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# The $k$-tuple twin domination in generalized de Bruijn and Kautz networks☆

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ABSTRACT

Given a digraph (network) $G = (V, A)$, a vertex $u$ in $G$ is said to out-dominate itself and all vertices $v$ such that the arc $(u, v) \in A$; similarly, $u$ in-dominates both itself and all vertices $w$ such that the arc $(w, u) \in A$. A set $D$ of vertices of $G$ is a *$k$-tuple twin dominating set* if every vertex of $G$ is out-dominated and in-dominated by at least $k$ vertices in $D$, respectively. The $k$-tuple twin domination problem is to determine a minimum $k$-tuple twin dominating set for a digraph. In this paper we investigate the $k$-tuple twin domination problem in generalized de Bruijn networks $G_B(n, d)$ and generalized Kautz $G_K(n, d)$ networks when $d$ divides $n$. We provide construction methods for constructing minimum $k$-tuple twin dominating sets in these networks. These results generalize previous results given by Araki [T. Araki, The $k$-tuple twin domination in de Bruijn and Kautz digraphs, Discrete Mathematics 308 (2008) 6406–6413] for de Bruijn and Kautz networks.



## 1. Introduction

In this paper we deal with digraphs (networks) which admit self-loops but no multiple arcs. Specifically, let $G = (V, A)$ be a digraph with *vertex set* $V$ and *arc set* $A$. For a vertex $u \in V$, the *out-neighborhood* of $u$ is $N^+(u) = \{v \mid (u, v) \in A\}$ and the *in-neighborhood* of $u$ is $N^-(u) = \{v \mid (v, u) \in A\}$. The *closed out-neighborhood* and *closed in-neighborhood* of $u$ are $N^+[u] = N^+(u) \cup \{u\}$ and $N^-[u] = \{u\} \cup N^-(u)$, respectively. Note that if $u$ has a self-loop, the out-neighborhood and in-neighborhood of $u$ contain $u$ itself. For a subset $S \subseteq V$, write $N^+(S) = \cup_{u \in S} N^+(u)$ and $N^-(S) = \cup_{u \in S} N^-(u)$. The *out-degree* and *in-degree* of $u$ are $\deg^+(u) = |N^+(u) \setminus \{u\}|$ and $\deg^-(u) = |N^-(u) \setminus \{u\}|$, respectively. Denote by $\delta^+(G)$ and $\delta^-(G)$ the minimum out-degree and in-degree of $G$, respectively.

Domination in digraphs has received more attention in recent years since it has many applications. A vertex $u$ in $G$ is said to *out-dominate* the vertices in $N^+[u]$ and *in-dominate* the vertices in $N^-[u]$. For a positive integer $k$, a set $D$ of vertices of $G$ is called a *$k$-tuple out-dominating set* if $|N^+[u] \cap D| \geq k$ for each vertex $u$ of $G$, while $D$ is called a *$k$-tuple in-dominating set* if $|N^-[u] \cap D| \geq k$ for each vertex $u$ of $G$. In particular, the 1-tuple out-dominating and in-dominating sets are respectively called the *dominating set* and *absorbant* of $G$ in [1,2]. A set $D$ of vertices in $G$ is *a $k$-tuple twin dominating set* of $G$ if $|N^+[u] \cap D| \geq k$ and $|N^-[u] \cap D| \geq k$ for each vertex $u$ of $G$. *The $k$-tuple twin domination number*, denoted by $\gamma^*_{\times k}(G)$, of $G$ is the minimum cardinality of a $k$-tuple twin dominating set of $G$. When $k = 1$, it is a usual twin domination. Note that a digraph $G$ has a $k$-tuple twin dominating set if and only if $k \leq \delta^+(G) + 1$ and $k \leq \delta^-(G) + 1$. The concept of $k$-tuple twin domination in digraphs was recently introduced by Araki [3].

