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Biperfect Hopf Algebras

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1. INTRODUCTION

Recall that a finite group is called perfect if it does not have non-trivial one-dimensional representations (over $\mathbb{C}$). By analogy, let us say that a finite-dimensional Hopf algebra H over $\mathbb{C}$ is *perfect* if any one-dimensional H-module is trivial. Let us say that H is *biperfect* if both H and H^* are perfect. Note that by [R], H is biperfect if and only if its quantum double $D(H)$ is biperfect.

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It is not easy to construct a biperfect Hopf algebra of dimension > 1. The goal of this note is to describe the simplest such example we know.

The biperfect Hopf algebra H we construct is semisimple. Therefore, it yields a negative answer to [EG, Question 7.5]. Namely, it shows that [EG, Corollary 7.4], stating that a triangular semisimple Hopf algebra over $\mathbb{C}$ has a non-trivial group-like element, fails in the quasitriangular case. The counterexample is the quantum double $D(H)$.

2. BICROSSPRODUCTS

Let G be a finite group. If G_1 and G_2 are subgroups of G such that $G = G_1 G_2$ and $G_1 \cap G_2 = 1$, we say that $G = G_1 G_2$ is an *exact factorization*. In this case G_1 can be identified with G/G_2, and G_2 can be identified with G/G_1 as sets, so G_1 is a G_2-set and G_2 is a G_1-set. Note that if $G = G_1 G_2$ is an exact factorization, then $G = G_2 G_1$ is also an exact factorization by taking the inverse elements.

Following Kac [K] and Takeuchi [T], one can construct a semisimple Hopf algebra from these data as follows. Consider the vector space $H := \mathbb{C}[G_2]^* \otimes \mathbb{C}[G_1]$. Introduce a product on H by

$$(\varphi \otimes a)(\psi \otimes b) = \varphi(a \cdot \psi) \otimes ab \tag{1}$$

for all $\varphi, \psi \in \mathbb{C}[G_2]^*$ and $a, b \in G_1$. Here $\cdot$ denotes the associated action of G_1 on the algebra $\mathbb{C}[G_2]^*$, and $\varphi(a \cdot \psi)$ is the multiplication of φ and $a \cdot \psi$ in the algebra $\mathbb{C}[G_2]^*$.

Identify the vector spaces

$$\begin{aligned} H \otimes H &= (\mathbb{C}[G_2] \otimes \mathbb{C}[G_2])^* \otimes (\mathbb{C}[G_1] \otimes \mathbb{C}[G_1]) \\ &= \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}[G_2] \otimes \mathbb{C}[G_2], \mathbb{C}[G_1] \otimes \mathbb{C}[G_1]) \end{aligned}$$

in the usual way, and introduce a coproduct on H by

$$(\Delta(\varphi \otimes a))(b \otimes c) = \varphi(bc) a \otimes b^{-1} \cdot a \tag{2}$$

for all $\varphi \in \mathbb{C}[G_2]^*$, $a \in G_1$, and $b, c \in G_2$. Here $\cdot$ denotes the action of G_2 on G_1.

THEOREM 2.1 [K, T]. *There exists a unique semisimple Hopf algebra structure on the vector space $H := \mathbb{C}[G_2]^* \otimes \mathbb{C}[G_1]$ with the multiplication and comultiplication described in* (1) *and* (2).

The Hopf algebra H is called the *bicrossproduct* Hopf algebra associated with G, G_1, G_2 and is denoted by $H(G, G_1, G_2)$.

THEOREM 2.2 [M]. *$H(G, G_2, G_1) \cong H(G, G_1, G_2)^*$ as Hopf algebras.*

We are ready now to prove our first result.

THEOREM 2.3. *$H(G, G_1, G_2)$ is biperfect if and only if G_1, G_2 are self-normalizing perfect subgroups of G.*

Proof. It is well known that the category of finite-dimensional representations of $H(G, G_1, G_2)$ is equivalent to the category of G_1-equivariant vector bundles on G_2, and hence that the irreducible representations of $H(G, G_1, G_2)$ are indexed by pairs (V, x) where x is a representative of a G_1-orbit in G_2, and V is an irreducible representation of $(G_1)_x$, where $(G_1)_x$ is the isotropy subgroup of x. Moreover, the dimension of the corresponding irreducible representation is $\dim(V)|G_1|/|(G_1)_x|$. Thus, the one-dimensional representations of $H(G, G_1, G_2)$ are indexed by pairs (V, x) where x is a fixed point of G_1 on $G_2 = G/G_1$ (i.e., $x \in N_G(G_1)/G_1$), and V is a one-dimensional representation of G_1. The result follows now using Theorem 2.2. ∎

3. THE EXAMPLE

By Theorem 2.3, in order to construct an example of a biperfect semisimple Hopf algebra, it remains to find a finite group G which admits an exact factorization $G = G_1 G_2$, where G_1, G_2 are self-normalizing perfect subgroups of G. Amazingly the Mathieu group $G := M_{24}$ of degree 24 provides such an example! Once the example is found, it is not hard to verify. Still for the reader's convenience we will give a complete argument below.

We suspect that not only is M_{24} the smallest example but it may be the only finite simple group with a factorization with all the needed properties.

