) times. (A 1-walk is just a Hamilton path.) It is easy to see that if \(G\) has a \(k\)-walk, then \(G\) has a \((k + 1)\)-tree. Moreover, a vertex visited twice in a 2-walk \(W\) corresponds to a vertex of degree 3 in the 3-tree corresponding to \(W\). In [4], it was shown that every circuit graph has a 2-walk, and hence has a 3-tree. Moreover, this result has been extended to that every 3-connected planar graph \(G\) has a 2-walk \(W\) in which every vertex visited twice by \(W\) is included in a 3-cut of \(G\) [5]. (Since a 4-connected planar graph \(G\) has no 3-cut, this implies the existence of a Hamilton path in \(G\).) However, this result does not bound the number of vertices visited twice in 2-walks, and hence it is independent of our theorem.

One might expect a result for the number of vertices visited twice in 2-walks in a 3-connected planar graph, similarly to our theorem for 3-trees.

2. Examples

In this section, we construct examples of 3-connected graphs on surfaces and circuit graphs each of whose 3-tree must have many vertices of degree 3.

Let \(F_\chi\) be a surface with Euler characteristic \(\chi \geq 0\). That is, \(F_\chi\) is either the sphere, the projective plane, the torus or the Klein bottle depending on \(\chi = 2, 1, 0, 0\), respectively. Let \(G\) be a triangulation on \(F_\chi\) with \(k\) vertices. Let \(M\) be the face subdivision of \(G\), that is, the one obtained from \(G\) by putting a new vertex in each face of \(G\) and joining it with all three vertices of the corresponding boundary cycle.

By Euler’s formula, \(G\) has \(2k - 2\chi\) faces, and hence \(M\) has \(k + (2k - 2\chi)\) vertices. Let \(n = 3k - 2\chi\). Let \(X = V(G)\) and \(Y = V(M) - X\). Since \(Y\) is independent in \(M\), each edge of a 3-tree \(T\) of \(M\) is incident to a vertex of \(X\). Hence we have

\[
\sum_{v \in X} \deg_T(v) \geq |E(T)| = n - 1 = 3k - 2\chi - 1 = 2|X| + k - 2\chi - 1.
\]

Therefore, at least \(k - 2\chi - 1 = \frac{n - 4\chi - 3}{3}\) vertices of \(X\) have degree 3 in \(T\). Similarly, considering the graph obtained from the above example by subdividing one or two faces more, we have the following proposition:

**Proposition 4.** Let \(F_\chi\) be a surface with Euler characteristic \(\chi \geq 0\). For each \(n \geq 4\chi + 3\), \(F_\chi\) admits a 3-connected graph with \(n\) vertices each of whose 3-tree has at least \(\lceil \frac{n - 4\chi - 3}{3}\rceil\) vertices of degree 3. ■

By Proposition 4, the bounds on the number of vertices of degree 3 in Theorems 2 and 3 are best possible, except the spherical case. One may ask whether every 3-connected graph on the sphere with \(n\) vertices has a 3-tree with at most \(\frac{n - 11}{3}\) vertices of degree 3.
Now we turn our attention to circuit graphs. Let \( G \) be a spherical triangulation with \( k \) vertices, and let \( L \) be the face subdivision of \( G \). Let \( L' \) be the graph obtained from \( L \) by removing a vertex of \( G \), and let \( |V(L')| = n \). Then \( L' \) is a circuit graph. By the same computation as above, we have \(|V(L')| = n = 3k - 5\). Let \( X' = V(L') \cap V(G) \). For any 3-tree \( T \) of \( L' \),
\[
\sum_{v \in X'} \deg_T(v) \geq |E(T)| = n - 1 = 3k - 6 = 2|X'| + k - 4.
\]
Therefore, at least \( k - 4 = \frac{n - 7}{2} \) vertices of \( X' \) have degree 3 in \( T \). Similarly, considering the graph obtained by subdividing one or two faces more, we have the following proposition:

**Proposition 5.** For each \( n \geq 7 \), there exists a circuit graph with \( n \) vertices each of whose 3-tree has at least \( \left\lceil \frac{n - 7}{3} \right\rceil \) vertices of degree 3.

By Proposition 5, the estimation for the number of vertices of degree 3 in Theorem 1 is sharp.

