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Topology and its Applications 159 (2012) 360–370

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

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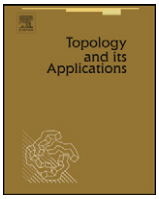

Topologically subordered rectifiable spaces and compactifications ☆

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ARTICLE INFO

Article history:
Received 14 April 2011
Received in revised form 6 August 2011
Accepted 6 October 2011

MSC:
54A25
54B05
54E20
54E35

Keywords:
Rectifiable spaces
Orderable spaces
Suborderable spaces
Remainders
Compactifications
P-spaces
Totally disconnected
Quasi-G_δ-diagonal
k-Networks
Ohio-complete

ABSTRACT

A topological space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. In this paper, we mainly discuss the rectifiable spaces which are suborderable, and show that if a rectifiable space is suborderable then it is metrizable or a totally disconnected P-space, which improves a theorem of A.V. Arhangel'skiĭ (2009) in [8]. As an application, we discuss the remainders of the Hausdorff compactifications of GO-spaces which are rectifiable, and we mainly concerned with the following statement, and under what condition Φ it is true.

Statement. Suppose that G is a non-locally compact GO-space which is rectifiable, and that $Y = bG \setminus G$ has (locally) a property-Φ. Then G and bG are separable and metrizable.

Moreover, we also consider some related matters about the remainders of the Hausdorff compactifications of rectifiable spaces.

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1. Introduction

Recall that a *topological group* G is a group G with a (Hausdorff) topology such that the product maps of $G \times G$ into G is jointly continuous and the inverse map of G onto itself associating x^{-1} with arbitrary $x \in G$ is continuous. A *paratopological group* G is a group G with a topology such that the product maps of $G \times G$ into G is jointly continuous. A topological space G is said to be a *rectifiable space* provided that there are a surjective homeomorphism $\varphi : G \times G \to G \times G$ and an element $e \in G$ such that $\pi_1 \circ \varphi = \pi_1$ and for every $x \in G$ we have $\varphi(x, x) = (x, e)$, where $\pi_1 : G \times G \to G$ is the projection to the first coordinate. If G is a rectifiable space, then φ is called a *rectification* on G. It is well known that rectifiable spaces and paratopological groups are all good generalizations of topological groups. In fact, for a topological group with the neutral element e, then it is easy to see that the map $\varphi(x, y) = (x, x^{-1}y)$ is a rectification on G. However, there exists a paratopological group which is not a rectifiable space; Sorgenfrey line [15, Example 1.2.2] is such an example. Also, the 7-dimensional sphere S_7 is rectifiable but not a topological group [33, §3]. Further, it is easy to see that paratopological groups and rectifiable spaces are all homogeneous. Recently, the study of rectifiable spaces has become an interesting topic in topological algebra, see [8,11,16,24,23].

☆ Supported by the NSFC (No. 10971185) and the Educational Department of Fujian Province (No. JA09166) of China.
E-mail address: linfucai2008@yahoo.com.cn.

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doi:10.1016/j.topol.2011.10.001

By a remainder of a Tychonoff space X we understand the subspace $bX \setminus X$ of a Hausdorff compactification bX of X.

2. Preliminaries

Let $(X, \leqslant)$ be a linearly ordered set. Then, a *linearly ordered topological space* (abbreviated LOTS) is a triple $(X, \tau(\leqslant), \leqslant)$, where $\tau(\leqslant)$ is the usual order topology (i.e., open-interval topology) of the order $\leqslant$. Moreover, a space X is a *generalized ordered space* (abbreviated GO-space) if X is a subspace of a LOTS Y, where the order of X is the one induced by the order of Y, see [22].

Recall that a space X is *orderable* (resp. *suborderable*) if X is homeomorphic to a LOTS (resp. GO-space) [26]. Thus, a space X is orderable if and only if the topology of X coincides with the order topology by some linear order of X [34]. The Sorgenfrey line [15] is a suborderable space which is not orderable. It is well known that every compact or connected subspace of a suborderable space is orderable, see [15].

If it will cause no confusion, for a GO-space (or LOTS) $(X, \tau, \leqslant)$, we shall omit "τ" or "$\leqslant$". Moreover, we shall sometimes use "LOTS" (resp. "GO-spaces") instead of "orderable spaces" (resp. "suborderable spaces").

Definition 2.1. ([7]) Let ζ and η be any families of non-empty subsets of X.

(1) The family ζ is called a *filterbase* on a space X if, whenever P_1 and P_2 are in ζ, there is a $P \in \zeta$ such that $P \subset P_1 \cap P_2$;
(2) A filterbase ζ on a space X is said to *converge to a point* $x \in X$ if every open neighborhood of x contains an element of ζ;
(3) If $x \in X$ belongs to the closure of every element of a filterbase ζ on X, we say that ζ *accumulates to* x or *a cluster point for* ζ;
(4) Two filterbases ζ and η are called to be *synchronous* if, for any $P \in \zeta$ and any $Q \in \eta$, $P \cap Q \neq \emptyset$;
(5) A space X is called *bisequential* if, for every filterbase ζ on X accumulating to a point $x \in X$, there exists a countable filterbase ξ on X converging to the same point x such that ζ and ξ are synchronous.

