



Conformal quantum mechanics as the CFT_1 dual to AdS_2

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ABSTRACT

A $(0+1)$ -dimensional candidate theory for the CFT_1 dual to AdS_2 is discussed. The quantum mechanical system does not have a ground state that is invariant under the three generators of the conformal group. Nevertheless, we show that there are operators in the theory that are not primary, but whose “non-primary character” conspires with the “non-invariance of the vacuum” to give precisely the correlation functions in a conformally invariant theory.

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1. Introduction

An elementary realization of the AdS/CFT correspondence [1] proceeds as follows. Consider a scalar field Φ on a $(d+1)$ -dimensional AdS space [in Poincaré coordinates (z, x^i) , $i = 1, \dots, d$, with boundary at $z = 0$]. The field is governed by the action $I(\Phi)$, which leads to equations of motion for Φ on the background AdS space. When the equations are solved, subject to the boundary condition $\Phi(z, \mathbf{x}) \rightarrow \phi(\mathbf{x})$ as $z \rightarrow 0$, the solution provides us with a functional of (the unspecified) ϕ : $\Phi(\phi)$. The action, evaluated on this particular solution results in a further functional of ϕ : $I(\Phi)|_{\phi=\phi(\phi)} \equiv W(\phi)$. In the AdS/CFT correspondence the functional $W(\phi)$ is identified with the generating functional in d dimensions for the n -point correlation functions of the operators $O(\mathbf{x})$ sourced by $\phi(\mathbf{x})$,

$$\langle O(\mathbf{x}_1) \cdots O(\mathbf{x}_n) \rangle = \frac{\delta}{\delta \phi(\mathbf{x}_1)} \cdots \frac{\delta}{\delta \phi(\mathbf{x}_n)} W(\phi)|_{\phi=0}. \quad (1.1)$$

The form of these operators and the theory governing them remain unknown. But the d -dimensional dynamics and the averaging state $\langle \cdots \rangle$ are taken to be conformally invariant. (Boldface coordinates refer to d -dimensional space–time.)

In a simple application of this procedure one finds a 2-point function [2],

$$G_2(\mathbf{x}, \mathbf{y}) \equiv \langle O(\mathbf{x}) O(\mathbf{y}) \rangle \propto \frac{1}{|\mathbf{x} - \mathbf{y}|^{2\Delta}} \quad (1.2)$$

and a 3-point function [2],

$$G_3(\mathbf{w}, \mathbf{x}, \mathbf{y}) \equiv \langle O_1(\mathbf{w}) O_2(\mathbf{x}) O_3(\mathbf{y}) \rangle \propto \frac{1}{|\mathbf{w} - \mathbf{x}|^{\Delta_1 + \Delta_2 - \Delta_3} |\mathbf{x} - \mathbf{y}|^{\Delta_2 + \Delta_3 - \Delta_1} |\mathbf{y} - \mathbf{w}|^{\Delta_3 + \Delta_1 - \Delta_2}}. \quad (1.3)$$

These expressions are consistent with the putative conformal invariance, where the operators O_i are conformal primaries carrying dimension Δ_i , while the state in which the correlations are taken is a conformally invariant “vacuum”.

The above development can be carried out for any dimension, but “the best understood... of AdS/CFT dualities is the case AdS_3/CFT_2 largely because the conformal group is infinite-dimensional (in two dimensions) and greatly constrains the dynamics... In lower dimensions—namely the AdS_2 case... very little is understood” [3]. Our goal is to describe in greater detail some features of the AdS_2/CFT_1 duality. In string theory AdS_2/CFT_1 is interesting because all known black holes have an AdS_2 factor in their horizon geometry ($AdS_2 \times K$). However, in our investigation the AdS_2 geometry stands alone and no reference is made to strings or black holes.

Here we specifically inquire whether the results (1.2) and (1.3) for the 2- and 3-point functions can arise in a conformal quantum theory defined on a 1-dimensional base space, i.e. time. Thus we work with quantum mechanics of a particle on a half-line subject to an inverse square interaction potential. The scale invariance of this model was identified in Ref. [4], and its properties were thoroughly analyzed by de Alfaro, Fubini and Furlan (dAFF) [5]. [Other conformally (= $SO(2, 1)$) invariant quantum mechanical models involve multicomponent variables [6], singular potentials [7] and/or

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various magnetic velocity-dependent interactions [8], but they offer no further insights.] Our arguments rely on the underlying $SO(2, 1)$ group structure, not on the specific dynamics.