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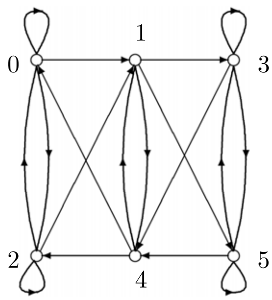


**Fig. 1a.** $G_B$ (6,3).

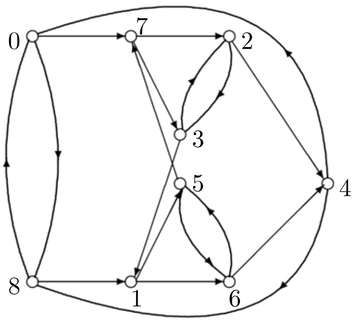


**Fig. 1b.** $G_K$(9,2).

This study is motivated by an application of $k$-tuple twin domination in networks suggested by Araki [3]. Let our graph be the model of a network. Each vertex in a $k$-tuple twin dominating set in digraphs provides a service (file-server, sensor and so on) for the network. In the network, there is a direct communication between every vertex and file-servers in both directions. It is reasonable to assume that this access is available even when some file-servers go down. A $k$-tuple twin dominating set provides the desired fault-tolerance for such cases because each vertex can access at least $k$ servers and each server can have at least $k-1$ backup servers. Since each backup copy may cost a lot, the number of duplicated copies has to be minimized.

Let $d, n$ be two positive integers and $n \geq d \geq 2$. The generalized de Bruijn digraph $G_B(n, d)$ is defined by congruence equations as follows:

$$\begin{cases} V(G_B(n, d)) = \{0, 1, 2, \ldots, n-1\} \\ A(G_B(n, d)) = \{(x, y) \mid y \equiv dx + i \,(\bmod\, n), 0 \leq i \leq d-1\}. \end{cases}$$

In particular, if $n = d^m$, then $G_B(n, d)$ is the de Bruijn digraph $B(d, m)$. The generalized Kautz digraph $G_K(n, d)$ is defined by the following congruence equation:

$$\begin{cases} V(G_K(n, d)) = \{0, 1, 2, \ldots, n-1\} \\ A(G_K(n, d)) = \{(x, y) \mid y \equiv -dx - i \,(\bmod\, n), 1 \leq i \leq d\}. \end{cases}$$

In particular, if $n = d^m + d^{m-1}$, then $G_K(n, d)$ is the Kautz digraph $K(d, m)$. The generalized de Bruijn and Kautz digraphs have been studied as interconnection network topologies because of various good properties [4,5]. The graphs $G_B(6, 3)$ and $G_K(9, 2)$ are exhibited in Figs. 1. For notational convenience, sometimes we simply write $G_B$ and $G_K$ instead of $G_B(n, d)$ and $G_K(n, d)$, respectively, if $n$ and $d$ are explicit from the context.

For generalized de Bruijn digraphs, their Hamiltonian property [6], diameter [7], connectivity [8], absorbant [2] and twin domination [9,10] have been studied. Also, several structural objects such as spanning trees, Eulerian tours [11], closed walks [12] and small cycles [13] have been counted. For generalized Kautz digraphs, their diameter [14], their connectivity [15,8] and the number of cycles [16] have been studied. Kikuchi and Shibata [1] considered the domination problem for generalized de Bruijn and Kautz digraphs. In [17] Tian and Xu further investigated the distance domination for these digraphs. Recently, Araki [18,3] studied the $k$-tuple domination and $k$-tuple twin domination in de Bruijn and Kautz digraphs. Wu et al. [19] considered the $k$-tuple domination for generalized de Bruijn and Kautz digraphs.

In [3] Araki presented the $k$-tuple twin domination number of de Bruijn and Kautz digraphs, separately, by constructing minimum $k$-tuple twin dominating sets in these digraphs.

**Theorem 1** (*Araki,[3]*). *For $d \geq 2$, $m \geq 1$, and $1 \leq k \leq d-1$, $\gamma^*_{\times k}(B(d, m)) = kd^{m-1}$.*

**Table 1**
The vertices of $G_B(n, d)$ or $G_K(n, d)$ when $d|n$.

| | | | | |
|---|---|---|---|---|
| $0$ | $\frac{n}{d}$ | $2\frac{n}{d}$ | $\cdots$ | $(d-1)\frac{n}{d}$ |
| $1$ | $\frac{n}{d}+1$ | $2\frac{n}{d}+1$ | $\cdots$ | $(d-1)\frac{n}{d}+1$ |
| $2$ | $\frac{n}{d}+2$ | $2\frac{n}{d}+2$ | $\cdots$ | $(d-1)\frac{n}{d}+2$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $i$ | $\frac{n}{d}+i$ | $2\frac{n}{d}+i$ | $\cdots$ | $(d-1)\frac{n}{d}+i$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\frac{n}{d}-1$ | $\frac{n}{d}+\left(\frac{n}{d}-1\right)$ | $2\frac{n}{d}+\left(\frac{n}{d}-1\right)$ | $\cdots$ | $n-1$ |

