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Non-ergodic maps in the tangent family

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ABSTRACT

We consider maps in the tangent family for which the asymptotic values are eventually mapped onto poles. For such functions the Julia set $J(f) = \overline{\mathbb{C}}$. We prove that for almost all $z \in J(f)$ the limit set $\omega(z)$ is the post-singular set and f is non-ergodic on $J(f)$. We also prove that for such f does not exist a f-invariant measure absolutely continuous with respect to the Lebesgue measure finite on compact subsets of $\mathbb{C}$.

1. INTRODUCTION

The *Fatou set* $F(f)$ of a meromorphic function $f : \mathbb{C} \to \overline{\mathbb{C}}$ is defined in exactly the same manner as for rational functions; $F(f)$ is the set of points $z \in \mathbb{C}$ such that all the iterates are defined and form a normal family on a neighborhood of z. The *Julia set* $J(f)$ is the complement of $F(f)$ in $\overline{\mathbb{C}}$. Thus, $F(f)$ is open, $J(f)$ is closed, $F(f)$ is completely invariant while $f^{-1}(J(f)) = J(f) \setminus \{\infty\}$ and $f(J(f) \setminus \{\infty\}) \subset J(f)$. For a general description of the dynamics of meromorphic functions see e.g. [1]. We would however like to note that it easily follows from Montel's criterion of normality that if $f : \mathbb{C} \to \overline{\mathbb{C}}$ has at least one pole which is not an omitted value then

$$(1.1) \qquad J(f) = \overline{\bigcup_{n \geq 0} f^{-n}(\infty)}.$$

The post-singular set of f i.e. the closure in $\overline{\mathbb{C}}$ of the forward orbit of the set of singularities of f^{-1}, is denoted by $P(f)$. If z is a point which belongs to the domain of the definition of each iterate of f we denote by $\omega(z)$ the set of cluster points of the sequence $(f^n(z))_{n\in\mathbb{N}}$ in $\overline{\mathbb{C}}$. Let $dist_\chi(z, A)$ denote the distance of the point z to the set $A \subset \overline{\mathbb{C}}$ with respect to the chordal metric χ. In ([3], Theorem 6.2) there is proved a very important characterization.

Theorem 1.1. *If f is a rational function of degree ≥ 2 or a transcendental meromorphic function then at least one of the following statements holds:*

(i) $\lim_{n\to\infty} dist_\chi(f^n(z), P(f)) = 0$ *for almost all* $z \in J(f)$;

(ii) $J(f) = \overline{\mathbb{C}}$ *and for all* $A \subset \overline{\mathbb{C}}$ *of positive measure the set* $\{n \in \mathbb{N} : f^n(z) \in A\}$ *is infinite for almost all* $z \in \overline{\mathbb{C}}$.

It may be difficult to decide which of statements (i) or (ii) applies to a given meromorphic function. It can also occur that both statements hold for a meromorphic function. Note that in the case (ii) f is ergodic and recurrent. For example if the set of singularities of f^{-1} is finite and each singularity is preperiodic but not periodic then statement (ii) holds for f and $\omega(z) = \overline{\mathbb{C}}$ for almost all $z \in \mathbb{C}$. The more general results giving sufficient conditions for maps to be ergodic is proved in [6] (see Theorem 1, p.133).

We consider a holomorphic family

$$\mathcal{F} = \{f_\lambda(z) = \lambda \tan z, \quad \lambda \in \mathbb{C}^*, z \in \mathbb{C}\},$$

where $\mathbb{C}^* := \mathbb{C} - \{0\}$, which is called the *tangent family*. The singular sets S_{f_λ} contain exactly two asymptotic values, the omitted values $\pm\lambda i$. The pre-image of a punctured neighborhood of λi (resp. $-\lambda i$) is an upper (resp. lower) half plane. It follows from the symmetry of the tangent function that the forward orbits of these asymptotic values are symmetric with respect to origin.

To simplify notation we write F_λ, J_λ, ω_λ for objects associated to functions in $\mathcal{F}$. All functions in the family have the same poles; we use the notation $p_k = \frac{\pi}{2} + k\pi$, $k \geq 0$ for the poles on the positive axis and $-p_k$ for the poles on the negative axis.

The symmetry of the maps with respect to 0 implies that the Fatou set F_λ and the Julia set J_λ are symmetric with respect to the origin. In [5], the stable behavior of functions in $\mathcal{F}$ was completely characterized. We define the sets:

$$\mathcal{C}_0 = \{\infty\}, \quad \mathcal{C}_p = \{\lambda : f_\lambda^p(\lambda i) = \infty\}, p > 0, \quad \mathcal{C} = \bigcup_0^\infty \mathcal{C}_p. \tag{1.2}$$

Points in $\mathcal{C}_p$ are called *virtual centers* of order p. The hyperbolic maps form a natural and important subset of $\mathcal{F}$. In this family these maps can be characterized as

$$\mathcal{H} = \{\lambda \in \mathbb{C}^* : f_\lambda \text{ has an attracting periodic cycle}\}.$$

We denote a connected component of $\mathcal{H}$ by Ω. In [5] section 8, the hyperbolic components are enumerated in terms of the sets defined in (1.2). We recall the following results proved in ([5], Proposition 8.11, Theorem 8.12)

Theorem 1.2. *The virtual centers $c_p \in C_{p-1}$ are in one to one correspondence with pairs of hyperbolic components (Ω_p, Ω'_p). In Ω_p each function has a pair of periodic cycles of period p and each attracts the orbit of an asymptotic value whereas in Ω'_p each function has a single attracting cycle of period 2p which attracts both asymptotic values. The virtual center $c_p \in C_{p-1}$ is a common boundary point of (Ω_p, Ω'_p). The virtual center c_0 is the point at infinity and it corresponds to the unique pair of components (Ω_1, Ω'_1); these are the only unbounded components and they are linked by hyperbolic components of period 2.*

Thus the parameters $\lambda \in \mathcal{C}$ play a very special role in the family $\mathcal{F}$. The other motivation to study maps corresponding to these parameters follows from [7], where it is shown that if $\lambda_0 \in \mathcal{C}_p, p \geq 1$, then some repelling periodic points of period $p+1$ of f_λ tend to a pole for $\lambda \to \lambda_0$ and for $\lambda = \lambda_0$ these cycles dissappear.

