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MEDIANS IN MEDIAN GRAPHS

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0. Introduction

A median of a family of vertices in a graph is any vertex whose distance-sum to that family is minimum. In the framework of metric spaces the problem of minimizing a distance-sum is often referred to as the Fermat problem. On the other hand, medians have been studied from a purely order-theoretic or combinatorial point of view (for instance, in statistics, or in Jordan's work [12] on trees). The aim of this paper is to investigate the mutual relationship of the metric and the ordinal/ combinatorial approaches to the median problem in the class of median graphs. A connected graph is a median graph if any three vertices admit a unique median (see Avann [1]). Note that trees and the covering graphs of distributive lattices are median graphs. Very little is known about medians in arbitrary graphs (cf. Slater [20]); so far, only trees (Zelinka [22], and many others) and the covering graphs of distributive lattices (Barbut [4]) have been considered. In both cases we get that (i) the medians of any family form an interval (a path in a tree, an order-theoretic interval in a distributive lattice), and (ii) medians of odd numbered families are unique (see Slater [19] for trees, and Barbut [4] for distributive lattices). These results point to the fact that (i) and (ii) must be true for any median graph.

After recalling some basic definitions and facts concerning median graphs and median semilattices (for further information, see Bandelt and Hedlíková [3]), we establish (i) and (ii) for arbitrary median graphs. Our results are based on theorems of Avann, Sholander, and Barbut. In trees medians have nice local properties (cf. [7]). Indeed, median sets are related to mass centers (Zelinka [22]) and security centers (Slater [18]). In Section 3 this is extended to median graphs.

The study of medians applies to social choice theory (see Barbut [5], and Barthélemy and Monjardet [8]). The median procedure is strongly related to the simple majority rule: the median of a family $(A_1, \dots, A_{2k+1})$ of subsets of a set X may be written as

$$\bigcup_{\substack{K\subseteq\{1,\dots,2k+1\}\\ |K|=k+1}} \bigcap_{i\in K} A_i \qquad \text{(Barbut's formula).}$$

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On the other hand, given any family π of vertices in a graph, one may consider vertices x such that for any other vertex y, a majority of vertices in π is nearer to x than to y. In general, such vertices x (called Condorcet vertices) need not exist, that is, we have a paradox of voting. For trees, however, Condorcet vertices of any family π exist and coincide with the medians of π (Wendell and McKelvey [21]). Hence in this case the lattice-theoretic (alias set-theoretic) and the metric (alias graph-theoretic) interpretations of the majority rule are equivalent. In the final section we show that this equivalence in fact extends to cubefree median graphs (and characterizes the latter).

1. Preliminaries

1.1. Graphs and posets

All graphs under consideration are simple, loopless, connected, and are not necessarily finite. Let $G=(X,E)$ be a graph with vertex set X and edge set E. A shortest path joining two vertices u and v is called a *geodesic* in G. The (*geodesic*) *distance* d is defined as usual:

$$d(u,v) = \text{length of a geodesic joining } u \text{ and } v.$$

For any two vertices u and v the following set is called an *interval* in G:

$$I(u,v)=\{t\in X \mid d(u,v)=d(u,t)+d(t,v)\}.$$

In other words, the interval $I(u,v)$ consists of all vertices on geodesics joining u and v.

Partially ordered sets (or posets, for short) are usually represented by diagrams. Recall that in a poset $(X,\leq)$ an element v is said to *cover* another element u if $u<v$ and $u<t<v$ for no element t. If $(X,\leq)$ is *discrete*, that is, there are no infinite bounded chains, then the *covering graph* $G=(X,E)$ of $(X,\leq)$ is the graph whose vertices are the elements of $(X,\leq)$ and whose edges are those pairs $\{u,v\}$, $u,v\in X$, satisfying u covers v or v covers u. Every bipartite graph G occurs as the covering graph of some poset: the *canonical order* $\leq_a$ of G, with respect to a fixed vertex a, is defined by

$$u\leq_a v \quad \text{if and only if} \quad u\in I(a,v);$$

then G is the covering graph of $(X,\leq_a)$, and for $u\leq_a v$, the order-theoretic interval $[u,v]$ coincides with $I(u,v)$. Note that a is the least element of $(X,\leq_a)$. Moreover, $(X,\leq_a)$ is a *graded* poset, that is, it admits a real-valued function h on X such that $h(v)=h(u)+1$ whenever v covers u; h is called a *rank function*. For further information on these matters we refer the reader to Duffus and Rival [9], and Mulder [15].