A challenge that we face in studying the conformally invariant quantum mechanics is that in its Hilbert space there is no invariant vacuum state that is annihilated by all the generators of the $SO(2, 1)$ group. We show, however, that this does not pose an obstacle to defining correlation functions of the form (1.2) and (1.3), provided one identifies the relevant state and operators in the correlation functions. We shall present two equivalent formulations of such states and operators, which give rise to correlation functions obeying the constraints of conformal symmetry.

This Letter is organized as follows. In Section 2 we review the symmetry properties of AdS_2 and CFT_1 and introduce the conformal invariant quantum mechanical model possessing $SO(2, 1)$ symmetry studied by dAFF. In Section 3 we discuss how states and operators, though not transforming according to the conformal symmetry, combine to give rise to conformally invariant correlation functions. In Section 4 we formulate an operator-state correspondence for the CFT_1 and discuss how it can account for correlation functions with conformal scaling behavior. Details of some calculations are presented in Appendix A.

2. Review

We begin by reviewing needed formulas. The AdS_2 (Euclidean) line interval reads

$$ds^2 = \frac{1}{z^2}(dz^2 + dt^2). \quad (2.1)$$

(The d -dimensional “ \mathbf{x} ” collapses to the 1-dimensional “ t .”) Killing vectors are conveniently presented with complex coordinates $x = t + iz$,

$$k^{(n)} = x^{(n-1)} \frac{\partial}{\partial x} + (x^*)^{(n-1)} \frac{\partial}{\partial x^*}, \quad n = 1, 2, 3. \quad (2.2)$$

They satisfy the $SO(2, 1)$ algebra,

$$[k^{(m)}, k^{(n)}] = (n - m)k^{(m+n-2)}. \quad (2.3)$$

The same algebra can be canonically realized in conformal quantum mechanics with operators H , D and K , which also follow the $SO(2, 1)$ commutators,

$$\begin{aligned} i[D, H] &= H, \\ i[D, K] &= -K, \\ i[K, H] &= 2D \end{aligned} \quad (2.4)$$

or in the Cartan basis

$$\begin{aligned} R &\equiv \frac{1}{2} \left(\frac{K}{a} + aH \right), & L_{\pm} &\equiv \frac{1}{2} \left(\frac{K}{a} - aH \right) \pm iD, \\ [R, L_{\pm}] &= \pm L_{\pm}, & [L_-, L_+] &= 2R. \end{aligned} \quad (2.5)$$

(The parameter “ a ”, with time dimensionality, is introduced for dimensional balance.) The coincidence between the $SO(2, 1)$ isometry of AdS_2 and the $SO(2, 1)$ symmetry of a conformal quantum system is the basis of the AdS_2/CFT_1 correspondence ($R \sim k^{(2)}$, $L_+ \sim k^{(3)}$, $L_- \sim k^{(1)}$).

R can be taken to be a positive operator. It generates a compact subgroup. According to representation theory for $SO(2, 1)$ the spectrum of R is discrete,

$$\begin{aligned} R|n\rangle &= r_n|n\rangle, \\ r_n &= r_0 + n, \quad n = 0, 1, \dots, r_0 > 0, \\ \langle n'|n\rangle &= \delta_{n',n}. \end{aligned} \quad (2.6)$$

Ladder operators L_{\pm} act as

$$L_{\pm}|n\rangle = \sqrt{r_n(r_n \pm 1) - r_0(r_0 - 1)}|n \pm 1\rangle. \quad (2.7)$$

Eq. (2.7) implies that

$$|n\rangle = \sqrt{\frac{\Gamma(2r_0)}{n!\Gamma(2r_0 + n)}}(L_+)^n|0\rangle. \quad (2.8)$$

The r_0 eigenvalue of the lowest state—the R “vacuum” $|0\rangle$ —is connected to the Casimir invariant \mathcal{C} ,

$$\begin{aligned} \mathcal{C} &\equiv \frac{1}{2}(HK + KH) - D^2 = R^2 - L_+L_-, \\ \mathcal{C}|n\rangle &= r_0(r_0 - 1)|n\rangle. \end{aligned} \quad (2.9)$$