### 3. Lemmas

In this section, we shall give lemmas to prove Theorem 1. We begin with describing a nice recursive property of circuit graphs. Let \( B_1, B_2, \ldots, B_r \) be circuit graphs or \( K_2 \)'s. Suppose that for \( i = 1, \ldots, r - 1 \), \( B_i \) intersects only \( B_{i+1} \) at one common outer vertex \( v_i \), where \( v_1, \ldots, v_{r-1} \) are all distinct. Then, \( \mathcal{D} = B_1 \cup \cdots \cup B_r \) is said to be a linear chain of circuit graphs of length \( r \), where possibly \( r = 1 \). In this case, we use the expression \( \mathcal{D} = B_1, v_1, \ldots, v_{r-1}, B_r \). Note that each \( B_i \) is a block of \( \mathcal{D} \), and \( B_1 \) and \( B_r \) are end blocks. Each \( v_i \) is a separating vertex of \( \mathcal{D} \). Clearly, a linear chain \( \mathcal{D} \) of circuit graphs of length \( r \) is 2-connected if and only if \( r = 1 \) and \( \mathcal{D} \neq K_2 \).

**Proposition 6 ([4], Lemma 3).** If \( (G, C) \) is a circuit graph and \( v \in V(C) \), then \( G - v \) is a linear chain of circuit graphs of length \( r \geq 1 \). Moreover, if \( r \geq 2 \), then the neighbors of \( v \) in \( C \) are non-separating vertices lying on the distinct end blocks of \( G - v \).

We point out an important fact on circuit graphs which will be used in our argument later. Let \( G \) be a 3-connected plane graph on a surface and let \( C \) be any cycle of \( G \). Then the subgraph \( G' \) consisting of all vertices and edges lying on \( C \) and contained in the region bounded by \( C \) must be a circuit graph with boundary \( C \). (The three path condition of \( G' \) clearly holds by the 3-connectedness of \( G \).

Let \((G, C)\) be a circuit graph. A \( C \)-path of \( G \) is a path \( P \) joining a vertex \( u \in V(C) \) and a vertex \( v \in V(C) \) such that \( V(P) \cap V(C) = \{u, v\} \) and \( E(P) \cap E(C) = \emptyset \). An edge \( e \in E(G) \) is said to be removable in \( G \) if \( G - e \) (with the embedding induced by \( G \)) is also a circuit graph. Note that an edge \( e = xy \in E(C) \) is removable if and only if there exists a \( C \)-path joining \( x \) and \( y \). (Equivalently, \( e \in E(C) \) is not removable if and only if \( G - e \) is a linear chain of circuit graph of length at least 2.) Also, an edge \( e \in E(G) - E(C) \) is not removable if and only if there exists an inner vertex \( v \) such that any three disjoint paths from \( v \) to \( C \) must pass through the edge \( e \). A circuit graph \((G, C)\) is said to be edge-minimal if \( G \) has no removable edge.

**Lemma 7.** Let \((G, C)\) be an edge-minimal circuit graph and let \( v \in V(C) \) be a vertex of degree at least 3. Then \( G - v \) is a linear chain of circuit graphs of length at least 2.

**Proof.** By Proposition 6, \( G - v \) is a linear chain of circuit graphs of length \( r \) for some \( r \geq 1 \). To show the lemma, we shall prove that \( r \geq 2 \). Suppose that \( r = 1 \), that is, \( G - v \) is 2-connected. By the assumption, there are at least three edges incident to \( v \). Hence, if we let \( f \) be an edge in \( C \) incident to \( v \), then \( G - f \) is a circuit graph, which is contrary to the edge-minimality of \( G \).

**Lemma 8.** Let \((G, C)\) be an edge-minimal circuit graph and let \( xy \in E(C) \). If \( x \) and \( y \) have degree at least 3, then \( G/xy \) is also an edge-minimal circuit graph.