**Table 2**
The vertices of $G_B(n, d)$ when $d|n$.

| | | | | |
|---|---|---|---|---|
| $0$ | $1$ | $2$ | $\cdots$ | $(d-1)$ |
| $d$ | $d+1$ | $d+2$ | $\cdots$ | $2d-1$ |
| $2d$ | $2d+1$ | $2d+2$ | $\cdots$ | $3d-1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $id$ | $id+1$ | $id+2$ | $\cdots$ | $(i+1)d-1$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $\left(\frac{n}{d}-1\right)d$ | $\left(\frac{n}{d}-1\right)d+1$ | $\left(\frac{n}{d}-1\right)d+2$ | $\cdots$ | $n-1$ |

**Theorem 2** (*Araki, [3]*). *For $d \geq 2$ and $1 \leq k \leq d-1$,*

$$\gamma^*_{\times k}(K(d, m)) = \begin{cases} k & m = 1, \\ k(d^{m-1} + d^{m-2}) & m \geq 2. \end{cases}$$

One natural problem arising is that of what the exact values of the $k$-tuple twin domination numbers in generalized de Bruijn and Kautz digraphs are. It seems to be difficult to determine the $k$-tuple twin domination numbers for these general digraphs. Our purpose here is to give the $k$-tuple twin domination numbers for $G_B(n, d)$ and $G_K(n, d)$ when $d$ divides $n$. Since the vertex 0 has a self-loop in any $G_B(n, d)$, $\delta^+(G_B(n, d)) = d-1$. This means that $G_B(n, d)$ has a $k$-tuple twin dominating set if and only if $k \leq d$. For $G_K(n, d)$, note the fact that $G_K(n, d)$ contains no self-loop iff $(d+1)$ divides $n$ (see [20, pp. 112–131]). Then $\delta^+(G_K(n, d)) = d-1$ or $d$. So $G_K(n, d)$ has a $k$-tuple twin dominating set if and only if $k \leq d+1$ when $(d+1)$ divides $n$ or else $k \leq d$.

In this paper, by applying a distinct technique with that of Araki [3], we obtain the following generalized results.

**Theorem 3.** *For $d \geq 2$, $1 \leq k \leq d-1$, where d divides n, $\gamma^*_{\times k}(G_B(n, d)) = \frac{kn}{d}$.*

**Theorem 4.** *For $d \geq 2$, $1 \leq k \leq d-1$, where d divides n, $\gamma^*_{\times k}(G_K(n, d)) = \frac{kn}{d}$.*

Recalling that $G_B(d^m, d) = B(d, m)$ when $n = d^m$, while $G_K(d^m, d) = B(d, m)$ when $n = d^m + d^{m-1}$, we see that Theorems 1 and 2 are special cases of Theorems 3 and 4, respectively.

## 2. Proof of Theorem 3

For any positive integers $m, n$, we denote as $(m, n)$ the greatest common divisor of $m$ and $n$. $m|n$ means that $m$ divides $n$. When $d$ divides $n$, an easy observation is that the vertex set $V(G_B)$ of $G_B(n, d)$ can be represented as shown in Tables 1–2.

**Proof of Theorem 3.** As shown in Tables 1–2, we have

$$V(G_B) = \bigcup_{i=0}^{\frac{n}{d}-1} \bigcup_{j=0}^{d-1} \{id + j\}, \quad \text{or} \quad \bigcup_{i=0}^{\frac{n}{d}-1} \bigcup_{j=0}^{d-1} \left\{ j\frac{n}{d} + i \right\}.$$

Let $I_i = \bigcup_{j=0}^{d-1} \{id + j\}$ and $P_i = \bigcup_{j=0}^{d-1} \{j\frac{n}{d} + i\}$. Note that the set of $d$ elements in every row in Table 2 is exactly the out-neighborhood of each vertex in the same row in Table 1, that is, $N^+(i) = N^+(\frac{n}{d} + i) = \cdots = N^+((d-1)\frac{n}{d} + i) = I_i$. Then $N^-(id) = N^-(id + 1) = \cdots = N^-((i+1)d - 1) = P_i$. Let $T$ be a minimum $k$-tuple twin dominating set of $G_B(n, d)$.

