Advances in Mathematics **163**, 206–257 (2001)
doi:10.1006/aima.2001.1999, available online at http://www.idealibrary.com on IDEAL®

Invariant Theory and Calculus for Conformal Geometries

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Communicated by Charles Fefferman

Received March 13, 2001; accepted April 20, 2001

1. INTRODUCTION

In dimensions greater than 3 conformal geometries have local structure

geometries are the most well known of structures among the family of so called parabolic geometries. A programme of describing the local scalar invariants of such structures was initiated by C. Fefferman [9] who sought to understand the invariants of CR structures, another class in the parabolic family. With Graham, conformal structures were drawn into the programme [10] and then projective geometries included [18]. Recently there has been some major progress on these problems. In combination with the geometric constructions of [9] and [11], results of Bailey, Eastwood and Graham [2] completely solved the problem for odd dimensional conformal structures and presented progress for the even dimension conformal case and the closely related CR geometries. This latter progress was significant in the sense that it enabled an extension of the description in [9] of the asymptotic expansion of the Bergman kernel of a strictly pseudoconvex domain in $\mathbb{C}^n$ up to the logarithmic term. This result has since been extended by Hirachi [20] who provides a description of the log term. However for CR structures and even dimensional conformal structures the underlying constructions in [9] and [11] are obstructed at finite order and so cannot yield more than a finitely generated set of invariants. Thus in these cases the problem has remained for the most part open. This article presents an alternative approach to the invariant theory of conformal structures which solves the problem for even dimensional geometries

[1] The author gratefully acknowledges support from the Australian Research Council.

0001-8708/01 $35.00

save for a finitely generated "window" of exceptional invariants. In low dimensions this gap (which is described in more detail below) is very small and can realistically be closed by direct calculation or some elementary software and so in these cases the theory is effectively complete. The approach here is, in part, a development of [14] which solves the problem for projective geometries. One may regard the results here as progress in the invariant theory of CR structures since the conformal invariants for even dimensional geometries of dimensions $n \geqslant 4$ determine CR invariants via Fefferman's conformal metric construction [8]. In fact it is expected that the approach presented here can be adapted to deal directly with the CR case.

The geometrical constructions developed in [9, 10] and alluded to above, involve the Fefferman and Fefferman–Graham ambient metric constructions which build (at least formally) from the original parabolic geometry an auxiliary higher dimensional manifold equipped with, respectively, a pseudo-Hermitian or pseudo-Riemannian metric. The invariants of the auxiliary manifold are also invariants of the underlying parabolic structure. By classical theory it is known how to construct all invariants of general pseudo-Hermitian and pseudo-Riemannian structures. To the extent that these ambient constructions work the remaining problem is then to determine whether, or to what extent, these invariants give all the invariants of the parabolic structure. This leads to an algebraic problem which involves the representation theory of a parabolic subgroup P of a semi-simple group G, where G/P is the flat model of the structure under consideration ([18] provides an explicit description of these for the cases mentioned). Much progress has been made on the latter algebraic problems and the related geometric problems of the flat models. For projective geometries the relevant problems were solved in [12, 13]. The work of Bailey, Eastwood and Graham [2] (see also [3]) adapted that approach and introduced new ideas to produce a theory for the conformal and CR problems which extended and was simpler than Fefferman's pioneering work.

As mentioned above the ambient metric constructions are obstructed at finite order in the CR and even dimensional conformal cases and so to complete the parabolic invariant theory problems a rather different approach is required. Here, for the conformal case the role of the ambient metric construction is replaced by certain objects from a conformally invariant calculus that has been termed tractor calculus (see Section 4). (In fact there is a rather close relationship between the tractor calculus and the ambient constructions. However this will not be described here.) This calculus is based around a basic bundle (a tractor bundle) with connection which may be viewed as an induced bundle of the canonical Cartan bundle equipped with its normal connection. Invariant operators between this bundle and its tensor powers are described and in terms of these and their "curvature" (for example W_{ABCD} in (30)) invariants can again be proliferated in a classical way. That is, as complete contractions of expressions made from these objects. These are the Weyl invariants and generalised Weyl invariants described in Section 5.1. The inducing representation of the basic tractor bundle is the defining representation of G (which may be taken to be the identity connected component of $O(p+1, q+1)$ where $p+q=n$) restricted to the appropriate parabolic P. At each point such complete contractions may be viewed as arising from classically constructed G invariants. The key to the progress here is the observation that there is another way to construct invariants that exploits (albeit indirectly) the fact that the underlying structure group is the parabolic subgroup of G. These are the quasi-Weyl (q-Weyl) invariants of Section 5.2. As with the Weyl invariants and generalised Weyl invariants the set of q-Weyl invariants has the property that one can simply list a basic linearly generating set without performing calculations.

The main result is Theorem 5.5 which in rough terms claims that almost all invariants are q-Weyl. (Of course the main interest is the even dimensional case. However with little extra effort and for completeness the odd dimensional case has also been treated.) To be more specific we need some notation. In this paper a conformal invariant $I(\mathbf{g})$ is a polynomial function in the jets of the metric $\mathbf{g}$ and its inverse which simply scales under a rescaling of the metric. That is, for any smooth positive function Ω, $I(\Omega^2\mathbf{g}) = \Omega^u I(\mathbf{g})$ and u is said to be the weight of the invariant. A notion of the lowest degree of any such polynomial can be defined and is denoted d_0. Then k_0 is the number such that $u = -(2d_0 + k_0)$. In this notation, and dividing invariants into even invariants and odd invariants according to whether their sign is unchanged or changed under orientation reversal, here is a summary of the invariants missed by Theorem 5.5.

- Odd invariants can fail to be quasi-Weyl only in even dimensions n and only if $d_0 = n/2$ and $k_0 \leqslant n$.
- Even invariants can fail to be quasi-Weyl in even dimensions only if the following hold simultaneously $\max(|u|, 4d_0) \leqslant 2n-2$, $k_0 \geqslant 2$ and either $\frac{n+2}{2} \leqslant |u| < n$ and $4d_0 \leqslant n$ or $n \leqslant |u|$.
- Even invariants can fail to be quasi-Weyl in odd dimensions only if the following hold simultaneously, $n \geqslant 5$, $\max(|u|, 4d_0) \leqslant 2n-2$, $k_0 \geqslant 2$ and $n \leqslant |u|$.

From this array of conditions two main points should be made. Firstly the problem for odd invariants is essentially solved. Secondly the condition that $|u| \leqslant 2n-2$ for an invariant to fail to be quasi-Weyl is very restrictive and already limits the window to a finitely generated set. The condition that $2d_0 \leqslant n-1$ is also severe. For example, since there are no non-trivial invariants with $d_0 = 1$ it follows immediately that in dimension 4 all even invariants are quasi-Weyl. In higher dimensions the results here are complemented by the results of [2] (with [11]). As mentioned above their results completely solve the odd dimensional case while in even dimensions their results enable the construction of invariants of weight u where $|u| < n$. (In fact one can recover the complementary results of [2] for conformal invariants via tractor calculus but using rather different arguments than those presented here—see Remark 8.2.)

Invariants and the notation introduced above are defined more precisely in Section 3. This follows some preliminaries introducing other notation and a review of aspects of Riemannian, pseudo-Riemannian and conformal geometry in the next section. The methods of constructing invariants and the main theorem are presented in Section 5. The remaining sections are concerned with proving the main theorem bar the last which is primarily concerned with establishing that in odd dimensions and in some other cases the invariants obtained as q-Weyl invariants can also be obtained as Weyl invariants.