**Proof.** Since \( G \) has no \( C \)-path joining \( x \) and \( y \), \( G/xy \) satisfies the three path condition and hence is a circuit graph. Note also that every three disjoint paths from an inner vertex \( v \) to \( C/xy \) in \( G/xy \) corresponds to three disjoint paths from \( v \) to \( C \) in \( G \). This implies that every edge in \( E(G/xy) - E(C/xy) \) is not removable in \( G/xy \).
Suppose that an edge $st \in E(C/xy)$ is removable in $G/xy$. Then there exists a $C/xy$-path $P$ joining $s$ and $t$. Let $P'$ be the path in $G$ corresponding to $P$. Since $st$ is not removable in $G$, the endvertices of $P'$ are not consecutive in $C$. This implies that one of the endvertices of $P'$ is $x$ or $y$, say $y$, and the other endvertex, say $s$, is a neighbor of $x$ in $C$.

Let $C'$ be the cycle $P' \cup \{sx, xy\}$. Since $\deg_G(x) \geq 3$, there exists an inner vertex $v \in N_G(x)$. If $v \in V(P') - \{s, y\}$, then we find a $C$-path joining $x$ and $y$ in $G$, which contradicts that $xy$ is not removable. Thus $v$ lies in the interior of the region bounded by $C'$. By the three path condition, there exists a path $Q$ joining $v$ and $C'$ which avoids $s$ and $x$. Then, $Q \cup P' \cup \{xy\}$ contains a $C$-path joining $x$ and $y$, and hence $xy$ is removable, a contradiction. ■

The following lemma is essential to prove Theorem 1.

**Lemma 9.** Let $(G, C)$ be an edge-minimal circuit graph with $n \geq 4$ vertices, and let $u, v$ be any distinct vertices in $C$. Then $G$ has a spanning connected subgraph $H$ with $\Delta(H) \leq 3$ such that

(i) $C \subset H$,
(ii) $\deg_H(u) = \deg_H(v) = 2$,
(iii) $|E(H)| = n$, and
(iv) $|V_3(H)| \leq \frac{n-4}{3}$.

By (i) and (iii), for any $e \in E(C)$, the graph $H - e$ is a 3-tree of $G$.

**Proof of Lemma 9.** We use induction on $n$. An edge-minimal circuit graph with exactly four vertices is a 4-cycle and it obviously satisfies the lemma. This verifies the first step of induction. So we assume that $n \geq 5$.

**Claim 1.** We may assume $\deg_G(u) \geq 3$ and $\deg_G(v) \geq 3$.

**Proof.** If $V(G) = V(C)$, then by the edge-minimality of $G$, $G$ is just the cycle $C$. Then, the lemma clearly holds with $H = C$. Hence we may suppose that $V(G) \neq V(C)$. By the three path condition, there are at least three vertices of degree at least 3 on $C$. Assume that $\deg_G(v) = 2$ for example. Then, specifying one of the other vertices, say $w(\neq u)$, instead of $v$, we suppose to obtain a required $H$ with $\deg_H(w) = 2$. In this $H$ obtained, we must have $\deg_H(v) = 2$ since $H \supset C$ and $\deg_G(v) = 2$. Therefore, we may suppose that $\deg_G(v) \geq 3$. The same argument follows for the other vertex $u$. ■

**Claim 2.** We may assume that no two vertices of degree at least 3 are adjacent in $C$.

**Proof.** Suppose that there is an edge $xy$ in $C$ such that $\deg_G(x) \geq 3$ and $\deg_G(y) \geq 3$. We shall show that we can easily find a required $H$ in $G$.

By **Lemma 8**, $G/xy = G'$ is also an edge-minimal circuit graph with $n' = n - 1$ vertices. By induction hypothesis, $G'$ has a spanning subgraph $H'$ with $H' \supset \partial G'$, $\Delta(H') \leq 3$, $|E(H')| = n'$ and $|V_3(H')| \leq \frac{n'-4}{3}$. From $H'$, we construct a required spanning subgraph $H$ of $G$ by splitting the vertex $[xy]$, where $[xy]$ is the image of an edge $xy$ by the contraction. Clearly, $H$ satisfies the conditions (i) and (iii). If one or both of $x$ and $y$ are specified as $u$ or $v$ in $G$, then we can make $\deg_H([xy]) = 2$, by specifying $[xy]$ in the induction hypothesis for $G'$. By splitting $[xy]$ in $H'$, we obtain $H$ with $\deg_H(x) = \deg_H(y) = 2$. In other cases, the degree of $u$ and $v$ in $H$ are the same as in $H'$. Therefore, $H$ satisfies (ii). Since $\deg_H([xy]) \leq 3$, we can make at least one of $x$ and $y$ have degree 2 in $H$ by splitting $[xy]$. Therefore, the number of vertices of degree 3 does not increase by splitting $[xy]$, and hence we have