The above $SO(2, 1)$ structure is realized by dAFF in a canonical model, with

$$\begin{aligned} H &= \frac{1}{2} \left(p^2 + \frac{g}{q^2} \right), \quad g > 0, \\ D &= tH - \frac{1}{4}(pq + qp), \\ K &= -t^2H + 2tD + \frac{1}{2}q^2, \\ i[p(t), q(t)] &= 1, \\ \mathcal{C} &= \frac{g}{4} - \frac{3}{16}, \\ r_0 &= \frac{1}{2} \left(1 + \sqrt{g + \frac{1}{4}} \right). \end{aligned} \quad (2.10)$$

H , D and K are time-independent; q has scale dimension $-1/2$ and is a conformal primary,

$$\begin{aligned} i[H, q(t)] &= \frac{d}{dt}q(t), \\ i[D, q(t)] &= t \frac{d}{dt}q(t) - \frac{1}{2}q(t), \\ i[K, q(t)] &= t^2 \frac{d}{dt}q(t) - tq(t). \end{aligned} \quad (2.11)$$

In fact we shall mainly utilize the group structure summarized in (2.4)–(2.9). The specific realization (2.10) and (2.11) plays a secondary role.

3. Puzzle

Since dAFF present an explicit $SO(2, 1)$ -invariant CFT_1 model, we inquire whether states and operators in that model reproduce the conformally invariant correlation functions determined by the AdS_2 correspondence.

Now we can state our puzzle about the AdS_2/CFT_1 duality. In the dAFF model $SO(2, 1)$ invariant states are not normalizable, so forming diagonal matrix elements is problematical. Moreover, no state is invariant under all three $SO(2, 1)$ transformations. On the other hand, normalizable, non-invariant states interfere with derivations of conformal constraints. Furthermore, the AdS_2 calculation indicates that the averaged operators carry (unspecified) arbitrary dimensions, while the canonical model involves operators with fixed rational dimensions.

Nevertheless it is intriguing that dAFF find amplitudes that match precisely the forms found in the AdS_2 calculation. These are constructed in dAFF as follows. States “ $\langle t|$ ” are introduced on which the action of the $SO(2, 1)$ generators is realized with “ t ”-derivation,

$$\begin{aligned}
\langle t|H &= i \frac{d}{dt} \langle t|, \\
\langle t|D &= i \left(t \frac{d}{dt} + r_0 \right) \langle t|, \\
\langle t|K &= i \left(t^2 \frac{d}{dt} + 2r_0 t \right) \langle t|.
\end{aligned} \tag{3.1}$$

These “ t ”-based generators satisfy the Lie algebra (2.4) and lead to the correct Casimir (2.9). Explicit formulas are found for $\langle t|n\rangle$ by solving the equation

$$\langle t|R|n\rangle = r_n \langle t|n\rangle = \frac{i}{2} \left[\left(a + \frac{t^2}{a} \right) \frac{d}{dt} + 2r_0 \frac{t}{a} \right] \langle t|n\rangle, \tag{3.2}$$

$$\langle t|n\rangle \equiv \beta_n(t) (-1)^n \left[\frac{\Gamma(2r_0 + n)}{n!} \right]^{1/2} \left(\frac{a - it}{a + it} \right)^{r_n} \frac{1}{(1 + \frac{t^2}{a^2})^{r_0}}. \tag{3.3}$$

With these dAFF find

$$F_2(t_1, t_2) = \sum_n \beta_n(t_1) \beta_n^*(t_2) \equiv \langle t_1|t_2\rangle \propto \frac{1}{|t_1 - t_2|^{2r_0}}, \tag{3.4}$$

$$F_3(t_1, t, t_2) \equiv \langle t_1|B(t)|t_2\rangle \propto \frac{1}{|t - t_1|^\delta |t_2 - t|^\delta |t_1 - t_2|^{-\delta + 2r_0}}. \tag{3.5}$$

Here B is an unspecified primary with dimension δ .