I am deeply indebted to Michael Eastwood and Robin Graham for many suggestions and helpful conversations. This research was carried out at the Queensland University of Technology and the University of Adelaide under the support of an Australian Research Council QEII Fellowship and an Australian Research Council grant.

2. PRELIMINARIES

We shall work on real conformal n-manifolds M where $n \geqslant 3$. That is, for each such n, we will consider a smooth n-manifold equipped with an equivalence class $[\mathbf{g}]$ of metrics $\mathbf{g}$, of an arbitrary fixed signature (p, q), that will be called a *conformal equivalence class*. The condition of equivalence here is that two metrics $\mathbf{g}$ and $\hat{\mathbf{g}}$ are equivalent if $\hat{\mathbf{g}}$ is a positive scalar function multiple of $\mathbf{g}$. In this case we will say the two are *conformally equivalent* or just *conformal* and write $\hat{\mathbf{g}} = \Omega^2 \mathbf{g}$ for some positive smooth function Ω. (The transformation $\mathbf{g} \mapsto \hat{\mathbf{g}}$, which changes the choice of metric from the conformal class, is termed a *conformal rescaling*.) We shall only be interested in local properties of the conformal manifolds.

We will write $\mathcal{E}$ for the sheaf of germs of smooth functions on M. In line with this we will often write $\mathcal{E}^a$ and $\mathcal{E}_a$ for, respectively, the tangent and cotangent bundles to M (which we will not distinguish from the respective sheaves of germs of smooth sections). Tensor products of these bundles will be indicated by adorning the symbol $\mathcal{E}$ with appropriate indices. For example, in this notation $\bigotimes^2 T^*M$ is written $\mathcal{E}_{ab}$. Unless otherwise indicated, our indices will be *abstract indices* in the sense of Penrose [22]. An index which appears twice, once raised and once lowered, indicates a contraction. In case a frame is chosen and the indices are concrete, use of the Einstein summation convention (to implement the contraction) is understood. The symmetric tensor product of the cotangent bundle to some power ℓ, $\bigodot^\ell \mathcal{E}_a$, will usually be written

$$\mathcal{E}_{\underbrace{(ab\cdots c)}_{\ell}},$$

and $\mathcal{E}_{(ab\cdots c)_0}$ indicates the completely trace-free subbundle. Similarly $\mathcal{E}_{[ab\cdots c]}$ means the completely skew tensor product $\bigwedge^\ell \mathcal{E}_a$. These notations will be also used to indicate the projections onto these bundles. For example $T_{(ab\cdots c)_0}$ means the symmetric trace-free part of the tensor field $T_{ab\cdots c}$. Finally we will extend these conventions to the indexed tractor bundles defined in Section 4 below without further comment.

Density bundles $\mathcal{E}[w]$ will be defined on $(M, [\mathbf{g}])$ as follows. The bundle whose smooth sections are metrics from the conformal class is a ray subbundle $\mathcal{L}$ (i.e., a fibre subbundle with fibre $\mathbb{R}_+$) of $\mathcal{E}_{ab}$. We may view $\mathcal{L}$ as a principal bundle with group $\mathbb{R}_+$, so there are natural line bundles on $(M, [g])$ induced from the irreducible representations of $\mathbb{R}_+$. We write $\mathcal{E}[w]$ for the line bundle induced from the representation of weight $-w/2$ on $\mathbb{R}$ (that is $\mathbb{R}_+ \notin y \mapsto y^{-w/2} \in \mathrm{End}(\mathbb{R})$). Thus a section of $\mathcal{E}[w]$ is a real valued function f on $\mathcal{L}$ with the homogeneity property $f(\Omega^2 g, x) = \Omega^w f(g, x)$ where Ω is a positive function on M, $x \in M$ and g is a metric from the conformal class $[\mathrm{g}]$. We will use the notation $\mathcal{E}_a[w]$ for $\mathcal{E}_a \otimes \mathcal{E}[w]$ and so on.

Let $\mathcal{E}_+[w]$ be the fibre subbundle of $\mathcal{E}[w]$ corresponding to $\mathbb{R}_+ \subset \mathbb{R}$. Choosing a metric $\mathbf{g}$ from the conformal class defines a function $f\colon \mathcal{L} \to \mathbb{R}$ by $f(\hat{\mathbf{g}}, x) = \Omega^{-2}$, where $\hat{\mathbf{g}} = \Omega^2 \mathbf{g}$, and this clearly defines a smooth section of $\mathcal{E}[-2]_+$. Conversely, if f is such a section, then $f(\mathbf{g}, x)\, \mathbf{g}$ is constant up the fibres of $\mathcal{L}$ and so defines a metric in the conformal class. So $\mathcal{E}_+[-2]$ is canonically isomorphic to $\mathcal{L}$, and the *conformal metric* g_{ab} is the tautological section of $\mathcal{E}_{ab}[2]$ that represents the map $\mathcal{E}_+[-2] \cong \mathcal{L} \to \mathcal{E}_{(ab)}$. On the other hand, for a section $\mathbf{g}_{ab}$ of $\mathcal{L}$ consider the map $\phi_{ab} \mapsto \mathbf{g}^{cd}\phi_{cd}\mathbf{g}_{ab}$. This is clearly independent of the choice of $\mathbf{g}$. Thus, we get a canonical section g^{ab} of $\mathcal{E}^{ab}[-2]$ such that $g_{ab}\, g^{bc} = \delta^c_a$.

The conformal metric gives a canonical isomorphism of $\mathcal{E}^a[w]$ with $\mathcal{E}_b[w+2]$ which is expressed by writing $V_b = g_{ab} V^a$, and so on. We shall often use this isomorphism implicitly by raising and lowering indices without comment. Since we are working locally we may assume that our manifold is oriented and so is equipped with a conformal volume form

$$\varepsilon = \varepsilon_{ab\cdots c} \in \Gamma\mathcal{E}_{\underbrace{[ab\cdots c]}_{n}}[n],$$

compatible with the conformal metric.

Given a non-vanishing section ξ of $\mathcal{E}[1]$ a corresponding metric from the conformal class $\mathbf{g}_{ab} = g^{\xi}_{ab}$ is given by

$$g^{\xi}_{ab} = \xi^{-2}\mathbf{g}_{ab}.$$

We will call such a section ξ, a *conformal scale*. Note that under a change of conformal scale $\xi \mapsto \hat{\xi} = \Omega^{-1}\xi$ (we take Ω smooth and positive) we have $g^{\xi} \mapsto g^{\hat{\xi}} = \Omega^2 g^{\xi}$. For the purposes of calculating and producing explicit formulae it is often useful to make such a choice of scale and work with the Riemannian or pseudo-Riemannian structure given by g^{ξ}. In this case the manifold is equipped with the corresponding canonical Levi–Civita connection ∇^{ξ} or ∇^{ξ}_a. This is the unique torsion free connection on the tangent space and its tensor powers (and so also on the density bundles) which preserves the metric g^{ξ}. With the choice of scale ξ understood we will usually omit the ξ superscript and just write ∇ or ∇_a. If ∇ is the connection corresponding to another choice of conformal scale $\hat{\xi} = \Omega^{-1}\xi$ then

$$\begin{aligned}
\widehat{\nabla_a} f &= \nabla_a f + w\Upsilon_a f\\
\widehat{\nabla_a} U^b &= \nabla_a U^b + \Upsilon_a U^b - U_a \Upsilon^b + U^k \Upsilon_k \delta_a{}^b\\
\widehat{\nabla_a}\, \omega_b &= \nabla_a \omega_b - \Upsilon_a \omega_b - \Upsilon_b \omega_a + \Upsilon^k \omega_k \mathbf{g}_{ab},
\end{aligned} \tag{1}$$

where δ is the Kronecker delta, $\Upsilon_a = \Omega^{-1}\nabla_a\Omega$, and the quantities f, U^a, ω_a are sections of $\mathcal{E}[w]$, $\mathcal{E}^a$, $\mathcal{E}_a$, respectively. The corresponding formulae for ∇ acting on an arbitrary weighted tensor is easily obtained from these formulae using the Leibniz rule.