In this paper we describe the metric properties of the Julia set J_λ for these λ's. For $\lambda \in \mathcal{C}_p, p \geq 1$, the post-singular set $P_\lambda = \{\pm\lambda i, f_\lambda(\pm\lambda i), \ldots, f_\lambda^{p-1}(\pm\lambda i), f_\lambda^p(\pm\lambda i) = \infty\}$ and $J_\lambda = \overline{\mathbb{C}}$. Set

$$I_n(f) := \{z \in \mathbb{C} : \lim_{m \to \infty} f^{mn}(z) = \infty\}, \quad n \geq 1.$$

It follows from Theorem 1.1 that, if $I_n(f)$ has positive measure, then (i) holds. One can check that for $\lambda \in \mathcal{C}_p$, $p \geq 1, f_\lambda$ satisfies the assumptions of Proposition 8.1 and Theorem 8.2 in [3], so

$$\text{meas}(I_1(f_\lambda)) = 0 \quad \text{and} \quad \text{meas}(I_{p+1}(f_\lambda)) > 0,$$

where meas denotes 2-dimensional Lebesgue measure. Thus by Theorem 1.1 $\omega(z) \subset P_\lambda$ for almost all $z \in \mathbb{C}$. We prove that for these maps a stronger property holds, namely,

Theorem A. *Let $\lambda \in \mathcal{C}_p$, $p \geq 1$. Then $\omega(z) = P_\lambda$ for almost all $z \in \mathbb{C}$.*

To prove ergodicity of transcendental function on its Julia set one assumes that post-singular set is a compact repeller. If this assumption is not satisfied the map does not have to be ergodic. For example, for $f(z) = e^z$ the post-singular set is unbounded. In [9] it is proved that the post-singular set is a metric attractor for almost all $z \in J(f) = \mathbb{C}$ and f is not ergodic. One of the other possibility when the post-singular sets is not a compact repeller is the case when the singular values are prepoles i.e. some their iterates are equal to ∞. The considered maps satisfied this property. Our second main result is

Theorem B. *Let $\lambda \in \mathcal{C}_p$, $p \geq 1$, then f_λ is not ergodic on J_λ. There is a wandering set of positive measure in J_λ.*

Theorem 1 in [8] provide the sufficient conditions for existence of σ-finite ergodic conservative f- invariant measure μ equivalent with the Lebesgue measure. In our case the assumptions of that Theorem are not satisfied. We also show that

Theorem C. *Let $\lambda \in \mathcal{C}_p$, $p \geq 1$. Then for f_λ does not exist a f-invariant measure absolutely continuous with respect to the Lebesgue measure finite on compact subsets of $\mathbb{C}$.*

2. PRELIMINARIES

We need the following version of Koebe's Theorem (compare [4])

Theorem 2.1. *Let $f : B(z_0, \varrho) \to \mathbb{C}$ be a holomorphic univalent map, $0 < \eta < 1$. Then for $z \in S(z_0, \eta\varrho) = \{z \in \mathbb{C} : |z - z_0| = \eta\varrho\}$*

(i) $\frac{|f'(z)|\eta\varrho}{(1+\eta)^2} < |f(z) - f(z_0)| < \frac{|f'(z)|\eta\varrho}{(1-\eta)^2}$

(ii) $\frac{1-\eta}{(1+\eta)^3} < \frac{|f'(z)|}{|f'(z_0)|} < \frac{1+\eta}{(1-\eta)^3}$

(iii) $\left|\arg(\frac{f'(z)}{f'(z_0)})\right| \leq 2\ln(\frac{1+\eta}{1-\eta})$

and its straightforward corollaries.

Lemma 2.2. *Let $f : B(z_0, \varrho) \to \mathbb{C}$ be a holomorphic univalent map, $0 < \eta < 1$. Then for all $A \subset B \subset B(z_0, \eta\varrho)$ holds*

$$L^{-2}\frac{\text{meas}(A)}{\text{meas}(B)} \leq \frac{\text{meas}(f(A))}{\text{meas}(f(B))} \leq L^2\frac{\text{meas}(A)}{\text{meas}(B)},$$

where $L = \frac{\sup_{z\in B(z_0,\eta\varrho)}|f'(z)|}{\inf_{z\in B(z_0,\eta\varrho)}|f'(z)|} \leq \left(\frac{1+\eta}{1-\eta}\right)^4.$

Lemma 2.3. *Let $f : B(z_0, \varrho) \to \mathbb{C}$ be a holomorphic univalent map. Then*

$$B(f(z_0), |f'(z_0)|\varrho/8) \subset f(B(z_0, \varrho/2)) \subset B(f(z_0), 2|f'(z_0)|\varrho).$$

We recall lemma stated in [10]

Lemma 2.4. *For each $k \in \mathbb{N}$ let E_k be a finite collection of disjoint compact subsets of $\mathbb{R}^2$, each of them has positive 2-dimensional measure, and define*

$$\bar{E}_k = \bigcup_{F\in E_k} E_k, \quad E = \bigcap_{k=1}^{\infty} \bar{E}_k$$

$$density(\bar{E}_{k+1}, F) := \frac{\text{meas}(\bar{E}_{k+1} \cap F)}{\text{meas}(F)}.$$

Suppose also that for each $F \in E_k$, there exists $F' \in E_{k+1}$, and a unique $F'' \in E_{k-1}$ such that $F' \subset F \subset F''$ then

(1) *if for every $F \in E_k$,* $\text{density}(\bar{E}_{k+1}, F) \geq \Delta'_k$, *then* $\text{density}(\bar{E}, \bar{E}_1) \geq \prod_{k=0}^{\infty} \Delta'_k$

(2) *if for every $F \in E_k$,* $\text{density}(\bar{E}_{k+1}, F) \leq \Delta''_k$, *then* $\text{density}(\bar{E}, \bar{E}_1) \leq \prod_{k=0}^{\infty} \Delta''_k$

We will use the following property of infinite products.

Lemma 2.5. *Let $a_i > 0$, $i \in \mathbb{N}$ and $\sum_{i=0}^{\infty} a_i = s < 1/2$. Then*

$$\prod_{i=0}^{\infty} (1 - a_i) > (1 - 2s)$$

Proof. Define $b_n := \prod_{i=0}^{n}(1 - a_i)$. Then

$$-\ln b_n = \sum_{i=0}^{n} -\ln(1 - a_i) < 2\sum_{i=0}^{n} a_n.$$

The last inequality is a consequence of the property $-\ln(1 - x) < 2x$ which holds for $x \in (0, \frac{1}{2})$. Since the sequence $-\ln b_n$, $n \in \mathbb{N}$, is increasing there exists $b = \lim_{n\to\infty} b_n$ and $b < 2s$. Consequently $e^{-b} > e^{-2s}$. To finish the proof it is enough to observe that $e^{-2s} > 1 - 2s$ for $s \in (0, \frac{1}{2})$. $\square$

Assume $\lambda \in \mathcal{C}_p$, $p \geq 1$, and for readability omit the subscript λ. Then $J(f) = \overline{\mathbb{C}}$. We consider a new map $S(z) := f^{p+1}(z)$ defined on $J \setminus \bigcup_{k=0}^{p} f^{-k}(\infty)$. Set

$$g_1(z) := -\lambda i\left(\frac{z-1}{z+1}\right), \quad g_2(z) := e^{2iz} \quad \text{and} \quad \varphi(z) = g_2 \circ f^{p-1} \circ g_1(z)$$