These correlators are precisely of the form (1.2) and (1.3), obtained in the AdS₂ calculation. Upon comparing with (3.4) and (3.5), we see that $G_2 \sim F_2$ is the expected value of two operators, each with the effective dimension r_0 , while $G_3 \sim F_3$ involves these same two operators and a third operator B with dimension δ .

To resolve our puzzlement it remains to identify within CFT₁ the averaging states and the two operators that form G_2 . To this end we note that $\langle t|n\rangle = \beta_n(t)$ implies

$$\sum_n |n\rangle \langle n|t\rangle = |t\rangle = \sum_n \beta_n^*(t) |n\rangle. \tag{3.6}$$

With formulas (2.8) for $|n\rangle$ and (3.2) for $\beta_n(t)$, the summation may be performed and we find

$$\begin{aligned}
|t\rangle &= O(t)|0\rangle, \\
O(t) &= N(t) \exp(-\omega(t)L_+), \\
N(t) &= [\Gamma(2r_0)]^{1/2} \left(\frac{\omega(t) + 1}{2} \right)^{2r_0}, \\
\omega(t) &= \frac{a + it}{a - it} = e^{i\theta} \quad \text{with } t = a \tan \theta/2.
\end{aligned} \tag{3.7}$$

Thus

$$\begin{aligned}
F_2 &\sim G_2 \sim \langle 0|O^\dagger(t_1)O(t_2)|0\rangle, \\
F_3 &\sim G_3 \sim \langle 0|O^\dagger(t_1)B(t)O(t_2)|0\rangle.
\end{aligned} \tag{3.8}$$

We conclude that the averaging state is the R “vacuum” $|0\rangle$ and the operators are $O(t)$ and $O^\dagger(t)$. Note that as anticipated the averaging state is not conformally invariant. Also the operators O and O^\dagger do not respond to conformal transformation in the expected way; they are not primaries. But these “defects” conspire to validate the dAFF realization of the SO(2, 1) generators through the “ t ”-derivation, Eq. (3.1). For example, selecting D , we form

$$\begin{aligned}
D|t\rangle &= \left(\frac{L_+ - L_-}{2i} \right) |t\rangle \\
&= \left(\frac{L_+ - L_-}{2i} \right) N(t) \exp(-\omega(t)L_+) |0\rangle \\
&= \frac{i}{2} N(t) \left(\frac{d}{d\omega} e^{-\omega(t)L_+} + [L_-, e^{-\omega(t)L_+}] \right) |0\rangle.
\end{aligned} \tag{3.9}$$

The commutator in (3.9) gives $e^{-\omega(t)L_+} (-2\omega R + \omega^2 L_+)$. Thus

$$\begin{aligned}
D|t\rangle &= \frac{i}{2} N \left(-2\omega r_0 + (1 - \omega^2) \frac{d}{d\omega} \right) \frac{1}{N} |t\rangle \\
&= -i\omega r_0 |t\rangle + \frac{i}{2} (1 - \omega^2) \left(-\frac{1}{N} \frac{d}{d\omega} N + \frac{d}{d\omega} \right) |t\rangle \\
&= -i \left(r_0 + t \frac{d}{dt} \right) |t\rangle
\end{aligned} \tag{3.10}$$

in agreement with (3.1). In the last line we used the chain rule $t \frac{d}{dt} = \frac{1}{2} (1 - \omega^2) \frac{d}{d\omega}$. Similar arguments confirm (3.1) for H and K . In a sense $e^{-\omega(t)L_+}$, when acting on $|0\rangle$, behaves as a primary operator with dimension r_0 .

We now demonstrate how conformal constraints arise, even though the averaging state is not invariant and the operators do not transform simply.

Consider the expectation value of the commutator with Q : $\langle 0|[Q, O^\dagger(t_1)O(t_2)]|0\rangle$, where Q is any conformal generator. The following equality holds,

$$\begin{aligned}
\langle 0|Q O^\dagger(t_1)O(t_2)|0\rangle &- \langle 0|O^\dagger(t_1)O(t_2)Q|0\rangle \\
&= \langle 0|[Q, O^\dagger(t_1)]O(t_2)|0\rangle + \langle 0|O^\dagger(t_1)[Q, O(t_2)]|0\rangle.
\end{aligned} \tag{3.11}$$