A filterbase $\mathscr{U}$ in a topological space X is called an *open nest* [2] if $\mathscr{U}$ consists of open subsets of X and has the following property: For any $U, V \in \mathscr{U}$ either $U \subset V$ or $V \subset U$. A space X is *open π-nested* at a point $x \in X$ [2] if there exists an open nest in X converging to x, and X is *open nested* at x [2] if there exists an open nest in X which forms a local base for X at x. Finally, a space is *open nested* if it is open nested at each of its points.

Recall that a family $\mathcal{U}$ of non-empty open sets of a space X is called a *π-base* if for each non-empty open set V of X, there exists a $U \in \mathcal{U}$ such that $U \subset V$. The *π-character* of x in X is defined by $\pi\chi(x, X) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a local π-base at x in $X\}$. The *π-character of X* is defined by $\pi\chi(X) = \sup\{\pi\chi(x, X)\colon x \in X\}$. The *character* of x in X is defined by $\chi(x, X) = \min\{|\mathcal{U}|\colon \mathcal{U}$ is a local base at x in $X\}$. The *character of X* is defined by $\chi(X) = \sup\{\chi(x, X)\colon x \in X\}$.

A space X is *countably compact* if for each countably open cover of X has a finite subcover. A space X is *locally compact* if every point of G has a compact neighborhood. A Tychonoff space X is *pseudocompact* if every real-valued continuous function on X is bounded. A space X is *P-space* if each G_δ-set of X is open in X.

Theorem 2.2. *([13,16,32]) A topological space G is rectifiable if and only if there exists $e \in G$ and two continuous maps $p : G^2 \to G$, $q : G^2 \to G$ such that for any $x \in G$, $y \in G$ the next identities hold*

$$p\big(x, q(x, y)\big) = q\big(x, p(x, y)\big) = y \quad \text{and} \quad q(x, x) = e.$$

Given a rectification φ of the space G, we may obtain the mappings p and q in Theorem 2.2 as follows. Let $p = \pi_2 \circ \varphi^{-1}$ and $q = \pi_2 \circ \varphi$. Then the mappings p and q satisfy the identities in Theorem 2.2, and both are open mappings.

Let G be a rectifiable space, and let p be the multiplication on G. Further, we sometimes write $x \cdot y$ instead of $p(x, y)$ and $A \cdot B$ instead of $p(A, B)$ for any $A, B \subset G$. Therefore, $q(x, y)$ is an element such that $x \cdot q(x, y) = y$; since $x \cdot e = x \cdot q(x, x) = x$ and $x \cdot q(x, e) = e$, it follows that e is a right neutral element for G and $q(x, e)$ is a right inverse for x. Hence a rectifiable space G is a topological algebraic system with operations p and q, a 0-ary operation e, and identities as above. It is easy to see that this algebraic system need not to satisfy the associative law about the multiplication operation p. Clearly, every topological loop is rectifiable.

In Section 3, we mainly discuss the rectifiable spaces which are suborderable, and show that if a rectifiable space is suborderable then it is metrizable or a totally disconnected P-space, which generalizes a theorem of A.V. Arhangel'skiĭ in [8]. Moreover, we discuss the orderability of rectifiable spaces which are suborderable. In Section 4, we mainly discuss the remainders of the rectifiable spaces, and in particular, we discuss the remainders of the rectifiable spaces which are suborderable.

All spaces are T_1 and regular unless stated otherwise. The notation $\mathbb{N}$ denotes the set of all positive natural numbers. The letter e denotes the neutral element of a group and the right neutral element of a rectifiable space, respectively. Readers may refer to [7,15,17] for notations and terminology not explicitly given here.

3. Suborderable rectifiable spaces

In [8], A.V. Arhangel'skiĭ proved the following theorem.

Theorem 3.1. *([8]) If G is an orderable rectifiable space, then G is metrizable or P-space.*

In this section, we shall generalize this theorem, see Theorem 3.10. Firstly, we give some lemmas.

Lemma 3.2. *If a rectifiable space G is open π-nested at some point $a \in G$ then G is open nested.*

Proof. Since G is homogeneous, we may assume that a is the right neutral element e of the rectifiable space G. Take an open nest $\mathscr{A}$ converges to e. Put $\phi = \{q(U,U)\colon U \in \mathscr{A}\}$. Since the mappings π_2 and φ are open, the mapping q is open. Moreover, it is easy to see that, for each $x \in G$, $q(x,x) = \pi_2 \circ \phi(x,x) = \pi_2(x,e) = e$. Therefore, for each $U \in \mathscr{A}$, $q(U,U)$ is an open neighborhood of e. Then ϕ is an open nest. Next, we only prove that ϕ converges to e. Indeed, let V be an open neighborhood of e in G. Since q is continuous at point (e,e), there exists open neighborhood W of e such that $q(W,W) \subset V$. Then there is a $U \in \mathscr{A}$ such that $U \subset W$ since $\mathscr{A}$ converges to e. Thus we have $q(U,U) \subset q(W,W) \subset V$, which implies that ϕ is a base for G at the point e. By the homogeneity of the space G we have G is open nested. □

A space is *biradial* [2] if for each filterbase $\mathscr{F}$ in X accumulating to a point $x \in X$, there exists a chain $\mathscr{C}$ (i.e., a collection linearly ordered by $\supseteq$) in X converging to x and synchronous with $\mathscr{F}$.