Then $\varphi(0) = -1$, since $f^{p-1}(\lambda i)$ is a pole $p_k = \frac{\pi}{2} + k\pi$, and also $f^{p+1}(z) = g_1 \circ \varphi \circ g_2(z)$. Let $c > 0$ be large enough such that $S(z)$ is *well defined* for z satisfying $|\mathrm{Im} z| > c$ and

(2.1) $\quad \forall_{k=0,\dots,p-1} \quad |\mathrm{Im}(f_\lambda^k(\lambda i))| < c$

and φ is univalent on the disk $B(0, e^{-2c})$. Let $\varrho := e^{-2c}$, and $M := \frac{2|\lambda|}{|\varphi'(0)|}$. Choose y_0 such that

(2.2) $\quad y_0 > c,$

(2.3) $\quad y_0 > 1 + \dfrac{3}{M},$

(2.4) $\quad y_0 > \dfrac{4M}{\varrho} + 2|\lambda| + \dfrac{1}{\varrho^2},$

(2.5) $\quad y_0 > 4M,$

(2.6) $\quad y_0 > 32\sqrt{2}\pi^3$

and define the sequence of real numbers $y_k = e^{2y_{k-1}}, y_{-k} = -e^{2y_{k-1}}$ for $k \geq 1$. We also define for $k \neq 0$ the families of sets.

$$\begin{aligned}
V_k &:= \{z = x + iy;\ y_{|k|} + 2y_{|k|-1} \leq |y| \leq y_{|k|+1} - 2y_{|k|}, \mathrm{sgn} y = \mathrm{sgn} k\} \\
V_k^+ &:= \{z = x + iy \in V_k;\ (\mathrm{sgn} k)s + n\pi + 3/y_{|k|} \leq x \\
&\qquad \leq (\mathrm{sgn} k)s + \pi/2 + n\pi - 3/y_{|k|}\} \\
V_k^- &:= \{z = x + iy \in V_k;\ (\mathrm{sgn} k)s + \pi/2 + n\pi + 3/y_{|k|} \leq x \\
&\qquad \leq (\mathrm{sgn} k)s + (n+1)\pi - 3/y_{|k|}\},
\end{aligned}$$

where $s = \frac{\pi}{4}$ if

$$-\mathrm{Im}\lambda \mathrm{Re}\varphi'(0) + \mathrm{Re}\lambda \mathrm{Im}\varphi'(0) = 0 \quad \text{and} \quad \mathrm{Re}\lambda \mathrm{Re}\varphi'(0) + \mathrm{Im}\lambda \mathrm{Im}\varphi'(0) > 0,$$

or $s = \frac{3\pi}{4}$ if

$$-\mathrm{Im}\lambda\mathrm{Re}\varphi'(0) + \mathrm{Re}\lambda\mathrm{Im}\varphi'(0) = 0 \quad \text{and} \quad \mathrm{Re}\lambda\mathrm{Re}\varphi'(0) + \mathrm{Im}\lambda\mathrm{Im}\varphi'(0) < 0.$$

Otherwise s is chosen such that

$$\tan(2s) = \frac{\mathrm{Re}\lambda\mathrm{Re}\varphi'(0) + \mathrm{Im}\lambda\mathrm{Im}\varphi'(0)}{-\mathrm{Im}\lambda\mathrm{Re}\varphi'(0) + \mathrm{Re}\lambda\mathrm{Im}\varphi'(0)}$$

and $s \in (0, \pi/4) \cup (3\pi/4, \pi)$ if $-\mathrm{Im}\lambda\mathrm{Re}\varphi'(0) + \mathrm{Re}\lambda\mathrm{Im}\varphi'(0) > 0$, and $s \in (\pi/4, 3\pi/4)$ if $-\mathrm{Im}\lambda\mathrm{Re}\varphi'(0) + \mathrm{Re}\lambda\mathrm{Im}\varphi'(0) < 0$.

Theorem 2.6. *For $k > 0$*

(i) *if $z \in V_k^+$ then $S(z) \in V_{k+1}$ and if $z \in V_k^-$ then $S(z) \in V_{-(k+1)}$,*
(ii) *if $z \in V_{-k}^+$ then $S(z) \in V_{-(k+1)}$ and if $z \in V_{-k}^-$ then $S(z) \in V_{k+1}$.*

To prove Theorem 2.6 we need the following Lemmas.

Lemma 2.7. *For $z \in \mathbb{C}$ such that $y = \mathrm{Im}(z) > y_0$ the following inequalities are satisfied:*

$$|S(z) + \lambda i| \geq M\left(e^{2y} - \frac{2}{\varrho} + \frac{1}{\varrho^2 e^y}\right) \tag{2.7}$$

$$|S(z) + \lambda i| \leq M\left(e^{2y} + \frac{2}{\varrho} + \frac{1}{\varrho^2 e^y}\right) \tag{2.8}$$

Proof. Let $\eta = e^{-2y}/\varrho$ then $\eta \in (0, 1)$. Since φ is univalent on $B(0, \varrho)$, we can apply Theorem 2.1 to φ. Then for $|\xi - 0| = \eta\varrho$ we get

$$\frac{|\varphi'(0)|\eta\varrho}{(1+\eta)^2} \leq |\varphi(\xi) + 1| \leq \frac{|\varphi'(0)|\eta\varrho}{(1-\eta)^2},$$

where $\varphi(0) = -1$. Thus

$$|g_1(\varphi(\xi)) + \lambda i| = 2|\lambda|\frac{1}{|\varphi(\xi) + 1|},$$

and if $\xi = g_2(z)$ it follows

$$M\left(\frac{1 - 2\eta + \eta^2}{\eta\varrho}\right) \leq |S(z) + \lambda i| \leq M\left(\frac{1 + 2\eta + \eta^2}{\eta\varrho}\right). \quad \square$$

Lemma 2.8. *Let $z = x + iy$, $z' = x' + iy'$. Suppose $|y| \geq y_k$, $|y'| \geq |y| + 2y_{k-1}$, $k \geq 1$ and*

$$|\sin(\arg(Sz' + \lambda i))| > 1/y_k \tag{2.9}$$

then

$$|\mathrm{Im}Sz'| \geq |\mathrm{Im}Sz| + 2y_{k+1}.$$

Proof. We only will consider the case when $y, y' > 0$. Otherwise one can use property

$$|\mathrm{Im}(Sz)| = |\mathrm{Im}(S(-z))|$$

which holds as S is an odd function. Using (2.8) and (2.5), we can estimate as follows

$$\begin{aligned}|\mathrm{Im}Sz| &\le |Sz| \le |Sz + \lambda i| + |\lambda| \le M(e^{2y} + 2/\varrho + 1/(\varrho^2 e^y)) + |\lambda| \\ &\le Me^{2y} + 2M/\varrho + 1/\varrho^2 + |\lambda|.\end{aligned}$$