1.2. Medians

For any set X, let X^* denote the set $\bigcup_{n\in\mathbb{N}} X^n$ of all finite sequences of elements of X. X^* is a monoid with respect to concatenation $\circ$. In a graph $G=(X,E)$, the *distance* (alias *remoteness*) of a vertex u and a family $\pi=(a_1,\dots,a_p)\in X^*$ is given by

$$D(u,\pi)=\sum_{i=1}^{p} d(u,a_i).$$

Any vertex m that minimizes this sum is called a *median* of π:

$$D(m,\pi)=\min_{u\in X} D(u,\pi).$$

The set of all medians of π is called the *median set* of π and is denoted by $\mathrm{Med}(\pi)$.

In general, medians are not unique. Observe that for any pair $\pi=(a,b)$, the median set $\mathrm{Med}(\pi)$ coincides with the interval $I(a,b)$. Graphs in which medians of triples are unique will be the main subject of this paper: $G=(X,E)$ is a *median graph* if every family of three vertices admits just one median. Examples of median graphs are provided by trees and the covering graphs of discrete distributive lattices. Note that a graph G is median if and only if for any three vertices u, v, w the intersection $I(u,v)\cap I(v,w)\cap I(w,u)$ is a singleton, see Avann [1]. Further references concerning median graphs are Mulder and Schrijver [16], Mulder [15], Bandelt and Hedlíková [3], and Bandelt [2].

1.3. Median semilattices

For median graphs the canonical orders $\le_a$ play a crucial rôle. Recall that a *median semilattice* is a meet semilattice $(X,\le)$ such that (i) every principal ideal $\{x\mid x\le a\}$ is a distributive lattice, and (ii) any three elements have an upper bound whenever each pair of them does. Note that a median semilattice is discrete if and only if all its intervals are finite.

Proposition 1 (Avann [1]). *The covering graph of any discrete median semilattice is a median graph, and conversely, every median graph gives a discrete median semilattice with respect to any canonical order* $\le_a$.

Proposition 2 (Sholander [17]). *Every median semilattice* $(X,\le)$ *can be embedded in a distributive lattice* $(L,\le)$ *such that*

(i) *X is a lower set of $(L,\le)$, i.e. $u\le x\in X$ implies $u\in X$, and*
(ii) *each element of L is the join of finitely many elements of X.*

We refer to the above embedding $X\hookrightarrow L$ as the Sholander embedding of X in L. Note that if X is discrete, so is L. For a discrete median semilattice, the median m of three vertices u, v, w of the covering graph is given by

$$m=(u\wedge v)\vee(u\wedge w)\vee(v\wedge w).$$

Hence the Sholander embedding $X \hookrightarrow L$ preserves medians of triples. Therefore in view of Propositions 1 and 2 every median graph can be isometrically embedded in the covering graph of some discrete distributive lattice.

1.4. Barbut's theorems

In an old (1961), but recently (1980) published paper, Barbut [4] determines the median set of any family in a finite distributive lattice $(X, \leq)$. For a family $\pi = (a_1, \dots, a_p) \in X^*$ write

$$\alpha(\pi) = \bigvee_{\substack{K \subseteq \{1,\dots,p\} \\ |K| = [(p+2)/2]}} \bigwedge_{i \in K} a_i,$$

$$\beta(\pi) = \bigwedge_{\substack{K \subseteq \{1,\dots,p\} \\ |K| = [(p+2)/2]}} \bigvee_{i \in K} a_i = \bigvee_{\substack{K \subseteq \{1,\dots,p\} \\ |K| = [(p+1)/2]}} \bigwedge_{i \in K} a_i.$$

If p is an odd integer, then $\alpha(\pi) = \beta(\pi)$.