$$|V_3(H)| = |V_3(H')| \leq \frac{n'-4}{3} < \frac{n-4}{3}.$$  

Thus, $H$ satisfies (iv). ■

Since $v$ has degree at least 3 in $G$ by Claim 1, the graph $G' = G - v$ is a linear chain of circuit graphs of length at least 2, by **Lemma 7**. Let $G' = B_1, v_1, \ldots, v_{r-1}, B_r$, where each $B_i$ is a circuit graph or $K_2$, and $v_1, \ldots, v_{r-1}$ are distinct separating vertices of $G'$. Let $v_0$ and $v_r$ be the two neighbors of $v$ in $C$ belonging to $B_1$ and $B_r$, respectively. (See **Fig. 1**.) Let $k$ be the smallest integer such that $B_k$ contains the vertex $u$. We may assume that $k < r$, for otherwise we reverse the sequence of blocks of the linear chain. By Claims 1 and 2, $u$ and $v$ are not adjacent in $C$. Therefore we have $v_0 \neq u$.  


Consider the graph $\tilde{G}''$ induced by $V(B_{k+1}) \cup \cdots \cup V(B_r) \cup \{v\}$ with an additional edge joining $v_k$ and $v$ for the case when $vv_k \notin E(G)$. Observe that any inner vertex $w$ of $\tilde{G}''$ belongs to $B_j$ in $G$ for some $j \in \{k+1, \ldots, r\}$, and that $w$ has at least three disjoint paths to $\partial B_j \cap C$ and $v$. Moreover, since the outer cycle (denoted by $C''$) is a cycle, $\tilde{G}''$ must be a circuit graph. Let $G'' = \tilde{G}'' - v_k$ be a linear chain of circuit graphs of length $l - k \geq 1$. In particular, we put

$$G'' = D_{k+1}, u_{k+1}, D_{k+2}, u_{k+2}, \ldots, D_l, u_l, D_l,$$

where $u_k'$ and $v$ are the two neighbors of $v_k$ in $C''$ belonging to different end blocks $D_{k+1}$ and $D_l$, respectively, if $r \geq 2$.

For simpleness of notations, we rename $v_{i-1}, B_i, v_i$ to be $u_{i-1}, D_i, u_i$, for $i = 1, \ldots, k$. Then we have $V(G) = V(D_1 \cup \cdots \cup D_l)$. (See Fig. 2.)

**Claim 3.** Each $D_i$ $(i = 1, \ldots, l)$ is isomorphic to $K_2$ or has at least 4 vertices.

**Proof.** For contradictions, suppose that $D_m$ is isomorphic to $K_3$ for some $m$. In particular, we suppose that $u_{m-1}, u_m$ and another vertex $x$ form a 3-cycle. In this case, we can remove the edge $u_{m-1}u_m$ from $G$, and the resulting graph is easily verified to be a circuit graph. This contradicts the edge-minimality of $G$. ■

**Claim 4.** $D_1$ and $D_{k+1}$ are isomorphic to $K_2$.

**Proof.** By Claim 1, we have $\deg_G(v) \geq 3$. Therefore, we have $\deg_G(u_0) = 2$ by Claim 2, and hence $D_1 = K_2$. If $D_k$ is 2-connected, then $u_k$ has degree at least 3 in $G$. Otherwise, it follows that $u_k = u$, and hence we have $\deg_G(u_k) \geq 3$ by Claim 1. Thus in either case, $u_k$ has degree at least 3, and hence its neighbor $u_k'$ has degree 2 in $G$, by Claim 2. Therefore, $D_{k+1} = K_2$. ■

For each $i = 1, \ldots, l$ with $D_i \neq K_2$, we define $D'_i$ to be an edge-minimal spanning circuit subgraph of $D_i$. If $D_i = K_2$, then we set $D'_i = D_i$.