Analogously, using (2.9) and (2.3) along with (2.4) we get

$$\begin{aligned}|\mathrm{Im}Sz'| &\ge |\mathrm{Im}(Sz' + \lambda i)| - |\lambda| = |\sin(\arg(Sz' + \lambda i))| \cdot |Sz' + \lambda i| - |\lambda| \\ &\ge \frac{1}{y_k} M(e^{2y'} - 2/\varrho + 1/(\varrho^2 e^{y'})) - |\lambda| \ge \frac{1}{y_k} Me^{2(y+2y_{k-1})} - 2M/\varrho - |\lambda| \\ &\ge Me^{2y} y_k - 2M/\varrho - |\lambda| \ge Me^{2y}(1 + 3/M) - 2M/\varrho - |\lambda| \\ &\ge Me^{2y} + 2y_{k+1} + y_0 - 2M/\varrho - |\lambda| \ge Me^{2y} + 2M/\varrho + 1/\varrho^2 + |\lambda| + 2y_{k+1} \\ &\ge |\mathrm{Im}Sz| + 2y_{k+1}. \quad \square\end{aligned}$$

Lemma 2.9. *There is a constant b such that for all $k > 0$ and $y_0 > b$ it holds:*

(i) *if $z \in V_k^+$ then $\mathrm{Im}(Sz) > 0$ and $\arg(S(z) + \lambda i) \in (2/y_k, \pi - 2/y_k)$,*

(ii) *if $z \in V_k^-$ then $\mathrm{Im}(Sz) < 0$ and $\arg(S(z) + \lambda i) \in (\pi + 2/y_k, 2\pi - 2/y_k)$.*

Proof. Fix $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. We will show that each point z which belongs to the half-line

$$L_k = \{z = (s + n\pi + 3/y_k) + iy;\ y \in \mathbb{R},\ y \ge y_0\}$$

has the property that

$$\arg(Sz + \lambda i) > 2/y_k.$$

We also consider a half-line

$$L = \{z = s + n\pi + iy;\ y \in R,\ y \ge y_0\}.$$

Then $g_2(z) = e^{(2iz)} = e^{-2y} e^{i2x}$ maps these straight-lines onto rays starting at 0 i.e. $g_2(L) = e^{-2y} e^{i2s}$ and $g_2(L_k) = e^{-2y} e^{i2(s + \frac{3}{y_k})}$, $y \ge y_0$. Since $|x - (s + n\pi)| = \frac{3}{y_k}$, we get

$$|\arg(g_2(L)) - \arg(g_2(L_k))| = \frac{6}{y_k}.$$

The function $\varphi : B(0, \varrho) \to \mathbb{C}$ maps the rays $g_2(L)$ and $g_2(L_k)$ onto curves starting at $\varphi(0) = -1$. Let L' and L_k' be the half-lines tangent respectively to the curves $\varphi(g_2(L))$ and $\varphi(g_2(L_k))$ at $z_0 = -1$. Hence

$$\begin{aligned} L'(t) &= -1 + te^{i[\arg(\varphi'(0)) + 2s]}, \quad t \ge 0 \\ L_k'(t) &= -1 + te^{i[\arg(\varphi'(0)) + 2s + 6/y_k]}, \quad t \ge 0 \end{aligned} \tag{2.10}$$

Since φ is a univalent holomorphic map in $B(0,\rho)$, the angle between L' and L_k' is the same as the angle between $g_2(L)$ and $g_2(L_k)$ and is equal to $6/y_k$. Fix $z \in L_k$ and define $z_1 = g_2(z)$. Then

$$\varphi(z_1) = -1 + r_1 e^{i\theta}, \quad r > 0. \tag{2.11}$$

Define $\arg(L') := \arg(\varphi'(0)) + 2s$, $\arg(L_k') := \arg(\varphi'(0)) + 2s + 6/y_k$. We claim that there exists b such that if $y_0 > b$, then

$$|\theta - \arg(L_k')| < 2/y_k. \tag{2.12}$$

It implies that

$$|\theta - \arg(L')| > 2/y_k, \tag{2.13}$$

since

$$\frac{6}{y_k} = |\arg(L_k') - \arg(L')| = |\arg(L_k') - \theta + \theta - \arg(L')|.$$

The half-line L' is mapped by $g_1(z) := -\lambda i\left(\frac{z-1}{z+1}\right)$ onto half-line starting from $-\lambda i$. We choose s such that $g_1(L')$ is parallel to the real axis $\mathbb{R}$, i.e. $\operatorname{Im}(g_1(L')) = \operatorname{Im}(-\lambda i)$. One may check that $s = \frac{\pi}{4}$ if

$$-\operatorname{Im}\lambda\operatorname{Re}\varphi'(0) + \operatorname{Re}\lambda\operatorname{Im}\varphi'(0) = 0 \quad \text{and} \quad \operatorname{Re}\lambda\operatorname{Re}\varphi'(0) + \operatorname{Im}\lambda\operatorname{Im}\varphi'(0) > 0,$$

or $s = \frac{3\pi}{4}$ if

$$-\operatorname{Im}\lambda\operatorname{Re}\varphi'(0) + \operatorname{Re}\lambda\operatorname{Im}\varphi'(0) = 0 \quad \text{and} \quad \operatorname{Re}\lambda\operatorname{Re}\varphi'(0) + \operatorname{Im}\lambda\operatorname{Im}\varphi'(0) < 0.$$

Otherwise s is chosen such that

$$\tan(2s) = \frac{\operatorname{Re}\lambda\operatorname{Re}\varphi'(0) + \operatorname{Im}\lambda\operatorname{Im}\varphi'(0)}{-\operatorname{Im}\lambda\operatorname{Re}\varphi'(0) + \operatorname{Re}\lambda\operatorname{Im}\varphi'(0)}$$

and $s \in (0,\pi/4) \cup (3\pi/4,\pi)$ if $-\operatorname{Im}\lambda\operatorname{Re}\varphi'(0) + \operatorname{Re}\lambda\operatorname{Im}\varphi'(0) > 0$, and $s \in (\pi/4, 3\pi/4)$ if $-\operatorname{Im}\lambda\operatorname{Re}\varphi'(0) + \operatorname{Re}\lambda\operatorname{Im}\varphi'(0) < 0$. Consequently for $S(z) = g_1(\varphi(z_1))$ we get $S(z) = -\lambda i + te^{i\psi}$ for some $t > 0$ and $0 < \psi < 2\pi$. Then by (2.13)

$$\arg(S(z) + \lambda i) > 2/y_k \tag{2.14}$$

and we are done. One can check, using similar arguments, that each point z which belongs to the half-line

$$L_k^1 = \left\{ z = \left(s + (n+1/2)\pi - \frac{3}{y_k} \right) + iy;\ y \in R,\ y \geq y_0 \right\},$$

has property

$$\arg(S(z) + \lambda i) < \pi - 2/y_k. \tag{2.15}$$

We choose b such that for $y_0 > b$ the images $S(L_k)$, $S(L_k^1)$ are above the line parallel to the real axis, passing trough $-\lambda i$. Also the images of the points belonging to V_k^+ and contained in the strip bounded by lines L_k, L_k^1 and the line