When the vacuum is invariant the left side vanishes, because an invariant vacuum is annihilated by Q . The right side involves the variations of $O^\dagger(t_1)$ and $O(t_2)$. Thus one would conclude that the conformal variation of the correlation function vanishes. For us neither is true. The averaging state is not annihilated by Q , which fails to transform $O^\dagger(t_1)$ and $O(t_2)$ properly. But the two defects cancel against each other, thereby establishing the conventional result. This may also be seen by moving in (3.11) the left side to the right and canceling it against the same terms on the right. This leaves the obvious identity

$$0 = \langle 0|O^\dagger(t_1)Q O(t_2)|0\rangle - \langle 0|O^\dagger(t_1)Q O(t_2)|0\rangle. \tag{3.12}$$

In order to obtain the invariance constraint using our CFT₁ results, we let Q act on the left bra in the first term, and on the right ket in the second. With (3.1) this produces the invariance constraint.

The state $|t\rangle = N(t)e^{-\omega(t)L_+}|0\rangle$ is like a coherent state, but not quite: it is not an eigenstate of L_- but of $L_- + \omega R$,

$$(L_- + \omega R)|t\rangle = -r_0 \omega |t\rangle. \tag{3.13}$$

[A conventional coherent state may also be constructed [9],

$$\begin{aligned}
|\lambda\rangle &\equiv [\Gamma(2r_0)]^{1/2} \sum_n \frac{\lambda^n}{[n! \Gamma(2r_0 + n)]^{1/2}} |n\rangle \\
&= \Gamma(2r_0) \sum_n \frac{\lambda^n}{n! \Gamma(2r_0 + n)} (L_+)^n |0\rangle \\
&\text{with } L_- |\lambda\rangle = \lambda |\lambda\rangle.
\end{aligned} \tag{3.14}$$

We have used (2.8). The sum may be performed, but the result yielding a modified Bessel function with argument $2\sqrt{\lambda}L_+$ is not illuminating.]

An aspect of our construction is noteworthy. Consider the state $|t\rangle$ at $t = 0$, where $\omega = 1$, and form

$$|\Psi\rangle = e^{-Ha}|t=0\rangle = e^{-Ha}e^{-L+}|0\rangle. \quad (3.15)$$

In fact $|\Psi\rangle$ is proportional to the R “vacuum” $|0\rangle$. To prove this, act on (3.15) with R and use

$$\begin{aligned} Re^{-Ha} &= e^{-Ha} \left(\frac{K}{2a} + iD \right), \\ \left(\frac{K}{2a} + iD \right) e^{-L+} &= e^{-L+} \left(R - \frac{1}{4}L_- \right). \end{aligned} \quad (3.16)$$

It follows that

$$\begin{aligned} R|\Psi\rangle &= e^{-Ha}e^{-L+} \left(R - \frac{1}{4}L_- \right) |0\rangle = r_0 e^{-Ha}e^{-L+} |0\rangle \\ &= r_0 |\Psi\rangle, \end{aligned} \quad (3.17)$$

which establishes that $|\Psi\rangle$ is proportional to $|0\rangle$. In the present context, this is an example of the operator-state correspondence: [10] the operator e^{-L+} with effective scale dimension r_0 corresponds to the eigenstate of R with lowest eigenvalue r_0 .

4. Operator-state correspondence with neither an invariant vacuum nor a primary operator

In dimensions $d \geq 2$, CFT is a quantum field theory and one usually assumes that a normalized and invariant vacuum state exists. (This is also true in second quantized quantum mechanics.) Normal ordering ensures that group generators annihilate the vacuum. In other words for a field theory we are dealing with a Fock space built on an empty no-particle vacuum. However quantum mechanics resides in a Hilbert space, which is a fixed number subspace of the Fock space. This prevents us from finding a normalized $SO(2, 1)$ vacuum state $|\Omega\rangle$ that satisfies

$$H|\Omega\rangle = K|\Omega\rangle = D|\Omega\rangle = 0. \quad (4.1)$$

A simple way to see that (4.1) cannot be satisfied is by applying the Casimir defined in Eq. (2.9): $C|\Omega\rangle = r_0(r_0 - 1)|\Omega\rangle \neq 0$ generically.