The Riemann curvature is defined by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\, U^c = R_{ab}{}^c{}_d U^d \tag{2}$$

and satisfies

$$\begin{aligned}
R_{abcd} &= R_{[ab][cd]}\\
R_{[abc]\,d} &= 0\\
\nabla_{[a} R_{bc]\,de} &= 0.
\end{aligned} \tag{3}$$

The last two identities above will be referred to as the *Bianchi symmetry* and the *Bianchi identity* respectively. The totally trace-free part of R_{abcd} is the conformally invariant *Weyl curvature* C_{abcd}. The Riemann tensor can be expressed as

$$R_{abcd} = C_{abcd} + 2g_{c[a}\mathsf{P}_{b]d} + 2g_{d[b}\mathsf{P}_{a]c}, \tag{4}$$

where C_{abcd} is the totally trace-free part and $\mathsf{P}_{ab} = \mathsf{P}_{(ab)}$. By considering numbers of independent components it is easily seen that C_{abde} must vanish in dimension 3. Equation (4) defines the *rho-tensor* P_{ab} as a trace-adjusted multiple of the Ricci tensor $R_{bc} = R_{ab}{}^{a}{}_{c}$:

$$R_{ab} = (n-2)\,\mathsf{P}_{ab} + \mathsf{P}g_{ab} \qquad \text{where} \quad \mathsf{P} = \mathsf{P}_{a}{}^{a}.$$

Under conformal rescaling the rho-tensor transforms according to

$$\hat{\mathsf{P}}_{ab} = \mathsf{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \tfrac{1}{2}\Upsilon_c \Upsilon^c g_{ab}. \tag{5}$$

It follows from the Bianchi identity that

$$\nabla_c C_{ab}{}^{c}{}_{d} = 2(n-3)\,\nabla_{[a}\mathsf{P}_{b]d} \text{ and } \nabla^b \mathsf{P}_{ab} = \nabla_a \mathsf{P}. \tag{6}$$

In dimension three $C_{abd} := 2\nabla_{[a}\mathsf{P}_{b]d}$ is conformally invariant and is known as the *Cotton–York tensor*. Note that it follows from the second of Eqs. (6) that the Cotton–York tensor is trace-free.

A conformal geometry is *conformally flat* (or just *flat*) if there is a choice of scale ξ such that the corresponding metric g^{ξ} is flat as a Riemannian or pseudo-Riemannian metric. In dimensions $n \geqslant 4$ there exists such a scale if and only if the Weyl curvature C_{abcd} vanishes. Similarly, in dimension 3 a manifold is conformally flat if and only if the Cotton–York tensor vanishes.

It is well known that (3), along with

$$\begin{aligned} \nabla_a R_{bcde} &= \nabla_a R_{[bc][de]} \\ \nabla_a R_{[bcd]e} &= 0, \end{aligned}$$

give a complete set of identities for the Riemann curvature tensor up to first order. (See, for example, [24] for these results). The two identities displayed here follow from covariantly differentiating both sides of the first two identities (3) and so, in this sense, are not new identities. Similarly for higher derivatives of the curvature there are identities arising from (2). For example $[\nabla_a, \nabla_b]\, R_{cdef}$ can be re-expressed as tensor quadratic in the (undifferentiated) curvature

$$[\nabla_a \nabla_b]\, R_{cdef} = R_{abc}{}^{k} R_{kdef} + R_{abd}{}^{k} R_{ckef} + R_{abe}{}^{k} R_{kdkf} + R_{abf}{}^{k} R_{kdek}.$$

(Here and throughout the notation $[A, B]$ means the commutator of the operators A and B, i.e., $[A, B]=(AB-BA)$.) In fact all identities amongst higher derivatives of the curvature arise from the results already given and their covariant derivatives:

Proposition 2.1. *Any identity between the covariant derivatives of the Riemannian curvature can be established using just the identities* (2), (3) *and the identities which follow from covariantly differentiating both sides of these expressions.*

Proof. It is easily established via normal coordinates (and it is a classical result) that any Riemannian or pseudo-Riemannian invariant, which at each point depends only on the infinite-jet of the metric at that point, depends only on the infinite-jet of the curvature at that point. For $\dim(\mathscr{M})=n$ let $N(n, r)$ be the number of functionally independent invariants depending on just the r-jet of the metric. In view of this and (2) it follows immediately that the number of independent components in

$$\underbrace{\nabla_{(a}\nabla_b \cdots \nabla_{c)}}_{r-2} R_{defg}$$

gives an upper estimate for the number $N(n, r)-N(n, r-1)$. Allowing for this *and* for the identities (3), we can deduce

$$N(n, r)-N(n, r-1) \leqslant G(n, r),$$

where, for $r \geqslant 2$,

$$G(n, r) := \frac{n(n+r-1)!\,(r-1)}{2(n-2)!\,(r+1)!}.$$

Furthermore if there were to be any non-trivial identities satisfied by the curvature R_{abcd} and its covariant derivatives up to order $(r-2)$ which non-trivially involve the $(r-2)$th covariant derivative of the curvature and such that these identities cannot be deduced from (3) and (2), then

$$N(n, r)-N(n, r-1) < G(n, r).$$

However, from equation (75.3) in [24] it follows that, for $r \geqslant 3$,

$$N(n, r)-N(n, r-1) = G(n, r).$$

So there are no such identities for $r \geqslant 3$. With the results quoted above this completes the proof. ∎

Remark 2.2. In fact if one wants to calculate with the tensor symbols in a purely formal manner then one needs also to include the "Cayley–Hamilton identity," viz that for any tensor $T_{ab\cdots c}$ of valence $n+1$ one has $T_{[ab\cdots c]}=0$. This and its consequences, with the identities described above give a full set of identities for formal calculation (see [16]).

3. INVARIANTS

Here Riemannian or pseudo-Riemannian invariants will be collectively referred to as metric invariants and these are required to be polynomial in the jets of the metric and its inverse. More precisely we make the following definition.

DEFINITION 3.1. A metric invariant $E(g)$ is a polynomial in the variables $\partial_i\partial_j\cdots\partial_k\mathbf{g}_{lm}$ and $(\det \mathbf{g}_{ij})^{-1}$ which is independent of the coordinate system $\{x^i\}$ used. (Here $\mathbf{g}_{ij}$ means components of a metric $\mathbf{g}$ in the coordinates $\{x^i\}$ and $\partial_i := \partial/\partial x^i$.)

Some metric invariants have the special property of simply scaling under a conformal change of metric, these are the conformal invariants.

DEFINITION 3.2. A (conformal) invariant of weight u is a metric invariant $I(\mathbf{g})$ such that $I(\Omega^2\mathbf{g})=\Omega^u I(\mathbf{g})$ for any smooth positive function Ω.

The word "conformal" will usually be omitted and it will be understood that the term invariant on its own will mean a conformal invariant. We will use the term *coupled invariant* to mean a polynomial with properties as in Definition 3.2 except that it may also depend on the coordinate components of some tensor or density field and its coordinate derivatives.