Proposition 3 (Barbut [4], Monjardet [14]). *In the covering graph of a finite distributive lattice, the median set of any family π of vertices is the (order-theoretic) interval $[\alpha(\pi), \beta(\pi)]$.*

Proposition 4 (Barbut [4]). *For a finite lattice $(X, \leq)$ the following conditions are equivalent:*

(i) *$(X, \leq)$ is distributive.*

(ii) *There exists an odd integer $p \geq 3$ such that* $\mathrm{Med}(\pi)$ *is a singleton for all $\pi \in X^p$.*

(iii) *Each odd numbered family of elements in X admits just one median.*

2. General properties of medians in median graphs

2.1. Convexity

A subset C of the vertex set X of a graph G is called (*geodesically*) *convex* if $I(u, v) \subseteq C$ for all $u, v \in C$. The *convex hull* of a set $A \subseteq X$ is the least convex subset of X containing A. In a median graph, intervals are convex, and the convex hull of any finite set is finite. For a discussion of convexity in median graphs and median semilattices, see Mulder [15], Evans [10], Bandelt and Hedlíková [3].

Lemma 1. *Let $G = (X, E)$ be a median graph. Then for any $\pi = (a_1, \dots, a_p) \in X^*$ the set* $\mathrm{Med}(\pi)$ *is contained in the convex hull of $a_1, \dots, a_p$.*

Proof. Let C be the convex hull of $a_1, \dots, a_p$, and let u be any vertex of G. Then there exists a unique vertex $u' \in C$ whose distance to u is minimal; moreover, $d(u, a) = d(u, u') + d(u', a)$ for all $a \in C$. Indeed, the element $u' = a_1 \wedge \cdots \wedge a_p$ of the semilattice $(X, \leq_u)$ belongs to C and has the required property. Therefore $D(u, \pi) = p \cdot d(u, u') + D(u', \pi)$, whence $\mathrm{Med}(\pi) \subseteq C$. □

In view of this lemma, any result concerning median sets in finite median graphs extends to the infinite case.

2.2. Extending Barbut's theorems

Using the results of Avann and Sholander we are able to generalize Propositions 3 and 4 to the case of arbitrary median graphs and discrete median semilattices. First we prove two easy lemmas.

Lemma 2. *In a median semilattice $(X, \leq)$, the join of a finite set $A \subseteq X$ exists whenever each pair of elements of A is bounded above.*

Proof. Proceed by induction on $n = |A|$. If $n \leq 3$, then the assertion is true by the definition of a median semilattice. So, let $n \geq 4$. Pick any $x, y \in A$, $x \neq y$. By assumption $x \vee y$, $\bigvee(A - \{x\})$, and $\bigvee(A - \{y\})$ exist. Therefore, x, y, and $\bigvee(A - \{x, y\})$ are pairwise bounded above, whence $\bigvee A$ exists. □

Lemma 3. *Let $G = (X, E)$ be any graph. If $\pi, \pi' \in X^*$ such that* $\mathrm{Med}(\pi) \cap \mathrm{Med}(\pi') \neq \emptyset$, *then* $\mathrm{Med}(\pi \circ \pi') = \mathrm{Med}(\pi) \cap \mathrm{Med}(\pi')$.

Proof. Let $u \in \mathrm{Med}(\pi \circ \pi')$ and $v \in \mathrm{Med}(\pi) \cap \mathrm{Med}(\pi')$. Then $D(v, \pi) \leq D(u, \pi)$, $D(v, \pi') \leq D(u, \pi')$, and $D(u, \pi \circ \pi') \leq D(v, \pi \circ \pi')$. Hence $D(u, \pi) + D(u, \pi') = D(u, \pi \circ \pi') \leq D(v, \pi \circ \pi') = D(v, \pi) + D(v, \pi') \leq D(u, \pi) + D(u, \pi')$. Therefore u and v have equal distance to π, π', and $\pi \circ \pi'$, respectively. Consequently, $\mathrm{Med}(\pi \circ \pi') = \mathrm{Med}(\pi) \cap \mathrm{Med}(\pi')$. □

This lemma holds, of course, for medians (and central points) in any metric space (X, d), cf. [8].

Proposition 5. *For a graph $G = (X, E)$ the following conditions are equivalent:*

(i) *Each odd numbered family of vertices in G admits a unique median.*

(ii) *There exists an odd integer $p = 2k + 1 \geq 3$ such that each family $\pi \in X^p$ admits a unique median.*

(iii) *G is a median graph.*

Proof. (i) *implies* (ii): Trivial.