Now the definition of a primary field $\mathcal{O}_\Delta(t)$ with scaling dimension Δ is given by the commutation relations

$$\begin{aligned} i[H, \mathcal{O}_\Delta(0)] &= \dot{\mathcal{O}}_\Delta(0), \\ i[D, \mathcal{O}_\Delta(0)] &= \Delta \mathcal{O}_\Delta(0), \\ i[K, \mathcal{O}_\Delta(0)] &= 0, \end{aligned} \quad (4.2)$$

where the dot denotes derivative with respect to time. It would follow from Eqs. (4.1) and (4.2) that

$$\left(\frac{K}{2a} + iD \right) \mathcal{O}_\Delta(0)|\Omega\rangle = \Delta \mathcal{O}_\Delta(0)|\Omega\rangle. \quad (4.3)$$

We show in Appendix A that there is an operator $\mathcal{O}_\Delta(0)$ and a non-normalizable “state” $|\Omega\rangle$ that together conspire to satisfy Eq. (4.3), even though $\mathcal{O}_\Delta(0)$ fails to satisfy (4.2), and $|\Omega\rangle$ is annihilated only by H and not by D and K . This allows us to define the state $|\mathcal{O}_\Delta\rangle \equiv \mathcal{O}_\Delta(0)|\Omega\rangle$ and we can proceed as usual to obtain the correlation functions of the CFT₁. To do so, one deduces with (4.3) that

$$\begin{aligned} Re^{-Ha}|\mathcal{O}_\Delta\rangle &= e^{-Ha} \left(\frac{K}{2a} + iD \right) |\mathcal{O}_\Delta\rangle \\ &= \Delta e^{-Ha}|\mathcal{O}_\Delta\rangle, \end{aligned} \quad (4.4)$$

from which it follows that

$$e^{-Ha}|\mathcal{O}_{r_0}\rangle \propto |0\rangle. \quad (4.5)$$

We now show that correlation functions of fields computed in the state $|\Omega\rangle$ have the same scaling behavior as the matrix elements (3.4) and (3.5) computed in the t -representation of dAFF. Considering explicitly the case of the 2-point correlation function, we can define

$$\begin{aligned} G_2(t_1, t_2) &= \langle \Omega | \mathcal{O}_{r_0}(t_1) \mathcal{O}_{r_0}(t_2) | \Omega \rangle \\ &= \langle \Omega | \mathcal{O}_{r_0}(0) e^{-i(t_1-t_2)H} \mathcal{O}_{r_0}(0) | \Omega \rangle \\ &= \langle \Omega | \mathcal{O}_{r_0}(0) e^{-Ha} e^{[2a-i(t_1-t_2)]H} e^{-Ha} \mathcal{O}_{r_0}(0) | \Omega \rangle \\ &= \langle 0 | e^{[2a-i(t_1-t_2)]H} | 0 \rangle \end{aligned} \quad (4.6)$$

in which the time translation invariance of $G_2(t_1, t_2) = G_2(t_1 - t_2) \equiv G_2(t)$ is made explicit in the second line of (4.6) due to H annihilating $|\Omega\rangle$. It is then straightforward to show, by differentiation of the last line of (4.6) with respect to time, that $G_2(t)$ satisfies

$$\left(t \frac{\partial}{\partial t} + 2r_0 \right) G_2(t) = 0. \quad (4.7)$$

Solution of (4.7) yields

$$G_2(t) \sim |t|^{-2r_0}. \quad (4.8)$$

As anticipated, the 2-point correlation function defined in (4.6) displays the same scaling behavior as (3.4). Similarly, the three point function

$$G_3(t; t_2, t_1) = \langle \Omega | \mathcal{O}_{r_0}(t_2) B(t) \mathcal{O}_{r_0}(t_1) | \Omega \rangle \quad (4.9)$$

constructed with a primary operator $B(t)$ satisfying (4.2) with scaling dimension δ , has the form

$$G_3(t; t_2, t_1) \sim \frac{1}{|t-t_1|^\delta |t_2-t|^\delta |t_1-t_2|^{-\delta+2r_0}}. \quad (4.10)$$

This scaling behavior is the same as (3.5). From the form of the correlation functions (4.8) and (4.10) one learns that, despite the fact that neither the state $|\Omega\rangle$ is annihilated by all the generators of the conformal group nor the operator \mathcal{O}_{r_0} is primary, still the combination $\mathcal{O}_{r_0}|\Omega\rangle$ makes the correlation functions to have the scaling behaviors that are expected as if they were built out of a fully invariant vacuum state and a primary operator.