It is well known that the curvature R_{abcd} and its ∇-derivatives are *tensor valued* polynomials in the jets of the metric and its inverse (see e.g., [21]) which are independent of coordinates. Thus one way to construct metric invariants is simply to juxtapose such tensors with an appropriate number dual metrics and/or volume forms and then form a complete contraction. For example $g^{ab}_{(\xi)}g^{cd}_{(\xi)}\mathsf{P}_{ac}\mathsf{P}_{bd}$ is a metric invariant. Note that this can also be written as $\xi^4 g^{ab}g^{cd}\mathsf{P}_{ac}\mathsf{P}_{bd}$. Since this invariant is homogeneous in ξ we can drop the ξ's and regard $g^{ab}g^{cd}\mathsf{P}_{ac}\mathsf{P}_{bd}$ as an $\mathcal{E}(-4)$ valued metric invariant. Since all metric invariants considered below are homogeneous in this sense we shall always eliminate ξ in this way and each metric invariant will have some weight u, i.e., will take values in $\mathcal{E}(u)$ for some $u\in\mathbb{R}$ (in fact, $0>u\in\mathbb{Z}$). It is a classical result (the key ingredient of which is Weyl's invariant theory [25]) that all such metric invariants can be written in the form

$$E=E_{\text{even}}+E_{\text{odd}},$$

where E_{even} is a linear combination of complete contractions of the form

$$\operatorname{contr}(g^{-1} \otimes \cdots \otimes g^{-1} \otimes g^{-1} \otimes R^{(k_1)} \otimes R^{(k_2)} \otimes \cdots \otimes R^{(k_d)}), \tag{7}$$

and E_{odd} is a linear combination of complete contractions of the form

$$\operatorname{contr}(\varepsilon \otimes g^{-1} \otimes \cdots \otimes g^{-1} \otimes g^{-1} \otimes R^{(k_1)} \otimes R^{(k_2)} \otimes \cdots \otimes R^{(k_d)}), \tag{8}$$

where $R^{(\ell)}$ means the ℓth ∇-derivative of R_{abcd}, i.e.,

$$\underbrace{\nabla_a \nabla_b \cdots \nabla_c}_{\ell} R_{defg}$$

and "contr$(\cdots)$" indicates that some complete contraction has been taken. A metric invariant expressed as a linear combination of complete contractions, in this way, will be said to be in *standard form*. (Of course the numbers d, and k_i (for $i = 1, \ldots, d$) in (7) are unrelated to the corresponding numbers in (8).) A metric invariant E is said to be *even* if $E = E_{\text{even}}$ while E is said to be *odd* if $E = E_{\text{odd}}$. We say that the expressions (7) and (8) are of *degree* d and refer to $k := \sum_1^d k_i$ as the *total order*. These terms may be extended to linear combinations of such expressions provided each term in the linear combination shares the same value of d and k.

Often it will be important for us to distinguish formal expressions, such as (7) and (8), from the invariants they determine. For example while the parameters d and k may make sense for a linear combination of these expressions they are not in general well defined descriptions of the metric invariants that they determine. Rather, then, we should think of a metric invariant as an equivalence class of expressions. Observe that the curvature R_{abcd} and the ∇_a derivatives of R_{abcd} have weight 2 while $g^{-1} \in \mathscr{E}^{(ab)}[-2]$ and $\varepsilon \in \mathscr{E}_{[ab\cdots e]}[n]$ so both the *expression* (7) and the *expression* (8) are each assigned a weight

$$u = -(2d + k). \tag{9}$$

Again this term may be extended to linear combinations of such expressions provided each term in the linear combination shares the same weight u. We shall deal only with metric invariants which are either even or odd and of well defined weight, that is metric invariants which can be expressed as a linear combination of terms all of the form (7) or all of the form (8), respectively, and such that the expression has weight weight u for some $u \in \mathbb{Z}$. In this case we say the metric invariant, if not trivial, has weight u and, without loss of generality, we will assume that any expression for such an even (odd) invariant, is a linear combination of terms of the form (7) (respectively (8)), with a well defined weight u.

In an expression for a metric invariant as a linear combination of terms of the form (7) or of the form (8), different terms may have differing degree d. However using (9) we obtain an upper estimate on the degree d of terms which contribute to a metric invariant of weight u:

$$d \leqslant -\frac{u}{2}. \tag{10}$$

On the other hand we shall say that an expression for a metric invariants has *principal degree p* if p is the degree of a term of lowest degree in the expression. Of course different expressions for a metric invariant may have differing principal degree. So we make the following definition:

DEFINITION 3.3 [Principal Degree]. We say that a metric invariant of weight u has principal degree d_0 where

$$d_0 := \max\{p \text{ s.t. p is the principal degree of an expression for } A\}.$$

In view of (10) d_0 is well defined and we have the following.

PROPOSITION 3.3. *The principal degree d_0 of an odd metric invariant of weight u satisfies*

$$d_0 \geqslant n/2.$$

Proof. Proof Suppose there is a term of degree $d < n/2$ in some expression for the invariant as a linear combination of terms of the form (8). Then in that term one of the $R^{(\ell)}$ has at least three of its indices contracted into the conformal volume form. Consider the following tensor expression which is a partial contraction of $\varepsilon \otimes R^{(\ell)}$ for some ℓ

$$\varepsilon^{pqr**\cdots*} \nabla_{[p} \nabla_{|a} \nabla_b \cdots \nabla_{d|} R_{qr]ef},$$

where the $*$'s indicate indices. This differs from

$$\varepsilon^{pqr**\cdots*} \nabla_a \nabla_b \cdots \nabla_d \nabla_{[p} R_{qr]ef} = 0,$$

by a linear combination of terms each of which is a partial contraction of $\varepsilon \otimes R^{(\ell_1)} \otimes R^{(\ell_2)}$ for some ℓ_1 and ℓ_2. By repeated use of this and similar observations it is straightforward to rewrite any term with degree $< n/2$ as a linear combination of terms each of which has degree $\geqslant n/2$ and so the result follows. ∎

Since metric invariants may depend on a choice of conformal scale they generally give no information about the intrinsic structure of a conformal geometry. For example the metric invariant $g^{ab} g^{cd} \mathsf{P}_{ac} \mathsf{P}_{bd}$, introduced above, can be made to vanish at any single point by an appropriate choice of scale. On the other hand, since C_{abcd} is invariant under conformal rescaling, $\varepsilon^{abcd}C_{abef}C_{cd}{}^{ef}$ is a metric invariant in dimension 4 which depends only on the conformal structure. In the language above we may describe an invariant of a conformal geometry as a $\mathscr{E}(u)$-valued metric invariant which is invariant under change of conformal scale and one approach to the construction of invariants is to write down appropriate general linear combinations of expressions of the form (7) or of the form (8) and then solve for the coefficients so that the entire expression is invariant under conformal rescaling (i.e., use (1) etc.). Of course this approach is uninsightful and while it is tractable for the lowest degree and lowest order cases it rapidly fails to be so otherwise. On the other hand in Section 4 we introduce some basic conformally invariant operators which, as discussed in Section 5, lead to several ways to simply proliferate invariants without performing calculations. Toward understanding the key construction method and the extent to which it recovers all invariants we need a special way of presenting invariants. We now describe this.

3.1. *A normal form for invariants.* Suppose we fix a conformal structure $[\mathbf{g}]$ on M and a point $p \in M$. By considering formal power series and the transformation Eqs. (1) and (5) it is easy to see that we can choose a conformal scale so that

$$\underbrace{\nabla_{(a}\nabla_b\cdots\nabla_d}_{s}\mathsf{P}_{ef)}=0 \qquad \text{at} \quad p\in\mathscr{M} \tag{11}$$

for $s \in \{0, 1, 2, \ldots\}$ (see [2] for some discussion of a similar normalisation). We will describe such a choice of conformal scale as a *normal (conformal) scale* at $p \in M$.