(ii) *implies* (iii): It suffices to prove that if each member of X^{2k+1} admits a unique median, then so does each $\pi \in X^{2k-1}$. So, let $\pi \in X^{2k-1}$ and $a, b \in \mathrm{Med}(\pi)$. Then $\{a, b\} \subseteq \mathrm{Med}(\pi \circ (a, b))$ by Lemma 3, whence $a = b$ by assumption.

(iii) *implies* (i): By Proposition 1, G is the covering graph of some median semilattice $(X, \leq)$. This semilattice is embedded in a distributive lattice $(L, \leq)$ via

Sholander's embedding (Proposition 2). Let $\pi=(a_1,\dots,a_p)\in X^p$ where $p=2k+1$. The convex hull of $\{a_1,\dots,a_p\}$ in the covering graph of $(L,\le)$ is the interval $[a,b]$ where a and b are the meet and the join of all a_i, respectively. By Lemma 1, any median of π in the covering graph of $(L,\le)$ is contained in the interval $[a,b]$. By Proposition 3, the median of π in the finite distributive lattice $[a,b]$ is unique and given by the join $\alpha(\pi)$ of the elements $\bigwedge_{i\in K}a_i$ where K runs through all $(k+1)$-subsets of $\{1,\dots,p\}$. Hence $\alpha(\pi)$ is the unique median of π in the covering graph of $(L,\le)$. Since this graph contains G as an isometric subgraph, it just remains to show that $\alpha(\pi)\in X$. Now, any two $(k+1)$-subsets of $\{1,\dots,p\}$ intersect, and thus the set $\{\bigwedge_{i\in K}a_i \mid K\subseteq\{1,\dots,p\},\ |K|=k+1\}$ is pairwise bounded above in $(X,\le)$. Hence by Lemma 2, $\alpha(\pi)$ belongs to X. We conclude that every $\pi\in X^{2k+1}$ admits a unique median, completing the proof. □

By a result in [2], a discrete semilattice is median if and only if its covering graph is median. Combining this with Proposition 5, we arrive at the following generalization of Barbut's theorem (Proposition 4):

Corollary 1. *For a discrete semilattice $(X,\le)$ the following conditions are equivalent:*

(i) *$(X,\le)$ is a median semilattice.*

(ii) *There exists an odd integer $p\ge3$ such that* $\mathrm{Med}(\pi)$ *is a singleton for all $\pi\in X^p$.*

(iii) *Each odd numbered family of elments in X admits just one median.*

From the proof of Proposition 5 we obtain the following result.

Corollary 2. *Let $(X,\le)$ be a discrete median semilattice. Then for any family $\pi=(a_1,\dots,a_p)\in X^p$,*

$$m=\bigvee_{\substack{K\subseteq\{1,\dots,p\}\\ |K|=[(p+2)/2]}}\ \bigwedge_{i\in K}a_i$$

is a median of π in the covering graph of $(X,\le)$.

In order to obtain the corresponding generalization of Proposition 3, one cannot just use an arbitrary canonical order of the given median graph. In fact, $\beta(\pi)$ defined as in 1.4 may not exist in a median semilattice $(X,\le)$. Nevertheless, we still have the following theorem.

Proposition 6. *Let $G=(X,E)$ be a median graph. Then for any family $\pi=(a_1,\dots,a_p)$ of vertices the median set* $\mathrm{Med}(\pi)$ *is some interval $I(\alpha(\pi),\beta(\pi))$ in G. The elements $\alpha(\pi)$ and $\beta(\pi)$ are determined in the semilattice $(X,\le_{a_p})$ by the formulas*

$$\alpha(\pi) = \bigvee_{\substack{K \subseteq \{1, \ldots, p-1\} \\ |K| = [(p+2)/2]}} \bigwedge_{i \in K} a_i,$$

$$\beta(\pi) = \bigvee_{\substack{K \subseteq \{1, \ldots, p-1\} \\ |K| = [(p+1)/2]}} \bigwedge_{i \in K} a_i.$$

Proof. If $p \in K$, then $\bigwedge_{i \in K} a_i = a_p$ is the least element of $(X, \leq_{a_p})$. Hence all the joins may be taken over subsets K of $\{1, \ldots, p-1\}$. If p is odd, then $\alpha(\pi) = \beta(\pi)$ and $\mathrm{Med}(\pi) = \{\alpha(\pi)\}$ by Proposition 5. In order to prove the assertion for even integers p, use Proposition 3 and proceed similarly as in the proof of Proposition 5. The details are left to the reader. □