Lemma 3.3. *([2]) Every biradial space is open π-nested at every point.*

Lemma 3.4. *([2]) Let X be an open nested space and $\chi(x,X) = \kappa$ for all $x \in X$. Then for any family γ of open sets in X such that $|\gamma| < \kappa$ the set $\bigcap\gamma$ is open.*

Lemma 3.5. *([23]) Every bisequential rectifiable space is metrizable.*

Since every GO-space is biradial, we have the following theorem by Lemmas 3.2, 3.3 and 3.5.

Theorem 3.6. *If GO-space G is rectifiable, then G is metrizable or P-space.*

Proof. Since every GO-space is biradial, G is first-countable or P-space by Lemmas 3.2, 3.3 and 3.4. Therefore, G is metrizable or P-space by Lemma 3.5. □

Corollary 3.7. *If a GO-space G which is a rectifiable space, and also, is locally countably compact or is of point-countable type,*[1] *then G is metrizable.*

Let G be a rectifiable space with operations p and q as defined after Theorem 2.2. A subset $A \subset G$ is said to be p,q *stable* if $p(A,A) \subset A$ and $q(A,A) \subset A$.

Lemma 3.8. *Let G be a GO-space which is a rectifiable space. If C is the connected component contains the right neutral element of G and if C has at least two points, then C is open and p,q stable in G.*

Proof. Since G is rectifiable, the subspace C is homogeneous, and hence C is open in G by [21, Lemma 2.2]. Also, since C is the connected component contains the right neutral element of G, it easy to check that $p(C,C) \subset C$ and $q(C,C) \subset C$. Therefore, C is open and p,q stable. □

In [8], A.V. Arhangel'skiĭ proved that a connected and orderable rectifiable space is homeomorphic to the space $\mathbb{R}$ of the real numbers. He made a small mistake. His result is "a connected orderable space G is rectifiable and has more than one point, then G is homeomorphic to the space $\mathbb{R}$ of the real numbers". Therefore, we have the following lemma.

Lemma 3.9. *If a connected GO-space G is rectifiable and has more than one point, then G is homeomorphic to the space $\mathbb{R}$ of the real numbers.*

Theorem 3.10. *If a GO-space G is rectifiable, then G is metrizable or a totally disconnected*[2] *P-space.*

[1] Recall that a space X is of *point-countable type* [15] if every point $x \in X$ is contained in a compact subspace of X with a countable outer base of open neighborhoods in X.

[2] A space is *totally disconnected* if each component is one point.

Proof. Suppose that G is non-metrizable. Then it follows from Theorem 3.6 that G is a P-space. Let G be not totally disconnected. Then the connected component C containing the identity of G is open and p, q stable by Lemma 3.8. Then C is homeomorphic to the space $\mathbb{R}$ of the real numbers by Lemma 3.9. Then $G = \bigcup\{x \cdot C\colon\ x \in G\}$ since $e \in C$. Since C is connected, G is the topological sum of real lines. Moreover, since $\{x \cdot C\colon\ x \in G\}$ is a disjoint family since C is connected. Therefore, G is the topological sum of real lines. Then G is metrizable, which is a contradiction. Hence G is totally disconnected. □

Theorem 3.11. *Let a GO-space G be metrizable. If G is a rectifiable space, then G is orderable.*

Proof. If G is totally disconnected, then G is orderable by [31, Lemma 8]. Let G be not totally disconnected. Then G is the topological sum of real lines by the proof of Theorem 3.10, and therefore, G is orderable [19]. □

In [21], C. Liu, M. Sakai and Y. Tanaka proved the following theorem:

Theorem 3.12. *([21]) Let G be a topological group. Then the following* (1) *and* (2) *are equivalent. When G is non-metrizable, the following* (1), (2) *and* (3) *are equivalent.*

(1) *G is orderable*;
(2) *G is suborderable*;
(3) *G is a biradial space.*

Therefore, we have the following question:

Question 3.13. If a non-metrizable GO-space G which is a rectifiable space, is G orderable?

By Theorem 3.10, we have the following theorem.

Theorem 3.14. *Let G be a rectifiable space which is a GO-space. Then, G is metrizable if one of the following properties* (1)–(7) *holds*:

(1) *G is locally separable*;
(2) *all points of G are G_δ-sets*;
(3) *G is a quasi-k-space*[3];
(4) $t(G) \leqslant \omega$;
(5) *G is (locally) a Σ-space*[4];
(6) *some countably compact subset of G is infinite*;
(7) *some connected subset of G contains at least two points.*

Proof. Let G be non-metrizable. Then G is totally disconnected P-space by Theorem 3.10, which is obvious contradictory to (7). Also, it easily seen that every countably compact subset of G is finite which is a contradiction with (6). Moreover, if G satisfies one of properties (1)–(5), then it is easy to see that G is discrete space by [30, Proposition 1.3], which is a contradiction. Therefore, G must be metrizable if G satisfies one of the properties (1)–(7). □

Theorem 3.15. *Let G be a rectifiable space. Then G is metrizable if and only if G^ω is biradial.*

Proof. In [21, Corollary 2.7], C. Liu, M. Sakai and Y. Tanaka proved that G^ω is biradial if and only if G is bisequential. Since a bisequentially rectifiable space is metrizable by Lemma 3.5, it follows that G^ω is biradial if and only if G is metrizable. □

Theorem 3.16. *Let (G, τ) be any rectifiable space which is not totally disconnected. Then (G, τ) is orderable as a topological space if and only if (G, τ) contains an open connected and p, q stable subspace H topologically isomorphic with $\mathbb{R}$.*

Proof. *Necessity.* Suppose that (G, τ) is orderable as a topological space. Then the connected component C containing the identity of G is open and p, q stable by Lemma 3.8. Then C is homeomorphic to the space $\mathbb{R}$ of the real numbers by Lemma 3.9.