$y = y_0$ have the same property, i.e. $\mathrm{Im}(S(z) + \lambda i) > 0$. It follows from (2.14) and (2.15) that these points satisfy the assumption (2.9) of Lemma 2.8. Since $y_0 > |\mathrm{Im}(\lambda i)|$ by the assumption (2.1), Lemma 2.8 implies that $\mathrm{Im}(S(z)) > 0$.

Now we prove the claim. Choose r such that $1/e^r < \varrho$. Set $\eta = \frac{1}{e^r \varrho}$. Then by Theorem 2.1 (iii) for $\xi \in B(0, 1/e^r)$ we obtain

$$\left|\arg\left(\frac{\varphi'(\xi)}{\varphi'(0)}\right)\right| \leq 2\ln\frac{1+\eta}{1-\eta} = o(1/r).$$

where $\ln\frac{1+\eta}{1-\eta} = o(1/r)$ since $\lim_{\eta\to 0}\frac{\ln\frac{1+\eta}{1-\eta}}{\eta} = 2$. Hence, enlarging b if it is necessary, for $r > b$ and for all $\xi \in B(0, 1/e^{2r})$ we get inequality

$$\left|\arg\left(\frac{\varphi'(\xi)}{\varphi'(0)}\right)\right| \leq 1/r. \tag{2.16}$$

Set

$$\beta_1(t) = z_1 \cdot t, t \in [0,1],$$

where z_1 was defined above and consider the curve $\beta_2(t) = \frac{\varphi(\beta_1(t))}{\varphi'(0)z_1}, t \in [0, 1]$.

Since $\beta_2(1) = \frac{\varphi(z_1)}{\varphi'(0)z_1}$ and $\beta_2(0) = \frac{-1}{\varphi'(0)z_1}$ then $\beta_2(1) - \beta_2(0) = \frac{\varphi(e^{2iz})+1}{\varphi'(0)z_1}$, and consequently

$$\arg(\beta_2(1) - \beta_2(0)) = \arg\left(\frac{\varphi(e^{2iz}) + 1}{\varphi'(0)z_1}\right) = \arg(\varphi(z_1) + 1) - \arg(L_k'), \tag{2.17}$$

where $\arg(L_k')$ was defined above. By (2.16)

$$|\arg(\varphi'(\beta_1(t))/\varphi'(0))| \leq 1/y_k.$$

Using (2.17) and the following inequality

$$|\arg(\beta_2(1) - \beta_2(0))| \leq \max_{t\in[0,1]} |arg(\beta_2'(t))|,$$

we obtain

$$\left|\arg\left(\frac{\varphi(e^{2iz}) + 1}{\varphi'(0)z_1}\right)\right| \leq \max_{t\in[0,1]} |arg(\beta_2'(t))| =$$
$$\max_{t\in[0,1]} |arg(\varphi'(\beta_1(t))/\varphi'(0))| \leq 2/y_k,$$

This proves (2.12) and finishes the proof of part (i). The proof of part (ii) is analogous. □

From now we will assume that

$$y_0 > b. \tag{2.18}$$

Proof of Theorem.

Case (i). Suppose $z \in V_k^+ \cup V_k^-$, for $k > 0$, then $|y| \leq y_{k+1} - 2y_k$. We must show that

$$|\mathrm{Im}(Sz)| < y_{k+2} - 2y_{k+1}.$$

Since

$$|\mathrm{Im} Sz| \leq |Sz| \leq |Sz + \lambda i| + |\lambda|,$$

and using (2.8) together with (2.5) and (2.4) we get

$$\begin{aligned} |\mathrm{Im} Sz| &\leq y_{k+2}M/(y_{k+1})^2 + 2M/\varrho + 1/\varrho^2 + |\lambda| \\ &\leq y_{k+2}/2 - y_{k+2}/4 + 2M/\varrho + 1/\varrho^2 + |\lambda| \\ &\leq y_{k+2}/2 \leq y_{k+2} - 2y_{k+1}. \end{aligned}$$

If $z \in V_k^+ \cup V_k^-$, then $|y| \geq y_k + 2y_{k-1}$. It follows from the proof of Lemma 2.9 that if $y_0 > b$ then Property (2.9) holds for $S(z)$. Then the assumption of Lemma 2.8 are satisfied. So

$$|\mathrm{Im}(Sz)| > 2y_{k+1} > y_{k+1} + 2y_k$$

and consequently $S(z) \in V_{k+1} \cup V_{-(k+1)}$. The proof of case (ii) follows from (i) and the property that $S(z)$ is an odd function. □

3. PROOF OF THEOREM B

We define a family $\mathcal{G} = \{Q_{(n,m)};\ m, n \in \mathbb{Z}\}$, where $Q_{(n,m)}$ is a square bounded by straight-lines

$$x = s + n\pi, \quad x = s + (n+1)\pi, \quad y = m\pi, \quad y = (m+1)\pi$$

if $m \geq 0$ or

$$x = -s + n\pi, \quad x = -s + (n+1)\pi, \quad y = m\pi, \quad y = (m+1)\pi$$

if $m < 0$. Let $Q \in \mathcal{G}$ be such that $Q \subset V_k \cup V_{-k}$ for some $k \in \mathbb{N}$, $k \neq 0$. Let $\mu \in \{-1, 1\}$,

$$\mathcal{Z}^{\mu}(Q) := \{Q' \in \mathcal{G} : Q' \subset \mathrm{Int} S(Q \cap V_k^{\mu})\},$$

$$P^{\mu}(Q) := \{z \in Q : Sz \in \bigcup \mathcal{Z}^{\mu}(Q)\}.$$

For each $i \geq 1$ define

$$\overline{\sigma} = (\sigma_0, \ldots, \sigma_{i-1}),\ \sigma_j \in \mathbb{Z}^2, j = 0, \ldots, i-1$$

$$\overline{\mu} = (\mu_0, \ldots, \mu_{i-1}),\ \mu_j \in \{-1, 1\}, j = 0, \ldots, i-1,$$