3. Local properties of medians in median graphs

3.1. Local medians

Let $G = (X, E)$ be any graph. For a vertex x of G, let $N(x)$ denote the set of all vertices y of G adjacent to x. We say that x is a *local median* of the family $\pi \in X^*$ if $D(x, \pi) \leq \min_{y \in N(x)} D(y, \pi)$. The set $\mathrm{Med}_{\mathrm{loc}}(\pi)$ of all local medians of π is called the *local median set* of π. Trivially, every median is a local median, but (in general) not vice versa. Consider, for instance, the graph of Fig. 1. Then the family π consisting of the four vertices indicated by "•" in the figure admits a local median which is not a median.

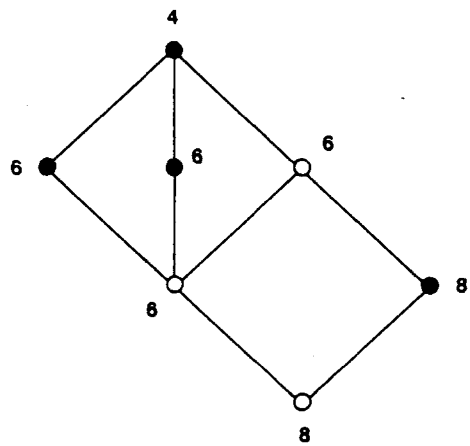

Fig. 1.

Local medians have been considered in finite trees [7] and the covering graphs of finite semimodular semilattices [6].

3.2. Local medians in discrete semimodular semilattices

A discrete (meet) semilattice $(X, \le)$ is *(lower) semimodular* if, for every $x, y \in X$, $x \vee y$ covers x and y implies x and y cover $x \wedge y$. If $(X, \le)$ is semimodular, then the geodesic distance d in the covering graph $G = (X, E)$ of $(X, \le)$ is determined by any rank function h on $(X, \le)$:

$$\begin{aligned} d(x, y) &= d(x, x \wedge y) + d(x \wedge y, y) \\ &= h(x) + h(y) - 2h(x \wedge y) \qquad \text{(cf. Monjardet [13]).} \end{aligned}$$

Given any semilattice $(X, \le)$, $\pi = (a_1, \dots, a_p) \in X^p$, and $x, y \in X$, we denote by $\pi[x, y]$ the number of indices $i = 1, \dots, p$ such that $x \wedge a_i = y \wedge a_i$.

Lemma 4 ([6]). *Let $(X, \le)$ be a discrete semimodular semilattice, and let $\pi \in X^p$. Then x is a local median of π in the covering graph G of $(X, \le)$ if and only if, for each $y \in N(x)$, $\pi[x, y] \ge \frac{1}{2}p$ whenever y covers x, and $\pi[x, y] \le \frac{1}{2}p$ whenever x covers y.*

Proof (cf. [6]). For adjacent vertices x and y of G we get

$$\begin{aligned} D(x, \pi) - D(y, \pi) &= p(h(x) - h(y)) - 2 \sum_{i=1}^{p} h(x \wedge a_i) - h(y \wedge a_i) \\ &= \begin{cases} -p - 2(\pi[x, y] - p) = p - 2\pi[x, y] & \text{if } y \text{ covers } x, \\ p - 2(p - \pi[x, y]) = 2\pi[x, y] - p & \text{if } x \text{ covers } y, \end{cases} \end{aligned}$$

whence the result. $\square$

3.3. Local medians in a median graph

In the covering graph of a modular lattice there may exist local medians which are not medians (cf. Fig. 1). However, in median graphs, this cannot occur:

Proposition 7. *Let $G = (X, E)$ be a median graph. Then for any family $\pi = (a_1, \dots, a_p)$ of vertices we get* $\mathrm{Med}(\pi) = \mathrm{Med}_{\mathrm{loc}}(\pi)$.