Sufficiency. It is well known that a non-trivial connected space is topological sum of real lines if and only if it is locally separable, metrizable, suborderable. Since H is an open connected subspace topologically isomorphic with $\mathbb{R}$, G is the topological sum of real lines, and hence (G, τ) is orderable [19, Theorem 9]. □

[3] A space is a *quasi-k-space* [27] if it is determined by a cover of countably compact subsets.

[4] A space X is a Σ-space, if there exist a σ-locally finite closed cover $\mathscr{F}$ in X , and a cover $\mathscr{C}$ of countably compact closed subsets in X such that, for $C \subset U$ with $C \in \mathscr{C}$ and U open in X, $C \subset F \subset U$ for some $F \in \mathscr{F}$.

Let $(X, \leqslant)$ be a linearly ordered set. A point $x \in X$ is *isolated from below* if it is not the supremum of the points strictly below it. Let α be a cardinal number. If x is not isolated from below, then x has *cofinality α from below* if x is the supremum of the set of α points strictly below it, and if α is the smallest cardinal with this property.

The concepts of "*isolated from above*" and "*cofinality from above*" are defined dually.

Lemma 3.17. *Let a rectifiable space G be orderable. If the right neutral element e of G is isolated neither from above nor from below, then its cofinality from above equals its cofinality from below.*

Proof. Let the cofinality of e from below be α, while its cofinality from above is strictly greater. Suppose that $\{U_\beta\colon\ \beta < \alpha\}$ is a system of neighborhoods of e satisfying the following conditions (1)–(3):

(1) $U_\beta = (a_\beta, b_\beta)$ for some points $a_\beta, b_\beta \in G$;
(2) $q(U_{\beta+1}, U_{\beta+1}) \subset U_\beta$ for each $\beta < \alpha$;
(3) $\lim a_\beta = e$.

Let $U = \bigcap_{\beta<\alpha} U_\beta$. Obviously, U contains an interval $[e, a]$ with $e \neq a$. Choose $b \in (e, a)$. For each $\beta < \alpha$, since $a_\beta \notin U_\beta$, we have $b \cdot a_\beta \notin U_{\beta+1}$ (if not, suppose that $b \cdot a_\beta \in U_{\beta+1}$, and hence $a_\beta = q(b, b \cdot a_\beta) \in q(U_{\beta+1}, U_{\beta+1}) \subset U_\beta$ by (2), which is a contradiction). Thus $b \cdot a_\beta \notin [e, a]$ for each $\beta < \alpha$, which implies that b is not in the closure of $\{b \cdot a_\beta\colon\ \beta < \alpha\}$. However, it follows from (3) that $b \cdot a_\beta \to b \cdot e = b$, which is a contradiction. □

Theorem 3.18. *Let a rectifiable space G be orderable. Then there is a totally ordered base for the neighborhoods of the right neutral element of G.*

Proof. If e is isolated from above or below, then the result is obvious. If e is isolated neither from above nor below, then it is easy to see that the result is true by Lemma 3.17. □

Lemma 3.19. *Let G be a rectifiable space in which the right neutral element has a totally ordered local base. Then G satisfies one of the following conditions* (1)–(2):

(1) *G is metrizable*;
(2) *G has a totally ordered local base at the right neutral consisting of clopen and p, q stable subspaces.*

Proof. Let G be non-metrizable. Suppose that $\{U_\beta\colon\ \beta < \alpha\}$ is a local base at e, where $\alpha > \omega$. Fix an arbitrary U_β. Then there exists a countable family $\{U_{\beta n}\colon\ n \in \mathbb{N}\}$ of neighborhoods at e such that $U_{\beta(n+1)} \cdot U_{\beta(n+1)} \subset U_{\beta n} \subset U_\beta$ and $q(U_{\beta(n+1)}, U_{\beta(n+1)}) \subset U_{\beta n}$ for each $n \in \mathbb{N}$. Indeed, let $U_{\beta 1} = U_\beta$. Since $U_{\beta 1}$ is an open neighborhood of e, there exists an open neighborhood $U_{\beta 2}$ of e such that $U_{\beta 2} \cdot U_{\beta 2} \subset U_{\beta 1}$ and $q(U_{\beta 2}, U_{\beta 2}) \subset U_{\beta 1}$. Now suppose for each $k \leqslant n$ has been defined well. For $k = n+1$, since $U_{\beta n}$ is an open neighborhood of e, there exists an open neighborhood $U_{\beta(n+1)}$ of e such that $U_{\beta(n+1)} \cdot U_{\beta(n+1)} \subset U_{\beta n}$ and $q(U_{\beta(n+1)}, U_{\beta(n+1)}) \subset U_{\beta n}$. By induction we have defined the countable family $\{U_{\beta n}\colon\ n \in \mathbb{N}\}$ of neighborhoods at e. Let $B_\beta = \bigcap_{n\in\mathbb{N}} U_{\beta n}$. Then B_β is open and p, q stable of G, and hence it is closed. By induction, it is easy to see that we can arrange a family of clopen and p, q stable subspaces $\{B_\gamma\colon\ \gamma \in \alpha\}$ which is totally ordered local base. □

By Theorem 3.18 and Lemma 3.19, we have the following theorem.