$$P_{\overline{\mu}}^{\overline{\sigma}}(Q) := \{z \in Q : S^j z \in P^{\mu_j}(Q_{\sigma_j}),\ j = 0, \ldots, i-1\},$$

$$E_0(Q) := \{Q\},$$

$$E_i(Q) := \{P_{\overline{\mu}}^{\overline{\sigma}}(Q) :\ \overline{\mu}, \overline{\sigma} \text{ have a length } i\},$$

$$E(Q) := \bigcap_{i=0}^{\infty} \overline{E_i(Q)}$$

where

$$\overline{E_i(Q)} := \bigcup E_i(Q)$$

$$E := \bigcup \{E(Q) : Q \in \mathcal{G}, \exists_k\; Q \subset V_k\}.$$

$$G := \bigcup_{i=-\infty}^{\infty} f^i(E).$$

Lemma 3.1. *Let $z \in \mathbb{C}$ be such that $|\mathrm{Im} f^n(z)| > y_k - 2y_{k-1}, k \geq 1$. Then*

$$|(f^n)'(z)| \geq \frac{y_k}{8\pi}.$$

Proof. Let ϕ be a holomorphic branch of the inverse map $(f^n)^{-1}$ mapping $f^n(z)$ onto z. Then ϕ is univalent in a ball $B(f^n(z), \frac{y_k}{4})$. By Koebe $\frac{1}{4}$-Theorem $\phi(B(f^n(z), \frac{y_k}{4})) \supset B(z, \frac{y_k}{16}|\phi'(f^n(z))|)$. Since $\phi(B(f^n(z), \frac{y_k}{4}))$ is contained in the strip of width π we obtain that $\frac{y_k}{16}|\phi'(f^n(z))| < \pi/2$. It implies that

$$(3.1) \qquad |\phi'(f^n(z))| < \frac{8\pi}{y_k}$$

and consequently $|(f^n)'(z)| \geq \frac{y_k}{8\pi}$. □

Lemma 3.2. *For every $Q \in \mathcal{G}$ contained in V_k*

$$\mathrm{meas}(Q \setminus (P^+(Q) \cup P^-(Q))) \leq \frac{13\pi}{y_k}$$

Proof. It follows directly from the definition that

$$(3.2) \qquad \mathrm{meas}(Q \setminus (V_k^+ \cup V_k^-)) \leq 4\frac{3\pi}{y_k} = \frac{12\pi}{y_k}$$

Since $S(Q \cap V_k^\mu) \setminus \bigcup Z^\mu(Q)$ is contained in $\sqrt{2}\pi$-neighborhood of $\partial S(Q \cap V_k^\mu)$ then $(Q \cap V_k^\mu) \setminus P^\mu(Q)$ is a subset of $\sqrt{2}\pi K$-neighbourhood of $\partial(Q \cap V_k^\mu)$, where

$$K := \max_{z \in S(Q \cap V_k^\mu) \setminus \bigcup Z^\mu(Q)} |(S^{-1})'(z)|$$

and S^{-1} means a holomorphic branch of the inverse function mapping $S(Q \cap V_k^\mu)$ onto $Q \cap V_k^\mu$. It together with (3.1) implies that $(Q \cap V_k^\mu) \setminus P^\mu(Q)$ is contained in $\frac{8\sqrt{2}\pi^2}{y_{k+1}}$-neighborhood of $\partial(Q \cap V_k^\mu)$. Therefore

$$\mathrm{meas}((Q \cap V_k^\mu) \setminus P^\mu(Q)) < 4\pi\frac{8\sqrt{2}\pi^2}{y_{k+1}} = \frac{32\sqrt{2}\pi^3}{y_{k+1}}$$

Thus by (3.2) and (2.6) we get

$$\mathrm{meas}(Q \setminus (P^+(Q) \cup P^-(Q))) \leq \frac{12\pi}{y_k} + \frac{32\sqrt{2}\pi^3}{y_{k+1}} < \frac{13\pi}{y_k} \qquad \square$$

Lemma 3.3. *There exists a constant $D_1 > 0$ such that for $y_0 > D_1$, $Q \in \mathcal{G}$, $Q \subset V_k$, $F \in E_i(Q)$ and $i \geq 0$ holds*

$$\text{density}(\bar{E}_{i+1}(Q), F) := \frac{\text{meas}(\bar{E}_{i+1}(Q) \cap F)}{\text{meas}(F)} \geq 1 - \frac{5}{y_{|k|+i}}.$$

Proof. Suppose $F \in E_i(Q)$ then $S^i(F)$ is a square contained in V_l with sides of lenght π, where $|l| = |k| + i$. By Lemma 3.2

$$\text{meas}(S^i(F) \setminus S^i(\bar{E}_{i+1}(Q))) \leq \frac{13\pi}{y_{|k|+i}}. \tag{3.3}$$

Let ϕ be a holomorphic branch of S^{-i} mapping $S^i(F)$ onto F. Let z_i be a center of $S^i(F)$. We consider a ball with center at $z_i \in S^i(F)$ and radius $r = \frac{y_{|l|}}{2}$. Then

$$S^i(F) \subset \bar{B}(z_i, \frac{\sqrt{2}}{2}\pi) \subset B(z_i, 5).$$

Since ϕ is univalent on $B(z_i, r)$ it has a bounded distortion on $B(z_i, 5)$. It follows from Lemma 2.2 and (3.3) that

$$\begin{aligned}\frac{\text{meas}(\phi(S^i(F) \setminus S^i(\bar{E}_{i+1}(Q))))}{\text{meas}(\phi(S^i(F)))} &\leq \left(\frac{1+\eta}{1-\eta}\right)^8 \frac{\text{meas}(S^i(F) \setminus S^i(\bar{E}_{i+1}(Q)))}{\text{meas}(S^i(F))} \\ &\leq \left(\frac{1+\eta}{1-\eta}\right)^8 \frac{13\pi}{\pi^2 y_{|k|+i}}.\end{aligned}$$

where $\eta := \frac{10}{y_{|k|+i}}$. So we can choose D_1 such that for $y_0 > D_1$ thesis holds. □

Lemma 3.4. *There exists a constant $D > D_1$ such that for $y_0 > D$, $Q \in \mathcal{G}$, $Q \subset V_k$ holds*

$$\text{density}(E(Q), Q) := \frac{\text{meas}(E(Q) \cap Q)}{\text{meas}(Q)} \geq 1 - \frac{12}{y_{|k|}}$$

Proof. Fix $k \neq 0$ and assume that $y_0 > D_1$. It follows from Lemma 2.4 and Lemma 3.3 that

$$\text{density}(E(Q), Q) \geq \prod_{i=0}^{\infty} \left(1 - \frac{5}{y_{|k|+i}}\right).$$