Let E be a metric invariant and suppose we fix a normal scale at $p \in M$. Suppose the invariant is expressed in standard form so that each term of the form (7) or (8), is of weight u and of degree at least d_0, where d_0 is the principal degree of E. It is easy to see that, by repeated use of (2), (4), (6), and (11), the metric invariant can be re-expressed, at p, as a linear combination of terms of the form (7) or (8), each with $d \geqslant d_0$, where now $R^{(\ell)}$ means

$$R^{(\ell)}=\begin{cases}\underbrace{\nabla_{(a}\nabla_b\cdots\nabla_{d)}}_{\ell}C_{efgh} & n\geqslant 4\\ \underbrace{\nabla_{(a}\nabla_b\cdots\nabla_{d)}}_{\ell-1}C_{fge} & n=3\end{cases}$$

A metric invariant expressed, at $p \in M$, in this final manner is said to be in *normal* form.

In fact this terminology is slightly misleading in the sense that starting with a metric invariant E, its "expression in normal form" is in general an expression for a different metric invariant. This is clear since there exist non-trivial metric invariants E which vanish at p when evaluated on any metric satisfying (11). The example $g^{ab}g^{cd}\mathsf{P}_{ac}\mathsf{P}_{bd}$ mentioned above is a case in point. Nevertheless, since ultimately we are concerned with conformal invariance, the "loss" or alteration of some metric invariants in this way causes no problems and the notion of a metric invariant expressed in normal form is quite a convenience. A metric invariant and its normal form differ by terms which vanish at p on metrics satisfying (11).

DEFINITION 3.5. Given a metric invariant E and a normal expression for E the principal part $E_{(d_0)}$ of this expression for E is obtained by removing those terms having degree greater than d_0, where d_0 is the principal degree of E.

Note that there is an equivalence relation on metric invariants given by

$$E \sim F$$

if and only if E and F are of the same principal degree d_0 and there is a normal form for E and a normal form for F such that $E_{(d_0)} = F_{(d_0)}$. We will write $[E]_{(d_0)}$ to describe the equivalence class of E. We may regard $[E]_{(d_0)}$ as the *principal part* of E and $E_{(d_0)}$ as a normal expression representing this. Often we will be less formal and describe $E_{(d_0)}$ as the principal part of E.

We observed above that $d_0 \geqslant n/2$ for odd invariants. It is clear that without loss of generality we may also assume the following:

$$\begin{aligned} &k \text{ is } \underline{\text{even}} \text{ if the invariant is even} \\ &k-n \text{ is } \underline{\text{even}} \text{ if the invariant is odd} \end{aligned} \tag{12}$$

and

$$d_0 \geqslant 2.$$

4. CONFORMAL TRACTOR CALCULUS

Many problems in Riemannian or pseudo-Riemannian geometry are most easily treated and discussed via tensor bundles and the Levi–Civita connection as a covariant differential operator on these. The conformal tractor calculus provides an analogous setting for conformal structures. The modern treatment of this was initiated in [1] but many of the key ideas were developed by Thomas some time earlier [23]. Here we will review the basic setup from a rather practical and calculational point of view. In addition a key new tool is introduced, namely the tractor "double-D" operator D_{AP}. This is a first order conformally invariant operator and in many senses is the conformal analogue (see [15]) of the Levi–Civita connection, as an invariant operator. (It should be pointed out that a version of the operator D_{AP} was found independently by Fefferman and Graham [19] in the context of their ambient metric construction.) Some extensions of the tractor calculus not discussed here are presented in [4]. In [17] a similar calculus was developed for quaternionic structures and their generalisations. More recently, the basic tractor calculus has been extended to all parabolic geometries in [5, 6] where the relationship with the Cartan connection is also described explicitly.

Given a choice of conformal scale ξ, the *tractor bundle* $\mathscr{E}^A$ is identified with the direct sum

$$[\mathscr{E}^A]_\xi = \mathscr{E}[1] \oplus \mathscr{E}^a[-1] \oplus \mathscr{E}[-1]$$

and under conformal rescaling this splitting is transformed according to

$$[U^A]_\xi = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} \mapsto [U^A]_{\hat{\xi}} = \begin{pmatrix} \hat{\sigma} \\ \hat{\mu}^a \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu^a + \Upsilon^a \sigma \\ \rho - \Upsilon_b \mu^b - \frac{1}{2} \Upsilon_b \Upsilon^b \sigma \end{pmatrix}. \tag{13}$$

This transformation is consistent with the composition of rescalings and so gives an equivalence relation (on the direct sum bundles) consistent with the equivalence relation on metrics in a conformal class. It follows that the bundle $\mathscr{E}^A$ is well defined on conformal manifolds. Henceforth we will drop the notation $[\bullet]_\xi$ when a choice of scale ξ is understood. The bundle $\mathscr{E}^A$ is naturally equipped with a non-degenerate symmetric form h_{AB}, the tractor metric, defined by

$$h_{AB} U^A V^B = \mu^a \beta_a + \sigma \gamma + \rho \alpha$$

for

$$U^A = \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix}, \qquad V^B = \begin{pmatrix} \alpha \\ \beta^b \\ \gamma \end{pmatrix}. \tag{14}$$

This metric has signature $(p+1, q+1)$ and is conformally invariant. The metric h_{AB} provides us with an isomorphism of $\mathcal{E}^A$ with its dual $\mathcal{E}_A$ which we will often use implicitly by raising and lowering indices. We will often use the term "(weighted) tractor" or "(weighted) tractor field" casually to refer to a section (or germ thereof) of an arbitrary tensor product of $\mathcal{E}^A$ and its dual (with $\mathcal{E}[w]$) and we will write $\mathcal{E}^{IJ\cdots L}{}_{MN\cdots P}[w]$ to denote $\mathcal{E}^{IJ\cdots L}{}_{MN\cdots P}\otimes\mathcal{E}[w]$. The tractor metric is used to *contract* tractor field indices in the same way as a Riemannian or pseudo-Riemannian metric is used to contract indices of tensors. The indices of a given tractor field may be suppressed according to convenience.

From the Definition of the tractor bundle and the transformation rule (13) it follows that there is a composition series

$$\mathcal{E}^A = \mathcal{E}[1] + \mathcal{E}^a[-1] + \mathcal{E}[-1]$$

meaning that there is an invariant injection $\mathcal{E}[-1]\to\mathcal{E}^A$ and that there is an injection of $\mathcal{E}^a[-1]$ into the quotient. We will describe the bundle $\mathcal{E}[-1]$ as the first composition factor and so on. (This is in slight contrast to [1] where for example $\mathcal{E}[1]$ is described as the primary part of $\mathcal{E}^A$.) Tensor products of the series above yield composition series for any tractor bundle. For example, it follows easily from elementary representation theory, is that $\mathcal{E}[-m]$ is the first composition factor of $\mathcal{E}_{\underbrace{(AB\cdots C)}_{m}}$. On the other hand beginning from the left hand end of the composition series, the first non-zero part of a given tractor is described as the projecting part.

We may regard the surjection $\mathcal{E}^A\to\mathcal{E}[1]$ as given by projection using a natural section X_A of $\mathcal{E}_A[1]$:

$$U^A \mapsto \sigma = U^A X_A.$$

Then one can show that X_A gives the invariant injection $\mathcal{E}[-1]\to\mathcal{E}_I$ by

$$\rho\mapsto\rho X_A.$$

In any choice of conformal scale, $X_A=(1\ 0\ 0)$ and so clearly is a *null tractor* in the sense that $X^A X_A = 0$.