5. Conclusion

Motivated by the conjectured AdS_{d+1}/CFT_d correspondence, and by the least understood but relevant case of the AdS_2/CFT_1 correspondence, we have studied the properties of conformally invariant quantum mechanics. Given the fact that in the CFT₁ one has to deal with a Hilbert space (instead of a Fock space when $d \geq 2$), we find that the usual operator-state correspondence needs to be modified. However, such a modification still allows one to build up correlation functions that behave as if they were constructed out of a fully invariant vacuum (i.e., annihilated by all the group generators) and primary operators carrying well defined scale dimension. This result is established with only implicit reference to a dynamical CFT₁ model. Rather, our derivation exploits the $SO(2, 1)$ group structure. However, we also have in hand an explicit dynamical model for the CFT side of the correspondence, so that many features can be evaluated in a controllable way. It would be interesting to construct the corresponding dual AdS_2 field theory. We leave this question to a future investigation [11].

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Appendix A

Here we clarify some of the steps related to the realization of the operator-state correspondence in the CFT₁ model defined in (2.10). Considering first Eq. (4.3) for $\langle q|\mathcal{O}_\Delta\rangle$:

$$\left[\frac{q^2}{4a} - \frac{1}{4} \left(\frac{d}{dq} q + q \frac{d}{dq} \right) \right] \langle q|\mathcal{O}_\Delta\rangle = \Delta \langle q|\mathcal{O}_\Delta\rangle, \quad (\text{A.1})$$

which gives

$$\langle q|\mathcal{O}_\Delta\rangle = e^{\frac{q^2}{4a}} q^{-\frac{1}{2}-2\Delta}. \quad (\text{A.2})$$

Now $\langle q|\Omega\rangle$ is found by using $H|\Omega\rangle = 0$ and solving

$$\langle q|H|\Omega\rangle = \frac{1}{2} \left(-\frac{d^2}{dq^2} + \frac{g}{q^2} \right) \langle q|\Omega\rangle = 0. \quad (\text{A.3})$$

The solution is non-normalizable,

$$\langle q|\Omega\rangle \sim q^{r_0}. \quad (\text{A.4})$$

Now from the definition

$$|\mathcal{O}_\Delta\rangle = \mathcal{O}_\Delta(0)|\Omega\rangle \quad (\text{A.5})$$

and assuming that the operator $\mathcal{O}_\Delta(0)$ is diagonal in the q representation, i.e.,

$$\langle q|\mathcal{O}_\Delta(0)|q'\rangle = V(q)\delta(q-q'), \quad (\text{A.6})$$

one gets, by (A.2) and (A.4)

$$V(q) = e^{\frac{q^2}{4a}} q^{-(\frac{1}{2}+2\Delta+r_0)}. \quad (\text{A.7})$$

The action of K and D on the state $|\Omega\rangle$ can be straightforwardly computed

$$\begin{aligned} \langle q|K|\Omega\rangle &= \frac{q^2}{2} \langle q|\Omega\rangle, \\ \langle q|D|\Omega\rangle &= \frac{i}{4} \left(\frac{d}{dq} q + q \frac{d}{dq} \right) \langle q|\Omega\rangle \\ &= \frac{i}{4} (2r_0 + 1) \langle q|\Omega\rangle. \end{aligned} \quad (\text{A.8})$$

Eqs. (A.8) show explicitly that $|\Omega\rangle$ is not annihilated either by K or D . Moreover, the commutation relations

$$\begin{aligned} i[K, \mathcal{O}_\Delta(0)] &= 0, \\ i[D, \mathcal{O}_\Delta(0)] &= \mathcal{O}_\Delta(0) \left[\left(\frac{1}{4} + \Delta + \frac{r_0}{2} \right) - \frac{K}{2a} \right] \end{aligned} \quad (\text{A.9})$$

make manifest that $\mathcal{O}_\Delta(0)$ is not a primary operator satisfying (4.2). Nevertheless, (A.8) and (A.9) combined yield (4.3).

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