We first show that $\gamma^*_{\times k}(G_B(n, d)) \geq \frac{kn}{d}$. If $|T \cap I_i| \geq k$ and $|T \cap P_i| \geq k$ for $0 \leq i \leq \frac{n}{d} - 1$, then $\gamma^*_{\times k}(G_B(n, d)) = |T| \geq \frac{kn}{d}$. Otherwise, there exists one set $I_i$ or $P_i$ such that $|T \cap I_i| \leq k-1$ or $|T \cap P_i| \leq k-1$. Suppose $|T \cap I_i| \leq k-1$. Since $I_i$ is the out-neighborhood of each vertex in $P_i$, we have $P_i \subseteq T$ and $|T \cap I_i| \geq k-1$ for otherwise $T$ could not $k$-tuple in-dominate vertices of $P_i$. So $|T \cap I_i| = k-1$ and $|T \cap P_i| = |P_i| = d \geq k+1$. Similarly, if $|T \cap P_i| \leq k-1$, then we can deduce that

**Table 3**
An example: $G_B(32, 4)$ and $k = 3$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 0 | 8 | 16 | 24 |
| 4 | 5 | 6 | 7 | 1 | 9 | 17 | 25 |
| 8 | 9 | 10 | 11 | 2 | 10 | 18 | 26 |
| 12 | 13 | 14 | 15 | 3 | 11 | 19 | 27 |
| 16 | 17 | 18 | 19 | 4 | 12 | 20 | 28 |
| 20 | 21 | 22 | 23 | 5 | 13 | 21 | 29 |
| 24 | 25 | 26 | 27 | 6 | 14 | 22 | 30 |
| 28 | 29 | 30 | 31 | 7 | 15 | 23 | 31 |

$|T \cap P_i| = k - 1$ while $|T \cap I_i| = |I_i| = d \geq k + 1$. Note that $|T| = |\bigcup_{i=0}^{\frac{n}{d}-1}(T \cap P_i)| = |\bigcup_{i=0}^{\frac{n}{d}-1}(T \cap I_i)|$. Consequently, $\gamma^*_{\times k}(G_B(n, d)) = |T| \geq \frac{kn}{d}$.

Next we prove that $\gamma^*_{\times k}(G_B(n, d)) \leq \frac{kn}{d}$. Note that if a set $T$ of vertices of $G_B$ satisfies that $|T \cap I_i| = k$ and $|T \cap P_i| = k$ for each $i = 0, 1, \ldots, \frac{n}{d} - 1$, then $T$ is a $k$-tuple twin dominating set of $G_B$. Therefore, it is sufficient to show that there exists a set $T$ of vertices of $G_B$ such that $|T \cap I_i| = k$ and $|T \cap P_i| = k$. Let $\left(\frac{n}{d}, d\right) = t$. We construct the set $T$ with $|T| = \frac{kn}{d}$ as follows:

$$T = \bigcup_{r=0}^{t-1} T_r, \quad \text{where } T_r = \bigcup_{s=0}^{\frac{n}{dt}-1} \bigcup_{j=0}^{k-1} \left\{ \left(\frac{n}{dt} r + s\right) d + r + j - d \left\lfloor \frac{r + j}{d} \right\rfloor \right\}.$$

We claim that $T$ is the desired set. Note that $0 \leq \frac{n}{dt} r + s \leq \frac{n}{d} - 1$ and $0 \leq r + j - d\lfloor \frac{r+j}{d} \rfloor \leq d - 1$. It is easy to check that $|T \cap I_i| = k$ for $0 \leq i \leq \frac{n}{d} - 1$. Let

$$T_j = \bigcup_{r=0}^{t-1} \bigcup_{s=0}^{\frac{n}{dt}-1} \left\{ \left(\frac{n}{dt} r + s\right) d + r + j - d \left\lfloor \frac{r + j}{d} \right\rfloor \right\},$$