**Claim 5.** For each $i = 1, \ldots, l$, $D'_i \subseteq (D_i \cap C)$.

**Proof.** By Claim 2, each edge of $C$ is incident with a vertex of degree two in $G$. Thus, we cannot remove any edge of $C$ when we obtain $D'_i$. ■
Note that each $D'_i$ is isomorphic to either a $K_2$ or an edge-minimal circuit graph with at least four vertices, by Claim 3. Let $n_i = |V(D'_i)|$ for $i = 1, \ldots, l$. Then we have
\[
\sum_{i=1}^{l} n_i = n + l - 2. \tag{1}
\]

Now we define a spanning tree $T_i$ of $D'_i$ for $i = 1, \ldots, l$. For $D'_i$ with $n_i = 2$, let $T_i = D'_i$. For $D'_i$ including $u$, if $n_k \geq 4$, let $H_k$ be a spanning connected subgraph with $\Delta(H_k) \leq 3$, including $\partial D'_k$, such that $|\partial(H_k)| = n_k$, $\deg_{H_k}(u_{k-1}) = \deg_{H_k}(u) = 2$ and $|V_3(H_k)| \leq (n_k - 4)/3$, whose existence is guaranteed by induction hypothesis. For each $D'_i$ with $n_i \geq 4$ and $i \neq k$, let $H_i$ be a spanning connected subgraph with $\Delta(H_i) \leq 3$, including $\partial D'_i$, such that $|\partial(H_i)| = n_i$, $\deg_{H_i}(u_{i-1}) = \deg_{H_i}(u_i) = 2$ and $|V_3(H_i)| \leq (n_i - 4)/3$. Note that the vertex $u_k$ in $D'_{k+1}$ is $u'_k$, and the vertex $u_l$ in $D'_l$ is $v$.

For each $i$ with $n_i \geq 4$, let $e_i \in E(\partial D'_i) - E(C)$ be the edge incident to $u_i$, and let $T_i$ be the 3-tree $H_i - e_i$. Let
\[
H = \left( \bigcup_{i=1}^{l} T_i \right) \cup \{vu_0, u_ku'_k\}.
\]

Then $H$ is connected and has maximum degree at most 3, and moreover, $|E(H)| = n$ and $\deg_H(u) = \deg_H(v) = 2$. Since $T_i \supset D_i \cap C$ for any $i$, we have $H \supset C$. Hence $H$ satisfies the condition (i), (ii) and (iii).

Now, in order to show (iv), we count the number of vertices of $H$ which have degree 3. For any $T_i$ with $n_i \geq 4$, we have $|V_3(T_i)| \leq \frac{n_i - 4}{3}$. Moreover, we might have $\deg_H(u_{i-1}) = 3$ for each $T_i$ with $n_i \geq 4$. (Note that $\deg_H(u_k) = 3$ holds if and only if $\deg_{H_k}(u_k) = 3$. Thus it is counted in $H_k$ as a vertex of degree 3 of $H_k$.) Therefore, by Claims 3 and 4, and Eq. (1), we have
\[
|V_3(H)| \leq \sum_{n_i \geq 4} \frac{n_i - 4}{3} + \sum_{n_i \geq 4} 1
\]
\[
= \sum_{i=1}^{l} \frac{n_i - 1}{3} - \sum_{n_i = 2}^{l} \frac{2 - 1}{3}
\]
\[
\leq \frac{1}{3}(n + l - 2) - \frac{l}{3} - 2 \cdot \frac{1}{3}
\]
\[
= \frac{n - 4}{3}.
\]

Thus, the lemma follows.

In Lemma 9, the edge-minimality of $G$ cannot be omitted, as explained below. Let $K$ be a maximal outerplane graph with precisely $k \geq 3$ vertices and let $G$ be the plane graph obtained from $K$ by adding a vertex to each finite face of $K$ and joining it to the three vertices of the corresponding boundary. Then, by Euler’s formula, $K$ has $k - 2$ finite faces, and hence $G$ has $k + (k - 2) = 2k - 2 \geq 4$ vertices. Let $n = |V(G)| = 2k - 2$. The subgraph $H$ of $G$ with $\Delta(H) \leq 3$ including $\partial G$ must have at least $k - 2 = \frac{4}{3} - 1$ vertices of degree 3, since $V(G) - V(K)$ is independent. In this case, all edges in $\partial G$ are removable in $G$.