Theorem 3.20. *Let a rectifiable space G be orderable. Then G satisfies one of the following conditions* (1)–(2):

(1) *G is metrizable*;
(2) *G has a totally ordered local base at the right neutral consisting of clopen and p, q stable subspaces of G.*

Remark. From Theorem 3.20 it follows that any non-metrizable, orderable rectifiable space is a P-space, and thus we obtain Theorem 3.1 again.

However, the following question is still open.

Question 3.21. Let G be a compact, totally disconnected rectifiable space. Does there exist a cardinal number τ such that G, regard as a topological space, is homeomorphic with the space $\{0, 1\}^\tau$? (That is, is every compact, totally disconnected rectifiable space a topological group?)

4. Compactification of rectifiable spaces

In this section, we assume that all spaces are Tychonoff. Firstly, we give some lemmas.

Lemma 4.1. *(Henriksen and Isbell [20]) A space X is of countable type*[5] *if and only if its remainder in any (in some) compactification of X is Lindelöf.*

Lemma 4.2. *([6]) If G is a topological group, and some remainder of G is not pseudocompact, then every remainder of G is Lindelöf.*

Lemma 4.3. *Let a GO-space G be rectifiable, and Y be the remainder of some Hausdorff compactification of G. If Y is Lindelöf, then G is metrizable.*

Proof. Since Y is Lindelöf, it follows from Lemma 4.1 that G is of countable type.

Case 1: There exists some countably compact subset F of G that is infinite.

Then G is metrizable by Theorem 3.14.

Case 2: There is no countably compact subset of G that is infinite.

Since G is of countable type, there exists a compact subset F such that the right neutral element $e \in F$ and F has a countable neighborhood base. The compact subset F is finite since there does not exist any compact subset of G that is infinite. Therefore, the point e has a countable neighborhood base, and thus G is metrizable by Lemma 3.5. □

Theorem 4.4. *Let a GO-space G be rectifiable, and Y be the remainder of some Hausdorff compactification of G. If Y is of countable pseudocharacter, then one of the following conditions holds*:

(1) *G is metrizable*;
(2) *Y is countably compact.*

Proof. It follows from Lemma 4.2 that G is Lindelöf or Y is pseudocompact.

By Lemma 4.3, we may assume that Y is pseudocompact, and therefore, Y is first-countable. Indeed, fix an arbitrary point $y_0 \in Y$. Since Y is a Tychonoff space with a countable pseudocharacter, it is easy to see that there exists a sequence $\eta = \{U_n\colon n \in \mathbb{N}\}$ of open neighborhoods at y_0 in Y such that $\bigcap_{n\in\mathbb{N}} \overline{U_n}^Y = \{y_0\}$ and $\overline{U_{n+1}}^Y \subset U_n$ for each $n \in \mathbb{N}$, where each $\overline{U_n}^Y$ denotes the closure of U_n in Y. Then η is a local base at y_0 in Y. Suppose that η is not a local base at y_0 in Y. Then there exists an open neighborhood U at y_0 in Y such that $U_n \setminus \overline{U}^Y \neq \emptyset$ for each $n \in \mathbb{N}$. Then $\{U_n \setminus \overline{U}\colon n \in \mathbb{N}\}$ is a decreasing sequence of non-empty open subsets of Y, and hence it follows from [15, Theorem 3.10.23] that $\bigcap_{n\in\mathbb{N}} \overline{U_n \setminus \overline{U}^Y}^Y \neq \emptyset$. However, $y_0 \notin \bigcap_{n\in\mathbb{N}} \overline{U_n \setminus \overline{U}^Y}^Y \subset \bigcap_{n\in\mathbb{N}} \overline{U_n} = \{y_0\}$, and hence $\bigcap_{n\in\mathbb{N}} \overline{U_n \setminus \overline{U}^Y}^Y = \emptyset$, which is a contradiction.

We assume that Y is non-countably compact. It follows, by a standard argument, that G has a countable π-base at some point which is an accumulation point of some countable subset of Y. Therefore, G is metrizable. □

Note. Since Sorgenfrey line is first-countable and non-metrizable, it is suborderable space which is not a rectifiable space. Moreover, it is well known that Sorgenfrey line is not countably compact. However, there exists a Hausdorff compactification of the Sorgenfrey line such that its remainder is homeomorphic to the Sorgenfrey line, such as the two-arrows space of P.S. Alexandroff and P.S. Urysohn [15, Exercise 3.10.C].

By the proof of Theorem 4.4, it is easy to see the following proposition holds since a rectifiable space is metrizable if it is of countable π-character [16].