To get a lower bound on $\prod_{i=0}^{\infty}\left(1 - \frac{5}{y_{|k|+i}}\right)$ it is enough to find an upper bound on $\sum_{i=0}^{\infty} \frac{5}{y_{|k|+i}}$. Since

$$\sum_{i=0}^{\infty} \frac{5}{y_{|k|+i}} \leq \sum_{i=0}^{\infty} \frac{5}{y_{|k|}^{i+1}} \leq \sum_{i=0}^{\infty} \frac{5}{y_{|k|}^{i+1}} \leq \frac{5}{y_{|k|}} \sum_{i=0}^{\infty} \frac{1}{y_0^i} = \frac{5y_0}{y_0 - 1} \frac{1}{y_{|k|}},$$

there exists $D > D_1$ such that $\sum_{i=0}^{\infty} \frac{5}{y_{|k|+i}} \leq \frac{6}{y_{|k|}}$ for $y_0 > D$. It follows from (2.6) that $\frac{6}{y_{|k|}} < 1/2$. Hence by Lemma 2.5 we get

$$\prod_{i=0}^{\infty}\left(1-\frac{5}{y_{|k|+i}}\right)>1-\frac{12}{y_{|k|}}. \quad \square$$

As a consequence of this Lemma we get the following Proposition.

Proposition 3.5. $\mathrm{meas}(E)>0$.

Further we will assume that

(3.4) $\quad y_0>D.$

Theorem B. *Let $\lambda\in\mathcal{C}_p$, $p\geq 1$, then f_λ is not ergodic on J_λ. There is a wandering set of positive measure in J_λ.*

Proof. We will show that there exist two sets Q_1', Q_2' contained in E which forward trajectories are disjoint. Fix $k>0$ and choose two squares $Q_1,Q_2\subset V_k\cup V_{-k}$ such that

(i) $\mathrm{meas}(Q_i\cap E)>0$,

(ii) for all $z_1\in Q_1$, $z_2\in Q_2$ holds $|\mathrm{Im}(z_1)|-|\mathrm{Im}(z_2)|>2y_{k-1}$.

Denote $Q_i':=Q_i\cap E$, $i=1,2$. We will show that for all integers $j\in N$ the following conditions are satsfied

(a) $S^jz_i\in V_{k+j}\cup V_{-(k+j)}$, $i=1,2$,

(b) $|\mathrm{Im}(S^j(z_1))|-|\mathrm{Im}(S^j(z_2))|>2y_{k+j-1}$.

The proof is by induction. For $j=0$ the conditions (a) and (b) are just the assumptions (i) and (ii). Let now assume that a), b) holds for $j=n$. Since E is forward invariant for S thus $S^n(z_i)\in E$ and by the assumption $S^n(z_i)\in V_{k+n}\cup V_{-(k+n)}$. Theorem 2.6 implies that $S^{n+1}(z_i)\in V_{k+n+1}\cup V_{-(k+n+1)}$. For $j=n$ it follows from (b) that

$$|\mathrm{Im}(S^n(z_1))|-|\mathrm{Im}(S^n(z_2))|>2y_{k+n-1}.$$

Since $S^n(z_i)\in V_{k+n}\cup V_{-(k+n)}$ thus analogously as in the proof of Lemma 2.9 we can show that

$$|\sin(arg(S^{k+n+1}(z_i)+\lambda i))|>\frac{1}{y_{k+n}}.$$

Thus the assumption (2.9) of Lemma 2.8 is satisfied. Therefore

$$|\mathrm{Im}(S^{n+1}(z_1))|-|\mathrm{Im}(S^{n+1}(z_2))|>2y_{k+n+1}>2y_{k+n}$$

what gives the condition (b) for $j=n+1$. Therefore the forward trajectories of Q_1' and Q_2' are disjoint. Their great orbits are also disjoint. So Q_1' and Q_2' are wandering subsets of J_λ of positive measure. Define $Q':=\bigcup_{j=1}^{\infty}S^j(Q_1')$, $Q'':=\bigcup_{j=1}^{\infty}S^j(Q_2')$. They are invariant, disjoint sets of positive measure. It implies that f_λ is not ergodic on $J(f_\lambda)=\overline{\mathbb{C}}$. $\square$

4. PROOF OF THEOREM A

Lemma 4.1. *For almost $z\in\mathbb{C}$ there is a sequence of integers $n_k\to\infty$ such that*

$$|\mathrm{Im} f^{n_k}(z)| \to \infty.$$

Proof. By Theorem 1.1(i) for almost all $z \in \mathbb{C}$ there is an asymptotic value $\alpha \in \{\lambda i, -\lambda i\}$ and a sequence of integers $n_1+1, n_2+1, \ldots, (n_k+1), \ldots$ such that $|f^{n_k+1}(z) - \alpha| \to 0$ and consequently $|\mathrm{Im} f^{n_k}(z)| \to \infty$. $\square$

Lemma 4.2.

$$\mathrm{meas}(\mathbb{C} \setminus G) = 0.$$

Proof. Suppose to the contrary that $\mathrm{meas}(\mathbb{C} \setminus G) > 0$. So there exists a density point z of $\mathbb{C} \setminus G$. By Lemma 4.1 there are two sequences of integers $n_k \in \mathbb{N}, n_k \to \infty$ and $l_k \in \mathbb{Z}, l_k \to \infty$ satisfying

$$y_{l_k} - 2y_{l_k-1} \le |\mathrm{Im} f^{n_k}(z)| < y_{l_k+1} - 2y_{l_k},$$

$l_k \to \infty$. Let ϕ_{n_k} denote the a holomorphic branch of the inverse map f^{-n_k} sending $z_{n_k} = f^{n_k}(z)$ to z. Let K_{n_k} be a square with center at z_{n_k} and sides of lengh $8y_{l_k-1}$. Then

$$\mathrm{meas}(K_{n_k} \cap \bigcup_{i \in \mathbb{Z}_*} V_i) \ge \frac{1}{2}\mathrm{meas}(K_{n_k}),$$

where $\mathbb{Z}_* := \mathbb{Z} \setminus \{0\}$. It implies that for

$$K'_{n_k} := \{z \in K_{n_k} : \exists Q \in \mathcal{G},\ z \in Q \subset K_{n_k} \cap \bigcup_{i \in \mathbb{Z}_*} V_i\}$$

the following inequality is true

$$\mathrm{meas}(K'_{n_k}) \ge \frac{1}{4}\mathrm{meas}(K_{n_k}).$$

From Lemma 3.4 we have that for $Q \subset \bigcup_{i \in \mathbb{Z}_*} V_i$

$$\mathrm{meas}(Q \cap E) \ge \frac{1}{2}\mathrm{meas}(Q).$$

Hence

$$\mathrm{meas}(K'_{n_k} \cap E) \ge \frac{1}{8}\mathrm{meas}(K_{n_k}).$$

The map ϕ_{n_k} is univalent on the disc $B(z_{n_k}, 2r_{n_k})$ where $r_{n_k} = \frac{1}{2}\mathrm{diam}(K_{n_k}) = 4\sqrt{2}y_{l_k-1}$. Also the distortion of ϕ_{n_k} on disc $B(z_{n_k}, r_{n_k})$ is universally bounded. By Lemma 2.3