where $j = 0, 1, \ldots, k - 1$. It is easy to verify that $T_i \cap T_j = \emptyset$ for $i \neq j$ with $0 \leq i, j \leq k - 1$. Thus, $\bigcup_{j=0}^{k-1} T_j = T$. Clearly, $|T_j| = \frac{n}{d}$ and $|T_j \cap I_i| = 1$ for $0 \leq i \leq \frac{n}{d} - 1$. Suppose that $|T \cap P_i| = k$ is not true for some $i$. Then there exists an $i$ such that $|T \cap P_i| < k$ and so there exists at least a set $T_j$ such that $T_j \cap P_i = \emptyset$. This implies that there must exist another set $P_{i'}$ such that $|T_j \cap P_{i'}| \geq 2$. That is, $T_j$ contains two distinct vertices $x_1 = (\frac{n}{dt} r_1 + s_1)d + r_1 + j - d\lfloor \frac{r_1+j}{d} \rfloor$ and $x_2 = (\frac{n}{dt} r_2 + s_2)d + r_2 + j - d\lfloor \frac{r_2+j}{d} \rfloor$ such that $x_1, x_2 \in P_{i'}$ where $0 \leq r_1 \leq r_2 \leq t - 1$, $0 \leq s_1, s_2 \leq \frac{n}{dt} - 1$. Thus there exist $l_1, l_2$ such that $x_1 = l_1 \frac{n}{d} + i'$ and $x_2 = l_2 \frac{n}{d} + i'$ where $0 \leq l_1, l_2 \leq d - 1$. Hence we have

$$\frac{n}{t}(r_2 - r_1) + (s_2 - s_1)d + (r_2 - r_1) + d\left(\left\lfloor \frac{r_1 + j}{d} \right\rfloor - \left\lfloor \frac{r_2 + j}{d} \right\rfloor\right) = (l_2 - l_1)\frac{n}{d}. \tag{1}$$

If $r_1 \neq r_2$, then $1 \leq r_2 - r_1 \leq t - 1$. But Eq. (1) implies that $t$ divides $r_2 - r_1$, a contradiction. If $r_1 = r_2$, then, by (1), we obtain

$$(s_2 - s_1)d = (l_2 - l_1)\frac{n}{d},$$

or equivalently

$$(s_2 - s_1)\frac{d}{t} = (l_2 - l_1)\frac{n}{dt}.$$

Since $x_1 \neq x_2$, $s_1 \neq s_2$. Thus $l_1 \neq l_2$. This implies that $\frac{n}{dt}$ divides $s_2 - s_1$. But $0 < |s_2 - s_1| \leq \frac{n}{dt} - 1$. This is a contradiction. So $|T_j \cap P_i| = 1$ for $0 \leq i \leq \frac{n}{d} - 1$ and $0 \leq j \leq k - 1$. Consequently, $\gamma^*_{\times k}(G_B(n, d)) \leq |T| = \frac{kn}{d}$. □

Theorem 3 is not true when $k = d$. For example, it is easy to check that $T = \{0, 1, 3, 4, 5, 6, 7\}$ is a minimum 2-tuple twin dominating set of $G_B(8, 2)$. So $\gamma^*_{\times 2}(G_B(8, 2)) = 7$.

In fact, the proof of Theorem 3 provides a construction method for constructing minimum $k$-tuple twin dominating sets in $G_B(n, d)$ when $d$ divides $n$.

**Example 1.** Table 3 gives two representations of the vertex set of $G_B(32, 4)$. By the construction method stated in Theorem 3, we can choose the minimum 3-tuple twin dominating set $T = \{0, 1, 2, 4, 5, 6, 9, 10, 11, 13, 14, 15, 16, 18, 19, 20, 22, 23, 24, 25, 27, 28, 29, 31\}$ of $G_B(32, 4)$, which is illustrated by bold numbers in Table 3.

## 3. Proof of Theorem 4

When $d$ divides $n$, the vertex set of $G_K(n, d)$ can be represented as follows:

$$V(G_K(n, d)) = \bigcup_{i=0}^{\frac{n}{d}-1} \bigcup_{j=0}^{d-1} \left\{ j\frac{n}{d} + i \right\}, \quad \text{or} \quad \bigcup_{i=0}^{\frac{n}{d}-1} \bigcup_{j=1}^{d} \{-id - j\} \pmod{n},$$

**Table 4**
The vertices of $G_K(n, d)$.

| | | | | |
|---|---|---|---|---|
| $n-1$ | $n-2$ | $n-3$ | $\cdots$ | $n-d$ |
| $n-1-d$ | $n-2-d$ | $n-3-d$ | $\cdots$ | $n-2d$ |
| $n-1-2d$ | $n-2-2d$ | $n-3-2d$ | $\cdots$ | $n-3d$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n-1-id$ | $n-2-id$ | $n-3-id$ | $\cdots$ | $n-(i+1)d$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n-1-\left(\frac{n}{d}-1\right)d$ | $n-2-\left(\frac{n}{d}-1\right)d$ | $n-3-\left(\frac{n}{d}-1\right)d$ | $\cdots$ | 0 |