Given any valence 1 tractor field Y_A, a tractor quantity $T_{IJ\cdots L}$ will be said to be *Y-saturated* if contraction of Y_A into *any index* of $T_{IJ\cdots L}$ results in annihilation. The case of X-saturation is important in respect of the whereabouts of the projecting part. For example if U^A, as in (14), is X-saturated then $\sigma=0$ and if $\mu^a\neq 0$ then it is the projecting part of U^A.

4.1. *Connections and invariant Operators* The bundle $\mathcal{E}^A$ has a natural conformally invariant *tractor connection* ∇ defined by

$$\nabla_b \begin{pmatrix} \sigma \\ \mu^a \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_b \sigma - \mu_b \\ \nabla_b \mu^a + \delta_b{}^a \rho + \mathsf{P}_b{}^a \sigma \\ \nabla_b \rho - \mathsf{P}_{ba} \mu^a \end{pmatrix} \tag{15}$$

for each choice of conformal scale. This determines a connection on the dual bundle $\mathcal{E}_A$, and tensor products in the usual way. Note that this connection preserves the tractor metric (i.e., $\nabla_k h_{IJ} = 0$) and so the raising and lowering indices with h_{IJ} commutes with the action of ∇. The use of the same symbol ∇ as for the Levi–Civita connection is no accident. In fact, more generally, we shall use ∇ to mean the coupled Levi–Civita-tractor connection. This is determined by the condition that it satisfy a Leibniz rule over tensor products of tractor bundles with weighted tensor bundles. For example if $v_a \in \Gamma\mathcal{E}_a$ and $W_B \in \Gamma\mathcal{E}_B$ then $\nabla_a(v_b \otimes W_C) = (\nabla_a v_b) \otimes W_C + v_b \otimes \nabla_a W_C$. The conformal transformation of ∇ on such tensor products follows easily from this definition, its invariance on tractor bundles, and the transformation formulae (1).

For a given choice of conformal scale we define a first order differential operator $\tilde{D}^A \colon \mathcal{E}^*[w] \to \mathcal{E}^A \otimes \mathcal{E}^*[w-1]$ by

$$\tilde{D}^A f := \begin{pmatrix} wf \\ \nabla^a f \\ 0 \end{pmatrix},$$

where $\mathcal{E}^*[w]$ indicates a tractor bundle of arbitrary valence and weight w. Note that

$$X^A \tilde{D}_A f = wf. \tag{16}$$

The operator $\tilde{D}_A$ is not invariant and under change of conformal scale

$$\widehat{\tilde{D}_A f} = \tilde{D}_A f + X_A \left(\Upsilon^i \nabla_i f + \frac{w}{2} \Upsilon^i \Upsilon_i f \right).$$

The main importance of $\tilde{D}_A$ lies in the next proposition which follows immediately.

Proposition 4.1. *The operator defined by*

$$D_{AP} := 2X_{[P} \tilde{D}_{A]} \tag{17}$$

is invariant on weighted tractor bundles.

Observe that the operator $\mathscr{E}[w] \to \mathscr{E}[w]$ given by $f \mapsto wf$ satisfies a Leibniz rule in the sense that if $f_1 \in \Gamma\mathscr{E}[w_1]$ and $f_2 \in \Gamma\mathscr{E}[w_2]$ then $f_1 f_2 \mapsto (w_1 + w_2) f_1 f_2 = (w_1 f_1) f_2 + f_1 (w_2 f_2)$. It follows that both $\tilde{D}_A$ and D_{AP} satisfy a Leibniz rule for tensor products of arbitrary weighted tractor bundles and $\mathscr{E}[w]$. Note also that

$$\tilde{D}_A h_{BC} = 0$$

so that index raising and lowering commutes with the operators $\tilde{D}_A$ and D_{AP}. (It turns out that a variation of the operator D_{AP} can be vastly generalised. This is called the fundamental D operator and is described and developed in [5, 6].)

There are other non-invariant tractor quantities which are very useful for calculations. For each choice of conformal scale ξ, define ξ_A by

$$\xi_A := \xi^{-1} \tilde{D}_A \xi.$$

Then,

$$X^A \xi_A = 1.$$

Furthermore, recall that $\nabla_a \xi = 0$, so

$$\xi_A = (0 \;\; 0 \;\; 1) \tag{18}$$

and so ξ_A is null and

$$\xi^A \tilde{D}_A f = 0 \tag{19}$$

for f any weighted tractor. Under change of conformal scale, $\xi \to \Omega^{-1}\xi$, ξ_A transforms to

$$\hat{\xi}_A = \xi_A - \Upsilon_A - \tfrac{1}{2} X_A \Upsilon_B \Upsilon^B,$$

where

$$\Upsilon_A = \Omega^{-1} \tilde{D}_A \Omega.$$

The operator $\tilde{D}$ is easily recovered from D_{AP},

$$\tilde{D}_A f = \xi^P D_{AP} f, \tag{20}$$

for f any weighted tractor field. For later use, note also that

$$\tilde{D}_B X^A = \delta_B^A - \xi^A X_B, \tag{21}$$

and so

$$D_{AP}X_B = 2X_{[P}h_{A]B}. \tag{22}$$

Given a choice of conformal scale, it is often useful to identify $\mathscr{E}_a$ with the sub-sheaf of $\mathscr{E}_A[-1]$ consisting of co-tractors which are both X-saturated and ξ-saturated. So we write

$$\mathscr{E}_a = \ker((\xi^I, X^I)\colon \mathscr{E}_A[-1] \to \mathscr{E}[-2] \oplus \mathscr{E}).$$

Thus we represent a 1-form w_a by a (weight -1) tractor field w_A,

$$w_A = (0 \;\; w_a \;\; 0),$$

and in this case write $w_a = w_A$. We call w_A the tractor expression for w_a. This extends in an obvious way to tensor products and we make the following definition.

Definition 4.2 [Tractor Expression]. Given a choice of ξ, the tractor expression for a tensor field $T_{ab\cdots d}$ is the unique tractor field $T_{AB\cdots D}$ which is ξ^A-saturated and has $T_{ab\cdots d}$ as projecting part (and so $T_{AB\cdots D}$ is also X^A-saturated). We will often identify tensors with their tractor expressions.

We will usually denote the tractor expression for a tensor by the same kernel symbol as used for the tensor (although in some instances adorning this with a tilde to distinguish it from conformally invariant tractors). So for example, with a conformal scale fixed, $\varepsilon_{AB\cdots D}$ is the tractor expression for the conformal volume form and g_{AB} is the tractor expression for the conformal metric. Note that

$$g_{AB} = h_{AB} - 2\xi_{(A}X_{B)}. \tag{23}$$

The tractor expression for a tensor depends on the choice of conformal scale. However using this it is easy to produce an invariant tractor field that contains the same information. For example for an arbitrary valence v tensor field $T_{ab\cdots d}$, with $T_{AB\cdots D}$ its tractor expression, an invariant tractor field associated with T is given by

$$T_{APBQ\cdots DS} := 2^v \text{ pair skew}(\underbrace{X_P X_Q \cdots X_S}_{v} T_{AB\cdots D}),$$

where by "pair skew" it is meant to simultaneously take the skew part over each of the index pairs AP, BQ, ..., DS.

If a tensor has symmetries then, often, there are more economical ways of forming an associated invariant tractor. For example, in view of the symmetries $C_{bdef} = C_{[bd][ef]}$ of the Weyl tensor we may form the invariant tractor object

$$C_{ABDEFG} := 9X_{[A}\tilde{C}_{BD][EF}X_{G]},$$

where $\tilde{C}_{BDEF}$ is the tractor expression for the Weyl curvature C_{bdef}. For the special case of the Weyl tensor we will deem this to be the *lifted expression* of the Weyl tensor. Similarly the Cotton–York tensor has the symmetries $C_{bde} = C_{[bd]\,e}$ and the lifted expression for the Cotton–York tensor will be taken as

$$C_{ABDEF} := 6X_{[A}\tilde{C}_{BD][E}X_{F]},$$

where $\tilde{C}_{BDE}$ is the tractor expression for the Cotton–York tensor.