THEOREM 3.1. *The group G contains a subgroup $G_1 \cong PSL(2, 23)$, and a subgroup $G_2 \cong (\mathbb{Z}_2)^4 \rtimes A_7$ where A_7 acts on $(\mathbb{Z}_2)^4$ via the embedding $A_7 \subset A_8 = SL(4, 2) = \mathrm{Aut}((\mathbb{Z}_2)^4)$. These subgroups are perfect self-normalizing and G admits an exact factorization $G = G_1 G_2$. In particular, $H(G, G_1, G_2)$ is biperfect.*

Proof. The order of G is $2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$, and G has a transitive permutation representation of degree 24 with point stabilizer $C := M_{23}$. It is known (see [AT]) that G contains a maximal subgroup $G_1 \cong PSL(2, 23)$ (the elements of $PSL(2, 23)$ are regarded as fractional linear transformations on the projective line $\mathbb{P}^1(F_{23})$) and that G_1 is transitive in the degree 24 representation. Thus, $G = G_1 C$.

LEMMA 1. *G_1 is perfect and self-normalizing.*

Proof. This is clear, since G_1 is maximal and not normal in the simple group G. ∎

It is known that C contains a maximal subgroup $G_2 \cong (\mathbb{Z}_2)^4 \rtimes A_7$ (see [AT]).

LEMMA 2. *G_2 is perfect.*

Proof. Note that $E := (\mathbb{Z}_2)^4$ is the unique minimal normal subgroup of G_2, E is noncentral, and G_2/E is simple. Thus, G_2 is perfect. ∎

LEMMA 3. *G_2 is self-normalizing.*

Proof. We note that G_2 is a subgroup of $F := E \rtimes A_8$ which is a maximal subgroup of G (see [AT]). Since E is the unique minimal normal subgroup of G_2, it follows that $N_G(G_2)$ is contained in $N_G(E)$. Since F normalizes E and is maximal, $F = N_G(E)$. Since G_2 is a maximal subgroup of F and is not normal in F, G_2 is self-normalizing. ∎

LEMMA 4. *$G = G_1G_2$ is an exact factorization.*

Proof. Since $|G| = |G_1||G_2|$, it suffices to show that $G = G_1G_2$. Let T be the normalizer of a Sylow 23-subgroup. So T has order $11 \cdot 23$ (T is at least this large since this is the normalizer of a Sylow 23-subgroup of G_1; on the other hand, this is also the normalizer of a Sylow 23-subgroup in A_{24} which contains G). The subgroup of order 23 has a unique fixed point which must be T-invariant in the degree 24 permutation representation of G. Moreover, T is also contained in some conjugate of G_1 (since the normalizer of a Sylow 23-subgroup of G_1 has the same form and all Sylow 23-subgroups are conjugate). So replacing G_1 and C by conjugates, we may assume that $T \leq G_1 \cap C$.

Since T and G_2 have relatively prime orders and $|C| = |T||G_2|$, it follows that $C = TG_2$. Thus, $G = G_1C = G_1TG_2 = G_1G_2$, as required.

Finally, by Theorem 2.3, $H(G, G_1, G_2)$ is biperfect. ∎

Remark 3.2. One characterization of the Mathieu group is that it is the automorphism group of a certain Steiner system. The group G_2 is the stabilizer of a flag in the Steiner system.

Remark 3.3. Given an example of a biperfect Hopf algebra H, one has also an example of a self-dual biperfect Hopf algebra. Indeed, $H \otimes H^*$ is such a Hopf algebra.

QUESTION 3.4. (1) *Does there exist a biperfect Hopf algebra which is not semisimple? Which has odd dimension?*

(2) *Do there exist biperfect Hopf algebras of dimension less than $|M_{24}|$?*

(3) *Does there exist a nonzero finite-dimensional biperfect Lie bialgebra (see, e.g.,* [*ES*, *Sects*. 2, 3] *for the theory of Lie bialgebras), i.e., a Lie bialgebra* $\mathbf{g}$ *such that both* $\mathbf{g}$ *and* $\mathbf{g}^*$ *are perfect Lie algebras?*

(4) *Does there exist a nonzero quasitriangular Lie bialgebra for which the cocommutator is injective?*

Remark 3.5. (1) A non-semisimple biperfect Hopf algebra H must have even dimension, since $S^4 = I$ and $\mathrm{tr}(S^2) = 0$. Note that an odd-dimensional biperfect Hopf algebra cannot be of the form $H(G, G_1, G_2)$ since groups of odd order are solvable.

(2) A positive answer to question (3) implies a positive answer to question (4) by the double construction.

(3) Questions (3) and (4) are equivalent to the same questions about QUE algebras, by the results of [EK].

REFERENCES

[AT] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "Atlas of Finite Groups," Clarendon, Oxford, 1985.

[EG] P. Etingof and S. Gelaki, The classification of triangular semisimple and cosemisimple Hopf algebras over an algebraically closed field, *Internat. Math. Res. Notices* **5** (2000), 223–234.

[EK] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras, II, *Selecta Math.* **4** (1998), 213–231.

[ES] P. Etingof and O. Schiffmann, "Lectures on Quantum Groups, Lectures in Mathematical Physics," International Press, Boston, MA, 1998.

[K] G. I. Kac, Extensions of groups to ring groups, *Math. USSR. Sb.* **5**, No. 3 (1968).

[M] S. Majid, Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction, *J. Algebra* **130** (1990), 17–64.

[R] D. E. Radford, Minimal quasitriangular Hopf algebras, *J. Algebra* **158** (1993), 285–315.

[T] M. Takeuchi, Matched pairs of groups and bismash products of Hopf algebras, *Comm. Algebra* **9**, No. 8 (1981), 841–882.