**Table 5**
An example: $G_K(32, 4)$ and $k = 3$.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 28 | 29 | 30 | 31 | 0 | 8 | 16 | 24 |
| 24 | 25 | 26 | 27 | 1 | 9 | 17 | 25 |
| 20 | 21 | 22 | 23 | 2 | 10 | 18 | 26 |
| 16 | 17 | 18 | 19 | 3 | 11 | 19 | 27 |
| 12 | 13 | 14 | 15 | 4 | 12 | 20 | 28 |
| 8 | 9 | 10 | 11 | 5 | 13 | 21 | 29 |
| 4 | 5 | 6 | 7 | 6 | 14 | 22 | 30 |
| 0 | 1 | 2 | 3 | 7 | 15 | 23 | 31 |

as shown in Tables 1 and 4. Let $I_i' = \bigcup_{j=1}^{d}\{-id-j\}$ and $P_i = \bigcup_{j=0}^{d-1}\{j\frac{n}{d}+i\}$. Note that the set of $d$ elements in every row in Table 4 is exactly the out-neighborhood of each vertex in same row in Table 1. That is, $N^+(i) = N^+(\frac{n}{d}+i) = \cdots = N^+((d-1)\frac{n}{d}+i) = I_i'$ and $N^-(-id-1) = N^-(-id-2) = \cdots = N^-(-id-d) = P_i$.

By using an argument analogous to that in the proof of Theorem 3, we can prove that Theorem 4 is true. Here we give an outline of the proof of Theorem 4.

**Proof of Theorem 4.** Let $T$ be a minimum $k$-tuple twin dominating set of $G_K(n, d)$. We can show that $\gamma^*_{\times k}(G_K(n, d)) = |T| \geq \frac{kn}{d}$.

To show that the converse inequality, we construct a $k$-tuple twin dominating set $T$ of $G_K(n, d)$ with $|T| = \frac{kn}{d}$ as follows:

$$T = \bigcup_{r=0}^{t-1} T_r,\ T_r = \bigcup_{s=1}^{\frac{n}{dt}}\bigcup_{j=0}^{k-1}\left\{n-\left(\frac{n}{dt}r+s\right)d-(r+j)+d\left\lceil\frac{r+j}{d}\right\rceil\right\},$$

where $t = \left(\frac{n}{d}, d\right)$. From proving that $|T \cap I_i'| = k$ and $|T \cap P_i| = k$, the assertion follows. □

**Example 2.** Table 5 gives two representations of the set of vertices of $G_K(32, 4)$. By the construction method stated in Theorem 4, we can choose the minimum 3-tuple twin dominating set $T = \{31, 31, 28, 27, 26, 24, 23, 22, 21, 19, 18, 17, 14, 13, 12, 10, 9, 8, 7, 5, 4, 3, 1, 0\}$ of $G_K(32, 4)$, which is illustrated by the bold numbers in Table 5.

**Observation 5.** *For $d \geq 2$, and $1 \leq k \leq d+1$ when $(d+1)|n$ or else $1 \leq k \leq d$, $\gamma^*_{\times k}(G_K(n, d)) \geq \lceil\frac{kn}{d+1}\rceil$.*

**Proof.** Let $T$ be a minimum $k$-tuple twin dominating set of $G_K(n, d)$. By definition, we have $2d|T| \geq 2k(n-|T|)+2(k-1)|T|$. So $\gamma^*(G_K(n, d)) = |T| \geq \lceil\frac{kn}{d+1}\rceil$. □

Theorem 4 is not true if $k = d$ or $d+1$. For example, it is easily checked that $T = \{1, 2, 3, 4, 5\}$ is a minimum 2-tuple twin dominating set in $G_K(6, 2)$. Hence $\gamma^*_{\times 2}(G_B(6, 2)) = 5$. If $k = d+1$, then, by Observation 5, we have $\gamma^*_{\times k}(G_K(n, d)) = n$.

Finally, the problem of determining the exact values of the $d$-tuple twin domination numbers for $G_B(n, d)$ and $G_K(n, d)$ with $d \nmid n$ remains open.

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