Proposition 4.5. *Let a non-locally compact GO-space G be rectifiable, and Y be the remainder of some Hausdorff compactification of G. If Y is of countable π-character and non-countably compact, then G is metrizable.*

Recall that the space X has a *quasi-G_δ-diagonal* provided there is a sequence $\{\mathcal{G}(n)\colon n \in \mathbb{N}\}$ of collections of open sets with property that, given distinct points $x, y \in X$, there is some n with $x \in st(x, \mathcal{G}(n)) \subset X \setminus \{y\}$. If each $\mathcal{G}(n)$ covers X, then X has a *G_δ-diagonal*. Obviously, "X has a G_δ-diagonal" implies "X has a quasi-G_δ-diagonal".

Lemma 4.6. *([24]) Let G be a non-locally compact, paracompact rectifiable space, and $Y = bG \setminus G$ have locally quasi-G_δ-diagonal. Then G and bG are separable and metrizable.*

[5] Recall that a space X is of *countable type* [15] if every compact subspace F of X is contained in a compact subspace $K \subset X$ with a countable outer base of open neighborhoods in X.

Lemma 4.7. *([12, Proposition 4.2]) If G is a GO-space which is a rectifiable space, then G is hereditarily paracompact.*

Therefore, it follows from Lemmas 4.6 and 4.7 that the following theorem is obtained.

Theorem 4.8. *Let a non-locally compact GO-space G be rectifiable, and Y be the remainder of some Hausdorff compactification of G. If Y has locally a quasi-G_δ-diagonal, then G, Y and bG are separable and metrizable.*

However, the following question is still open.

Question 4.9. Let a GO-space G be rectifiable, and Y be the remainder of some Hausdorff compactification of G. If Y is first-countable, is G metrizable?

Theorem 4.10. *Let a GO-space G be rectifiable, and Y be the remainder of some Hausdorff compactification of G. If Y is Ohio-complete,*[6] *then G is metrizable.*

Proof. Since Y is Ohio-complete, the space G is σ-compact or G is of countable type.

Case 1: There exists some countably compact subset F of G that is infinite.

Then G is metrizable by Theorem 3.14.

Case 2: No countably compact subset of G is infinite.

If G is σ-compact, G is finite, and hence G is discrete; if G is of countable type, then G is metrizable by the proof of Theorem 4.4. □

In [5], we know that each p-space is Ohio-complete, and therefore, we have the following corollary. The p-spaces are a class of generalized metric spaces [1]. It is well known that every metrizable space is a p-space, and every p-space is of countable type.

Corollary 4.11. *Let a GO-space G be rectifiable, and Y be the remainder of some Hausdorff compactification of G. If Y is a p-space, then G is separable and metrizable.*

Proof. It is easy to see by Lemma 4.1 and Theorem 4.10. □

Next, we consider the remainders with locally a point-countable k-network[7] of GO-spaces which are rectifiable. First, we give some technical lemmas.

Lemma 4.12. *([10]) Suppose that $B = X \cup Y$, where B is a compact space, and X, Y are dense nowhere compact subspaces of B. Suppose also that each compact subset of Y is contained in a compact G_δ-subset of Y. Then each locally finite (in X) family of non-empty open subsets of X is countable.*

Lemma 4.13. *Let G be a GO-space which is rectifiable. Then G is Lindelöf if and only if there exists a compactification bG such that for any compact subset $F \subset Y = bG \setminus G$ is contained in a G_δ-subset of Y.*

Proof. If G is Lindelöf, then Y is of countable type by Lemma 4.1. Therefore, we only need to show the sufficiency. It follows from Lemma 4.2 that Y is pseudocompact or Lindelöf.

Case 1: The space Y is Lindelöf.

It follows from Lemma 4.3 that G is metrizable, and hence G is a paracompact space. Therefore, the space G is Lindelöf by Lemma 4.12.

Case 2: The space Y is pseudocompact.

For each compact subset F of Y, there exists a compact subset L of Y such that $F \subset L$ and L is a G_δ-set in Y. Since Y is pseudocompact, it is well known that every compact subset L has a countably open neighborhood base, and hence Y is of countable type. Therefore G is Lindelöf by Lemma 4.1. □

[6] We call a space X is *Ohio complete* [5] if in each compactification bX of X there is a G_δ-subset Z such that $X \subset Z$ and each point $y \in Z \setminus X$ is separated from X by a G_δ-subset of Z.

[7] Let $\mathscr{P}$ be a family of subsets of a space X. The family $\mathscr{P}$ is called a *k-network* [28] if whenever K is a compact subset of X and $K \subset U \in \tau(X)$, there is a finite subfamily $\mathscr{P}' \subset \mathscr{P}$ such that $K \subset \bigcup \mathscr{P}' \subset U$.

Theorem 4.14. *Let G be a non-locally compact GO-space which is rectifiable, and $Y = bG \setminus G$ a remainder of G. Then the following conditions are equivalent*:

(1) *the space Y is of subcountable type*[8];
(2) *the space Y is of countable type.*

Proof. It is easy to see by Lemma 4.13 and Lemma 4.1. □

Theorem 4.15. *Let G be a non-locally compact GO-space which is rectifiable, and $Y = bG \setminus G$ a remainder of G. If Y is κ-perfect,*[9] *then Y is first-countable.*

Proof. It follows from Theorem 4.14 that Y is of countable type. Since every point of Y is a G_δ-point, the space Y is first-countable. □

The following lemma is an easy exercise.