4. Proof of the theorems

In this section, we shall prove our main theorems.

**Proof of Theorem 1.** Let $(G, C)$ be a 2-connected circuit graph with $n$ vertices. When $n = 3$, $(G, C)$ clearly has a 2-tree, that is, a 3-tree with no vertex of degree 3. Therefore, we may suppose that $n \geq 4$.

We may assume that $(G, C)$ is an edge-minimal circuit graph. Then, by Lemma 9, $G$ has a spanning connected subgraph $H$ with $n$ edges such that $H \supset C$, $\Delta(H) \leq 3$ and $|V_3(H)| \leq \frac{n - 4}{3}$. If $V_3(H) \neq \emptyset$, then there exists a vertex $w \in V(C)$ such that $\deg_H(w) = 3$. (For otherwise, i.e., all vertices on $C$ have degree 2 in $H$, then we have $H = C$, since $H$ is connected and $H \supset C$. This contradicts that $V_3(H) \neq \emptyset$.) Removing an edge $e$ of $H$ which is incident to $w$ and contained in $C$, we can reduce the number of vertices of $H$ whose degree are 3 at least by one. Therefore, we obtain a 3-tree $T$ of $G$ with at most $\frac{n - 4}{3} - 1 = \frac{n - 7}{3}$ vertices of degree 3, if $|V_3(H)| \geq 1$. ■
Proof of Theorem 2. Since a 3-connected graph on the sphere can be regarded as a circuit graph, we can apply Theorem 1. For the projective plane, Gao and Richter [4] proved that every 3-connected graph on the projective plane has a spanning circuit subgraph $G'$, and hence we can apply Theorem 1 to $G'$ directly. Proposition 4 guarantees the sharpness of the estimation for the number of vertices of degree 3 in the projective planar case. ■

In order to prove Theorem 3, we use the following fact, which is immediately obtained from Theorems 6.11 and 6.12 in [7].

Lemma 10 ([7]). Every 3-connected graph on the torus or the Klein bottle has a spanning subgraph which is a linear chain of circuit graphs.

Now we shall prove Theorem 3.

Proof of Theorem 3. Let $G$ be a 3-connected graph with $n$ vertices embedded in the torus or the Klein bottle. Since $G$ is 3-connected, we have $n \geq 4$. By Lemma 10, we can put

$$G' = B_1, v_1, B_2, v_2, \ldots, v_{r-1}, B_r,$$

where each $B_i$ is a circuit graph or $K_2$ and each $v_i$ is a separating vertex of $G'$. If $r = 1$, then the conclusion of the theorem immediately follows from Theorem 1.

Suppose $r \geq 2$. Take a vertex $v_0$ in $\partial B_1 - \{v_1\}$ and a vertex $v_r$ in $\partial B_r - \{v_{r-1}\}$. Then, the boundary $\partial G'$ consists of two paths both joining $v_0$ and $v_r$. Let $P$ be one of these paths. We define a new graph $\tilde{G}$ to be obtained from $G'$ by adding a new vertex $z$ in the infinite region so that $z$ is adjacent to all vertices of $P$. Then, it is easy to check that $\tilde{G}$ is a circuit graph of order $n + 1$ with $\partial \tilde{G} = P \cup \{v_r, zv_0\}$.

Let $\tilde{G}'$ be an edge-minimal spanning circuit graph of $\tilde{G}$. Note that $z$ is in $\partial \tilde{G}'$. By Lemma 9, $\tilde{G}'$ has a spanning connected subgraph $H$ with $n + 1$ edges such that $H \supset \partial \tilde{G}'$, $\deg_H(z) = 2$, $\Delta(H) \leq 3$ and $|V_3(H)| \leq \frac{(n+1) - 4}{3} = \frac{n-3}{3}$. Since $z$ is contained in a unique cycle in $H$ with $\deg_H(z) = 2$, it follows that $T = H - z$ is a connected spanning subgraph of $G$. Consequently, $T$ is a 3-tree of $G$ with $|V_3(T)| \leq |V_3(H)| \leq \frac{n-3}{3}$.

The sharpness of the bound has already been verified in Proposition 4. Therefore, the theorem holds. ■

References