Proof. Let $x \in \mathrm{Med}_{\mathrm{loc}}(\pi)$, and consider the semilattice $(X, \le_x)$. Suppose that there exists a $(k+1)$-subset K of $\{1, \dots, p\}$ where $k = [\frac{1}{2}p]$ such that $x \ne \bigwedge_{i \in K} a_i$ in $(X, \le_x)$. For any $y \in N(x)$ with $y \le \bigwedge_{i \in K} a_i$ we get

$$\begin{aligned} D(y, \pi) &\le D(x, \pi) - (k+1) + (p - k - 1) \\ &= D(x, \pi) + p - 2(k+1) < D(x, \pi), \end{aligned}$$

a contradiction. Therefore $x = \bigwedge_{i \in K} a_i$ for all $(k+1)$-subsets K of $\{1, \dots, p\}$, whence by Corollary 2, $x \in \mathrm{Med}(\pi)$. $\square$

Let $G=(X,E)$ be a median graph, and let $\pi=(a_1,\dots,a_p)\in X^p$. For $x,y\in X$, put

$$\pi(x,y)=|\{i \mid d(x,a_i)<d(y,a_i)\}|.$$

Since G is bipartite, we have $\pi(x,y)+\pi(y,x)=p$. If x and y are adjacent, then for any vertex a, $d(x,a)<d(y,a)$ if and only if $x\in I(a,y)$. The *π-branch weight* of a vertex x is defined by

$$W(x,\pi)=\max_{y\in N(x)} \pi(y,x).$$

Proposition 8. *Let $G=(X,E)$ be a median graph, and let $\pi=(a_1,\dots,a_p)$ be any family of vertices of G. For any vertex x the following conditions are equivalent:*
- (i) *x is a median of π.*
- (ii) *$\pi(x,y)\geq\frac{1}{2}p$ for all $y\in N(x)$.*
- (iii) *$W(x,\pi)\leq\frac{1}{2}p$.*
- (iv) *$W(x,\pi)$ is minimal.*

Proof. Let x be any vertex of G. Consider the median semilattice $(X,\leq_x)$. Then for any $a\in X$ and $y\in N(x)$, we get that $x=a\wedge x=a\wedge y$ if and only if $x\in I(a,y)$, whence $\pi[x,y]=\pi(x,y)$ in $(X,\leq_x)$. Now, median semilattices are semimodular, and therefore, by Lemma 4 and Proposition 7, $x\in\mathrm{Med}(\pi)$ if and only if $\pi(x,y)\geq\frac{1}{2}p$ for all $y\in N(x)$. Obviously, $W(x,\pi)\leq\frac{1}{2}p$ if and only if $\pi(x,y)\geq\frac{1}{2}p$ for all $y\in N(x)$. Hence (i), (ii), and (iii) are equivalent. If x is not a median of π, then $W(x,\pi)>\frac{1}{2}p$ by (iii), whence $W(x,\pi)$ is not minimal. So, if $|\mathrm{Med}(\pi)|=1$, we are done. Otherwise, for any adjacent vertices x and y in $\mathrm{Med}(\pi)$ we get $\pi(x,y),\pi(y,x)\geq\frac{1}{2}p$ by (ii), and thus $\pi(x,y)=\pi(y,x)=\frac{1}{2}p$. Consequently, $W(x,\pi)=\frac{1}{2}p=W(y,\pi)$. Since $\mathrm{Med}(\pi)$ is an interval and thus induces a connected graph, we infer that $W(x,\pi)=\frac{1}{2}p$ for all $x\in\mathrm{Med}(\pi)$ whenever $|\mathrm{Med}(\pi)|>1$. We conclude that, in any case, the medians of π are exactly the vertices with minimal π-branch weight. □

In the case of trees Proposition 8 summarizes some well-known characterizations of the centroid. For instance, Propositions 6 and 8 imply all the results in Section 2 of Slater's paper [19]. In particular, if G is a finite tree and π is the family of all vertices of G, then (iii)$\Leftrightarrow$(iv) of Proposition 8 gives the classical result of Jordan [12], while (i)$\Leftrightarrow$(iv) gives the theorem of Zelinka [22].

4. Medians vs. Condorcet vertices

4.1. Condorcet vertices

Let $G=(X,E)$ be a graph, and let $\pi=(a_1,\dots,a_p)\in X^p$. Then a vertex x of G is called a Condorcet vertex of π if $\pi(x,y)\geq\pi(y,x)$ for all $y\in X$. A related concept, the security center of a finite graph, was studied by Slater [18]. The *security center* of π consists of all vertices x of G for which

$$S(x,\pi)=\min_{\substack{y\in X\\ x\neq y}}(\pi(x,y)-\pi(y,x))$$

is maximal. Clearly, x is a Condorcet vertex of π if and only if $S(x,\pi)\geq 0$. Hence, if there exists a Condorcet vertex of π, then the security center of π coincides with the Condorcet set of π (i.e. the set of all Condorcet vertices of π).