$$B(z, \rho_{n_k}/4) \subset \phi(B(z_{n_k}, r_{n_k})) \subset B(z, 4\rho_{n_k}),$$

where $\rho_{n_k} = |\phi'_{n_k}(z_{n_k})| r_{n_k}$. Then, using Lemma 2.2,

$$\frac{\text{meas}\big(B(z,4\rho_{n_k})\cap G\big)}{\text{meas}\big(B(z,4\rho_{n_k})\big)} \geq \frac{\text{meas}\big(B(z,4\rho_{n_k})\cap G\big)}{16^2\,\text{meas}\big(B(z,\varrho_{n_k}/4)\big)} \geq \frac{\text{meas}\Big(\phi\big(B(z_{n_k},r_{n_k})\big)\cap G\Big)}{16^2\,\text{meas}\Big(\phi\big(B(z_{n_k},r_{n_k})\big)\Big)}$$

$$\geq \frac{\text{meas}\big(B(z_{n_k},r_{n_k})\cap G\big)}{81^2 16^2\,\text{meas}\big(B(z_{n_k},r_{n_k})\big)} \geq \frac{\text{meas}(K'_{n_k}\cap E)}{81^2 16^2(\pi/2)\text{meas}(K_{n_k})} \geq \frac{1}{8\cdot 81^2 16^2(\pi/2)} > 0.$$

Moreover by Lemma 3.1 we get

$$\rho_{n_k} < \frac{4\sqrt{2}y_{l_k-1}}{y_{l_k}/8\pi} \to 0 \quad \text{for} \quad k\to\infty$$

Since z is a density point of $\mathbb{C}\setminus F$ this leads to a contradiction. □

Theorem A. *Let $\lambda \in C_p, p \geq 1$. Then $\omega(z) = P_\lambda$ for almost all $z \in \mathbb{C}$.*

Proof. We will show that

$$\text{meas}(\{z \in E : \omega_S(z) \neq P_\lambda\}) = 0.$$

Let Q be a square in $\mathcal{G}$ such that $Q \subset V_k$. Denote:

$$E'_0 = \{Q\},$$
$$E_i^+ = \{P_{\bar{\mu}}^{\bar{\sigma}}(Q) : \bar{\mu}, \bar{\sigma}, \text{ have length i}, \ \mu_j = 1, j = 1,\dots,i-1\},$$
$$E_i^- = \{P_{\bar{\mu}}^{\bar{\sigma}}(Q) : \bar{\mu}, \bar{\sigma}, \text{ have length i}, \ \mu_j = -1, j = 1,\dots,i-1\},$$

Analogously as in Lemma 3.3 we can control distortion of S^{-n}. Since for every $Q' \subset V_l$ and each $\mu \in \{-1,1\}$

$$\frac{\text{meas}(Q'\cap V_l^\mu)}{\text{meas}(Q')} \leq \frac{1}{2}$$

we obtain for every $F \in E_i^\mu$

$$\text{density}(\bar{E}_{i+1}^\mu, F) \leq L_3\text{density}(Q'\cap V_l^\mu, Q') \leq \frac{L_3}{2}$$

for some $L_3 > 0$ and $|l| = |k| + i$. We can assume that the constant L_3 is less then 1.5. So

$$\text{density}(\bar{E}_{i+1}^\mu, F) < 3/4.$$

Consequently by Lemma 2.4

$$\text{density}\Big(\bigcap_{i=0}^{\infty} \bar{E}_i^\mu, \bar{E}'_0\Big) = 0. \tag{4.1}$$

Set $Z := \{z \in G : \ \omega(z) \neq P_\lambda\}$, $W(Q) := \bigcap \bar{E}_k^+ \cup \bigcap \bar{E}_k^-$. Then

$$Z \subset \bigcup \left\{ \bigcup_{i=-\infty}^{\infty} f^{-i}(W(Q)) : Q \in \mathcal{G}, \ Q \subset V_k \text{ for some } k \right\}.$$

Since $\text{meas}(W(Q)) = 0$ by (4.1) then also $\text{meas}(Z) = 0$. □

5. PROOF OF THEOREM C

Suppose there exists an invariant measure μ absolutely continuous with respect to the Lebesgue measure. It follows from Lemma 4.2 that meas$(\mathbb{C} \setminus G) = 0$. Thus also $\mu(\mathbb{C} \setminus G) = 0$, so $\mu(G) \neq 0$. Since $G = \bigcup_{k \in \mathbb{Z}} f^k(E)$, then $\mu(\bigcup_{k \in \mathbb{Z}} f^k(E)) \neq 0$. It implies that $\mu(E) \neq 0$. Suppose to the contrary that $\mu(E) = 0$. Since μ is f-invariant, then $\mu(f^k(E)) = \mu(E) = 0$ for all $k < 0$. For $k \geq 0$ we have $S(E) = f^{(p+1)k}(E) \subset E$, so $\mu(f^{(p+1)k}(E)) = 0$. Now we consider $f^{(p+1)k-q}(E) \subset f^{-q}(f^{(p+1)k}((E)) = f^{-q}(S^k(E))$ for $q = 1, \ldots, p$. Again since μ is f invariant, $\mu(f^{(p+1)k-q}(E)) = 0$. Thus $\mu(\bigcup_{k \in \mathbb{Z}} f^k(E)) = 0$ and we get a contradiction.

Denote $A_k = \{z : |\text{Im}(z)| > y_k\}, k \geq 1$. Then $E \subset A_1$ and $S^j(E) \subset A_j$ for $j \in \mathbb{N}$. Note that $S^{-k}(S^k(E)) \supset E$ so $\mu(A_k) \geq \mu(E)$. Since $f(A_k)$ is contained in a union of two balls $B(\lambda i, \rho_k) \cup B(-\lambda i, \rho_k)$, where $\rho_1 > \rho_1 > \ldots$ and $\rho_k \to 0$ for $k \to \infty$, then $\{\lambda i, -\lambda i\} = \bigcap_{k \geq 1} B(\lambda i, \rho_k) \cup B(-\lambda i, \rho_k)$ and $\mu(\{\lambda i, -\lambda i\}) = \lim_{k \to \infty} \mu(B(\lambda i, \rho_k) \cup B(-\lambda i, \rho_k)) \geq \mu(E)$. It leads to a contradiction with the definition of absolutely continouos measure with respect to the Lebesgue measure.

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