Another important tractor field is the conformally invariant *canonical tractor form*. This is the unique completely skew $(n+2)$-tractor $\eta_{IJA\cdots D} \in \Gamma\mathscr{E}_{[IJA\cdots D]}$ determined by

$$X^I\xi^J\eta_{IJAB\cdots D} = \varepsilon_{AB\cdots D}$$

for each choice of conformal scale. Equivalently

$$\eta_{IJAB\cdots D} = (n+1)(n+2)\; X_{[I}\xi_B\varepsilon_{AB\cdots D]}$$

from which its conformal invariance is easily seen since, under a change of conformal scale, the transformation of $\varepsilon_{AB\cdots D}$ is of the form

$$\varepsilon_{AB\cdots D} \mapsto \hat{\varepsilon}_{AB\cdots D} = \varepsilon_{AB\cdots D} + X_{[A}\gamma_{B\cdots D]}.$$

A short exercise reveals that

$$\tilde{D}_A\eta_{BC\cdots G} = 0.$$

Using the observation (18) we can recover the tractor connection from $\tilde{D}_A$ as

$$\nabla_a f = \tilde{D}_A f - w\xi_A f \tag{24}$$

for any weighted tractor field f. For example $\tilde{D}_A\xi_B + \xi_A\xi_B = \nabla_a\xi_B$. If we now use (15) to re-express the right hand side of this we obtain the equation

$$\tilde{D}_A\xi_B + \xi_A\xi_B = \mathsf{P}_{AB}$$

where the rho-tractor P_{AB} is the tractor expression for P_{ab}.

We can similarly give tractor formulae for the Levi–Civita connection corresponding to ξ. For example,

$$\begin{aligned}\nabla_a\phi &= \tilde{D}_A\phi - w\xi_A\phi\\ \nabla_a\phi_b &= \tilde{D}_A\phi_B + \xi_A\phi_B + \xi_B\phi_A + X_B \mathsf{P}_A{}^K\phi_K\\ \nabla_a\phi^b &= \tilde{D}_A\phi^B - \xi_A\phi^B + \xi^B\phi_A + X^B\mathsf{P}_A{}^K\phi_K,\end{aligned}\tag{25}$$

where $\phi\in\mathscr{E}[w]$ and ϕ_A is the tractor expression for $\phi_a\in\mathscr{E}_a$. The first of (25) is just (24) restricted to $\mathscr{E}[w]$. To obtain the second observe that, since $\phi_B = (0\ \phi_b\ 0)$, (15) gives

$$\nabla_a\phi_b = \nabla_a\phi_B + \xi_B\phi_A + X_B\mathsf{P}_{AK}\phi^K$$

and combining this with (24) gives the result. The last of (25) is obtained similarly or, alternatively, by raising an index in the second equation using $g^{ab}_{(\xi)} = \xi^2(h^{AB} - 2\xi^{(A}X^{B)})$. The tractor formula for ∇ acting on an arbitrary weighted tensor is easily obtained from the formulae (25) using the Leibniz rule.

The invariant operator D_{AP} can be used to construct other invariant operators and objects. As an elementary example of this consider $h^{AB}D_{A(Q}D_{|B|P)_0}f$ for f some weighted tractor. Expanding this out using the Definition (17) of D_{AP} we see that it may be re-expressed in the form

$$h^{AB}D_{A(Q}D_{|B|P)_0}f = -X_{(Q}D_{P)_0}f,$$

where D_P is some differential operator. We can deduce $D_P f$ is conformally invariant since the left-hand-side is invariant, X_Q is invariant and for any tractor field V_P, $X_{(Q}V_{P)_0} = 0$ only if $V_P = 0$ (see the next section). An explicit formula for D_P is easily extracted from this Definition using (21) and (20),

$$D_A f = (n+2w-2)\,\tilde{D}_A f - X_A \,\Box\, f,\tag{26}$$

where

$$\Box f := \tilde{D}_P\tilde{D}^P f = \nabla_p\nabla^p f + w\mathsf{P}f.$$

From this formula it is easily verified that D_A is in fact precisely the "D-operator" in [1]. Note the useful identities

$$X^A D_A f = w(n+2w-2)\,f,\tag{27}$$

$$D_A X^A f = (n+2w+2)(n+w)\,f,\tag{28}$$

and

$$[D_A, X_B]\, f = (n+2w)\, h_{AB} + 2D_{AB}\, f \tag{29}$$

for f a weighted tractor field of weight w.

4.2. *Curvature.* The *tractor curvature* $\Omega_{ab}{}^K{}_L$ of ∇ on $\mathcal{E}^K$ is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\, U^K = \Omega_{ab}{}^K{}_L U^L.$$

The tractor curvature is precisely the obstruction to a manifold being locally equivalent to a (conformally) flat conformal manifold and we say the tractor connection is *flat* if $\Omega_{ab}{}^K{}_L = 0$. Note that $\Omega_{abKL} = \Omega_{ab[KL]}$ and in a choice of conformal scale $\Omega_{ab}{}^K{}_L$ is represented by

$$\begin{pmatrix} 0 & 0 & 0 \\ 2\nabla_{[a} P_{b]}{}^k & C_{ab}{}^k{}_l & 0 \\ 0 & -2\nabla_{[a} P_{b]l} & 0 \end{pmatrix}.$$

It follows easily from the Jacobi identity and the Bianchi symmetry that

$$\nabla_{[a} \Omega_{bc]\,DE} = 0.$$

The tractor expression for the tractor curvature is denoted $\Omega_{AB}{}^K{}_L$ and we note that since, $\Omega_{AB}{}^K{}_L = \Omega_{[AB]}{}^K{}_L$, in any choice of scale this tractor quantity may be recovered in the obvious way from the invariant lifted expression

$$X_{[A} \Omega_{BC]}{}^K{}_L.$$

Observe that it is clear from the display above and the Definition of the tractor expression Ω_{ABCD} that it is X-saturated and trace-free. We make the definition

$$W_{AB}{}^K{}_L := \frac{3}{n-2}\, D^P X_{[P} \Omega_{AB]}{}^K{}_L.$$

Clearly this has the symmetry $W_{ABCD} = W_{[AB][CD]}$ and can be verified to be trace-free. One easily obtains that

$$W_{AB}{}^K{}_L = (n-4)\, \Omega_{AB}{}^K{}_L + 2X_{[A} \tilde{D}^P \Omega_{B]P}{}^K{}_L,$$

so, for $n \geqslant 5$, $W_{AB}{}^K{}_L$ invariantly extends $C_{ab}{}^k{}_l$ to a tractor. It also follows from this, (21) and the properties of Ω_{ABCD} that W_{ABCD} is X-saturated.

If $v^K \in \mathcal{E}^K[w]$ then it is straightforward to show that

$$[D_A, D_B]\, v^K = (n-2)[W_{AB}{}^K{}_L v^L + 6X_{[A}\Omega_{BP]}{}^K{}_L \tilde{D}^P v^L], \tag{30}$$

and the operators D_A commute amongst themselves if the tractor connection is flat.

5. CONSTRUCTING INVARIANTS AND THE MAIN THEOREM

Here we will employ the tractor calculus, described and developed above, to manufacture invariants. Our main interest will be in the quasi-Weyl invariants introduced in Section 5.2. However first we describe some more obvious approaches.