Lemma 4.16. *Suppose that X is a Lindelöf space with locally a point-countable k-network. Then X has a point-countable k-network.*

Lemma 4.17. *([23]) Suppose that X is a k-space with a point-countable k-network. Then each countably compact subset of X is a compact, metrizable, G_δ-subset of X.*

Lemma 4.18. *([5]) If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*

Lemma 4.19. *([24]) Let G be a non-locally compact rectifiable space, and $Y = bG \setminus G$ have locally a point-countable base. Then G and bG are separable and metrizable.*

Theorem 4.20. *Let a non-locally compact GO-space G be rectifiable, and $Y = bG \setminus G$. If Y is a k-space with locally a point-countable k-network, then G and bG are separable and metrizable.*

Proof. It follows from Lemma 4.2 that Y is Lindelöf or pseudocompact.

Case 1: Y is Lindelöf.

Then G is metrizable by Lemma 4.3, and hence G is paracompact. Since G is a k-space with locally a point-countable k-network, it follows from Lemma 4.17 that each compact subset of Y is a G_δ-subset of Y, and hence it follows from Lemma 4.12 that G is Lindelöf. Therefore, G is separable and metrizable, and hence Y is a Lindelöf p-space by Lemma 4.18. Since Y has a point-countable k-network by Lemma 4.16, the space Y is separable and metrizable by [18]. Then G and bG are separable and metrizable by Theorem 4.8.

Case 2: Y is pseudocompact.

By the proof of [23, Theorem 6.21], we see that Y is κ-perfect, and hence Y is first-countable by Theorem 4.15. Since Y is k-space with locally a point-countable k-network, Y has locally a point-countable base by [25, Corollary 2.1.12]. It follows from Lemma 4.19 that G and bG are separable and metrizable. □

However, the following is still open.

Question 4.21. Let G be a non-locally compact rectifiable space, and $Y = bG \setminus G$. If Y is a k-space with locally a point-countable k-network, are G and bG separable and metrizable?

Finally, we shall discuss some related problems about the remainders of Hausdorff compactifications of rectifiable spaces.

Lemma 4.22. *([17]) If a Tychonoff countably compact space X is the union of a countable family of D-spaces,*[10] *then X is compact.*

Lemma 4.23. *([9]) Let ϵ be the class of all topological spaces X such that the π-character of X is countable at a dense set of points. Then each Tychonoff space Y which is the union of a finite family of spaces belonging to ϵ also belongs to ϵ.*

[8] Recall that a space X is of *subcountable type* [10] if every compact subspace F of X is contained in a compact G_δ subspace K of X.

[9] Recall that a space X is of *κ-perfect* [10] if every compact subspace F is a G_δ subspace of X.

[10] A *neighborhood assignment* for a space X is a function φ from X to the topology of X such that $x \in \varphi(x)$ for each point $x \in X$. A space X is a *D-space* [14], if for any neighborhood assignment φ for X there is a closed discrete subset D of X such that $X = \bigcup_{d\in D} \varphi(d)$.

Theorem 4.24. *Assume that G is a non-locally compact rectifiable space, and that Y is the remainder of some Hausdorff compactification of G. If the following conditions hold, then G is metrizable*:

(1) *every ω-bounded*[11] *subspace of Y is compact*;
(2) *Y is of countable π-character at a dense set of points.*

Proof. By the proof of [9, Proposition 2.4], we can see that G is of countable π-character. Since G is a rectifiable space, it follows from [16] or Lemma 3.5 that G is metrizable. □

Theorem 4.25. *Assume that G is a non-locally compact rectifiable space, and that the remainder $bG \setminus G = \bigcup_{i=1}^{n} M_i$. If the following conditions hold, then G is metrizable*:

(1) *each M_i is a hereditarily D-space*;
(2) *each M_i is of countable π-character at a dense set of points.*

Proof. By Lemma 4.22, it is easy to see that each ω-bounded subspace of the remainder Y is compact. It follows from Lemma 4.23 that the π-character of Y is countable at a dense set of points. Therefore, G is metrizable by Theorem 4.24. □

It is well known that any Tychonoff space with a σ-discrete network is a hereditarily D-space [4], and that any Tychonoff spaces with a point-countable base is a hereditarily D-space [3]. Therefore, we have the following corollary by Theorem 4.25.

Corollary 4.26. *Assume that G is a non-locally compact rectifiable space, and that the remainder $bG \setminus G = \bigcup_{i=1}^{n} M_i$. If one of the following conditions holds, then G is metrizable*:

(1) *each M_i is first-countable and has a σ-discrete network*;
(2) *each M_i is metrizable*;
(3) *each M_i has a point-countable base.*

Proposition 4.27. *([24]) Let G be a rectifiable space with point-countable type. If O is an open neighborhood of e, then there exists a compact p, q stable subspace H of countable character in G satisfying $H \subset O$.*

Lemma 4.28. *([24]) Let G be a regular rectifiable space of countable pseudocharacter. Then G has a regular G_δ-diagonal.*[12]

Theorem 4.29. *Assume that G is a Moscow rectifiable space, and that Y is a remainder of G in some compactification bG of G. Then at least one of the following conditions holds*:

(1) *the space G contains a topological copy of D^{ω_1}, where $D = \{0, 1\}$ has the discrete topology*;
(2) *the space G has a regular G_δ-diagonal*;
(3) *the compactum bG of G is the Čech–Stone compactification of the space Y, and thus G is the Čech–Stone remainder of Y.*

Proof. *Case* 1: The space G is locally compact.