4.2. *Cubefree median graphs*

The Condorcet set of a finite tree is just the median set, see [21]. For a related result, see [11]. Hence the security center and the median set of a finite tree coincide, see [18]. On the other hand, by (i)$\Leftrightarrow$(ii) of Proposition 8, medians of any family in a median graph are 'local' Condorcet vertices, and vice versa. All this may suggest that for median graphs Condorcet sets and median sets are the same. However, this is not so: consider the cube with $\pi=(a,b,c,m,m)$ as indicated in Fig. 2. Then m is the unique median of π, while $S(x,\pi)=-1$ if $x\neq a,b,c$ and $=-3$ otherwise. Hence the security center of π properly contains $\mathrm{Med}(\pi)$, and there exists no Condorcet vertex of π. As our final result shows this 'Effet Condorcet' (paradox of voting) cannot occur in a *cubefree* median graph, i.e. a median graph which does not contain the graph of Fig. 2 as a subgraph.

First we need a lemma.

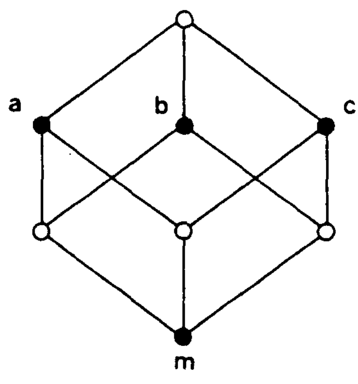

Fig. 2.

Lemma 5. *Let $(X,\leq)$ be a discrete median semilattice. Then the covering graph G of $(X,\leq)$ is cubefree if and only if every subset A of X with $|A|\geq 3$ contains two elements u and v* such that $\bigwedge A=u\wedge v$.

Proof. If G contains a cube, then $(X,\leq)$ contains a three-dimensional Boolean lattice, which violates the condition of the lemma. Conversely, assume that there exist three elements u, v, w of X whose pairwise meets are different from $u \wedge v \wedge w$. Then choose any elements $x \leq u \wedge v$, $y \leq u \wedge w$, and $z \leq v \wedge w$, which cover $u \wedge v \wedge w$. These elements are pairwise bounded above, and thus $x \vee y \vee z$ exists. It is clear that $I(u \wedge v \wedge w, x \vee y \wedge z)$ induces a cube in G. □

Proposition 9. *For a median graph $G = (X, E)$, the following conditions are equivalent:*

(i) *G is cubefree.*
(ii) Med(π) *is the security center of π for all $\pi \in X^*$.*
(iii) Med(π) *is the Condorcet set of π for all $\pi \in X^*$.*

Proof. As was mentioned in 4.1, (iii) implies (ii). If G contains some cube, then for π, choosen as in Fig. 2, we get that $S(x, \pi) \leq -1$ for all vertices x of G. Indeed, if x does not belong to the cube, then (since G does not contain $K_{2,3}$) x is adjacent to at most one of the vertices a, b, c, whence $S(x, \pi) \leq \pi(x, m) - \pi(m, x) \leq 1 - 2 = -1$. Therefore (ii) implies (i). Now assume that G is cubefree. Let x be any median of a family $\pi = (a_1, \ldots, a_p) \in X^p$. Suppose by way of contradiction that there exists a vertex y such that $\pi(x, y) < \pi(y, x)$. Then the set K of all indices i with $d(y, a_i) < d(x, a_i)$ contains more than $\frac{1}{2}p$ elements. Hence by Propositions 5 and 6 we must have $x = \bigwedge K$ in the semilattice $(X, \leq_x)$. By Lemma 5 there exist $i, j \in K$ such that $x = a_i \wedge a_j$, whence $d(a_i, a_j) = d(x, a_i) + d(x, a_j)$. Then $d(y, a_i) + d(y, a_j) < d(a_i, a_j)$ by the choice of K, which is absurd. We conclude that x is a Condorcet vertex of π. On the other hand, every Condorcet vertex of π is a median of π by Proposition 8, completing the proof. □

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