Since G is locally compact, G is of point-countable type. Then there exists a compact p, q stable subspace H of countable character in G by Proposition 4.27.

Subcase 1.1: The space G is metrizable.

Obviously, G has a regular G_δ-diagonal.

Subcase 1.2: The space G is non-metrizable.

Obviously, H is non-metrizable, and therefore, it follows from [15, 3.12.12] that H contains a topological copy of D^{ω_1}. Thus the space G contains a topological copy of D^{ω_1}.

Case 2: The space G is non-locally compact.

Obviously, G is nowhere locally compact since G is a rectifiable space. Then Y is dense in bG, that is, bG is a compactification of the space Y.

[11] A space X is said to be *ω-bounded* if the closure of every countable subset of X is compact.
[12] A space X is said to have a *regular G_δ-diagonal* if the diagonal $\Delta = \{(x, x): x \in X\}$ can be represented as the intersection of the closures of a countable family of open neighborhoods of Δ in $X \times X$.

Suppose that the condition (3) does not hold. Thus there exist closed subsets A and B of Y and a real-valued continuous function f on Y such that $f(A) = 0$ and $f(B) = 1$, while some point $z \in G$ belongs to the intersection of the closures of A and B in bG. It follows from the continuity of f that we can find open subsets U and V of Y containing A and B, respectively, such that the closures of U and V in Y are disjoint. Then there exist open subsets U_1 and V_1 of bG such that $U = U_1 \cap Y$ and $V = V_1 \cap Y$. Let F be the intersection of the closures of U_1 and V_1 in bG. Obviously, F is compact. Moreover, since Y is dense in bG, we have U and V are dense in U_1 and V_1, respectively. It follows that $F \subset G$. Put $W_1 = U_1 \cap G$ and $W_2 = V_1 \cap G$. Obviously, F is the intersection of the closures of W_1 and W_2 in G. Since G is a Moscow space, the subspace F is the union of closed G_δ-subsets of G. Then G contains a non-empty compact G_δ-subset P.

Subcase 2.1: The subspace P is metrizable.

Then each point of P is a G_δ-point in G, which implies G is of countable pseudocharacter. Then G has a regular G_δ-diagonal by Lemma 4.28.

Subcase 2.2: The subspace P is non-metrizable.

Since P is a compact G_δ-subset, the subspace P is dyadic compactum by [33, Corollary 5]. Since P is non-metrizable, it follows from [15, 3.12.12] that the subspace P contains a topological copy of D^{ω_1}. □

Corollary 4.30. *Let G be a rectifiable space and $Y = bG \setminus G$. If the following conditions hold, then the compactum bG is the Čech–Stone compactification of the space Y, and thus G is the Čech–Stone remainder of Y*:

(1) *the space G has no regular G_δ-diagonal*;
(2) *the space G is of countable tightness.*

Proof. Obviously, G is nowhere locally compact, since otherwise G would be metrizable by [16, Theorem 3.4].

Since G is of countable tightness, G is Moscow by [23, Theorem 5.10]. Since the tightness of G is countable, it follows from [15, 3.12.12] that G contains no copy of D^{ω_1}. Therefore, G is the Čech–Stone remainder of Y by Theorem 4.29. □

Theorem 4.31. *Let G be a rectifiable space. If for each $y \in Y = bG \setminus G$ there exists an open neighborhood $U(y)$ of y such that every ω-bounded subset of $U(y)$ is compact, then at least one of the following conditions holds*:

(1) *G is metrizable*;
(2) *bG can be continuously mapped onto the Tychonoff cube I^{ω_1}.*

Proof. *Case* 1: The space G is locally compact.

If G is not metrizable, then G contains a topological copy of D^{ω_1} by the proof of Theorem 4.29. Since the space G is normal, the space G can be continuously mapped onto the Tychonoff cube I^{ω_1}.

Case 2: The space G is not locally compact.

Obviously, both G and Y are dense in bG. Suppose that the condition (2) doesn't hold. Then, by a theorem of Šapirovskiĭ in [29], the set A of all points $x \in bG$ such that the π-character of bG at x is countable is dense in bG. Since G is dense in bG, it can follow that the π-character of G is countable at each point of $A \cap G$.

Subcase 2(a): $A \cap G \neq \emptyset$.

Since G is a rectifiable space, it follows that G is first-countable [16], which implies that G is metrizable.

Subcase 2(b): $A \cap G = \emptyset$.

Obviously, $A \subset Y$. For each $y \in Y$, there exists an open neighborhood $U(y)$ in Y such that $y \in U(y)$ and every ω-bounded subset of $U(y)$ is compact. Obviously, $A \cap U(y)$ is dense of $U(y)$. Also, it is easy to see that $A \cap U(y)$ is ω-bounded subset for $U(y)$. Therefore, $A \cap U(y)$ is compact. Then $A \cap U(y) = U(y)$, since $A \cap U(y)$ is dense of $U(y)$. Hence Y is locally compact, a contradiction. □

Acknowledgements

First, I have to thank my PhD advisor Shou Lin for his care, guidance and help in the past many years.

Moreover, I wish to thank the reviewers for the detailed list of corrections, suggestions to the paper, and all her/his efforts in order to improve the paper.

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