5.1. *Weyl invariants and generalised Weyl invariants.* Since D_P and W_{ABCD} are conformally invariant it is clear that one can use these to construct many invariants. For example

$$W^{ABCD}W_{ABCD} \qquad \text{and} \qquad D_I D_J (W^{ABIC}W_{AB}{}^J{}_C)$$

are invariants in all dimensions and are non-trivial in most dimensions (see [15] for a discussion of the second of these). Invariants such as these will be termed Weyl invariants.

Definition 5.1 [Weyl invariant]. Any complete contraction of a juxtaposition of tensor powers of $X_I \eta^{IJAB\cdots E}$, h^{AB}, W_{ABCD}, and various powers of the operator D_A acting on these ingredients and their juxtapositions is an invariant. Any linear combination of such invariants is called a Weyl invariant.

The term "Weyl invariant" here is borrowed from [2] where it is used for certain closely related algebraic invariants.

Of course one may also use the operator D_{AP} to form invariants. Let $f \in \Gamma\mathcal{E}[w]$. We have the coupled invariant, $D^A D^B h^{PQ}(D_{AP} f)\, D_{BQ} f = (n+2w-2)(n+4w)(n+2w-1)[(n+2w-2)\, \nabla^a f \nabla_a f - 2wf(\nabla^a \nabla_a f + w\mathsf{P} f)]$. The invariance of the left hand side is immediate from the invariance and definitions of the operators D_A and D_{AP}. The expansion on the right hand side demonstrates that the result is, in general, non-trivial. (More precisely it is non-trivial if $w \neq -\frac{n}{4}, \frac{1-n}{2}, \frac{2-n}{2}$.) By substituting, for example, $f = W^{ABCD}W_{ABCD}$ or $f = D_I D_J (W^{ABIC}W_{AB}{}^J{}_C)$ (from above), into this one obtains further invariants. It is clear that any complete contraction of a juxtaposition of tensor powers of $X_I \eta^{IJAB\cdots E}$, h^{AB}, W_{ABCD}, X_A and various powers of the operators D_A and D_{AP} acting on these ingredients and their juxtapositions is an invariant. (The required polynomial nature in the jets of the metric and its inverse is immediate from the definitions of these objects.) Let us call such invariants and their linear combinations *generalised Weyl invariants*.

An important feature of Weyl invariants and generalised Weyl invariants is that one can list a countable basic set of such invariants such that all invariants of this type (i.e., respectively Weyl or generalised Weyl) are linear combinations of this basic set. Furthermore it is clear the basic list is finite if we are only interested in invariants between specified finite weights. Thus it would be ideal if it turned out that all invariants could be shown to be either Weyl invariants or at least generalised Weyl invariants. While this is the case for odd dimensional structures (see Section 8 below) it is not the case for even dimensional structures. For example, in dimension 4, the invariant $C_{abcd}C^{abcd}$ is not a generalised Weyl invariant.

5.2. *Quasi-Weyl Invariants.* Here we describe another category of invariants which arise as linear combinations of a basic set that we can simply list without performing calculations. Again the Definition of these is based on their construction and the first step in this is to form a "weak expression" associated to any chosen metric invariant.

DEFINITION 5.2 [Weak Expression]. Let E be a metric invariant of principal degree d_0 and principal order k_0 (i.e., so $u=-(2d_0+k_0)$. We will formally associate with E a symmetric trace-free tractor

$$E_{\underbrace{PQ\cdots T}_{m}}, \tag{31}$$

as follows:

First the $n \geqslant 4$ case: If E is even then, in a normal expression $E_{(d_0)}$ for $[E]_{d_0}$, formally replace each C_{bcde} with its lifted expression C_{ABCDEF}, replace each ∇_a with D_{AP} and replace each g^{ab} with h^{AB}. Finally take the symmetric trace-free part. If E is odd carry out this procedure exactly as for even invariants except also replace $\varepsilon^{ab\cdots e}$ with $X_P\eta_Q^{P\,AB\cdots E}$. In either case the use of upper case indices corresponding to lower case ones (e.g., A corresponds to a) is judicious as each index contraction in E is to determine a corresponding index contraction in $E_{PQ\cdots T}$. In each case m is well defined and if E is odd then $m=k_0+2d_0+1$ in (31) otherwise $m=k_0+2d_0$. For all cases the result $E_{PQ\cdots T}$ is called a <u>weak expression</u> associated with E.

The $n=3$ case: In dimension 3 we proceed as above except that in this case it is the Cotton–York tensor C_{abc} (rather than the Weyl tensor) that we must replace by its lifted expression. In this case, if E is odd $m=d_0+k_0+1$, while if E is even then $m=d_0+k_0$.

Note that the weak expression depends only on the principal part $E_{(d_0)}$ of a normal expression for E. Thus for the task of listing possible weak expressions one only need to list metric invariants of well defined degree and in normal form. On the other hand for the theorems below it is important to be able to deal with more general metric invariants.

Observe that for any metric invariant E, with weak expression $E_{PQ\cdots T}$,

$$D^P D^Q \cdots D^T E_{PQ\cdots T}$$

is an invariant. (In dimensions greater than 4 it is easily demonstrated that this is a generalised Weyl invariant.) This provides a systematic way of building invariants associated with expressions for metric invariants. However there is an inefficiency in the construction which is easily eliminated. To see how this works let us digress for a moment.

For $f \in \mathscr{E}(1-\frac{n}{2})$ consider the coupled invariant

$$D^P D^Q g^{AB} f^2 D_{A(P} D_{|B|Q)_0} f = (2-n)^2 (3-n)(4-n) f^2 \Box f, \tag{32}$$

where, recall,

$$\Box f := \nabla_a \nabla^a f + \left(1 - \frac{n}{2}\right) \mathsf{P} f.$$

Note that (32) vanishes in dimension 3 and 4. We can improve on this result. By an easy calculation one obtains that, in any dimension,

$$g^{AB} f^2 D_{A(P} D_{|B|Q)_0} f = X_P X_Q f^2 \Box f.$$

So $f^2 \Box f$ is an invariant for all n. However this last result depends crucially on the fact that, for each n, the weight of f is $1-n/2$. In contrast using (22) one can show that

$$g^{AB} f^2 D_{A(P} D_{|B|Q)_0} f = X_{(Q} J_{P)_0} \tag{33}$$

for f *of any weight*. Since we know in advance that the left hand side has this form we may as well "remove" the X_Q and form $D^P J_P$. Now for $f \in \mathscr{E}(1-\frac{n}{2})$ we have

$$D^P J_P = (2-n)^2 f^2 \Box f$$

which compares favourably to (32). The observation (33) which allowed this improvement is typical of the general result that we wish to exploit. Thus we make the following definition.

DEFINITION 5.3. Suppose E is a metric invariant and $E_{PQ\cdots W}$ an associated weak expression. If this can be rearranged into the form

$$E_{PQ\cdots W} = \underbrace{X_{(P}X_{Q}\cdots X_{S}}_{x}\, E'_{TU\cdots W)_0} \tag{34}$$

then $E'_{TU\cdots W}$ will be termed a reduced weak expression for E.

Note that tractor fields of the form (34) form a linear subspace of the space of symmetric trace-free tractor fields. In fact this subspace is isomorphic to the space of symmetric trace-free tractors with the same valence as E' (and of the appropriate weight). To see this suppose $V_{TU\cdots W}$ is symmetric, trace-free and satisfies $X_{(P}X_{Q}\cdots X_{S}V_{TU\cdots W)_0}=0$. We have $\xi^P\xi^Q\cdots\xi^W X_{(P}X_{Q}\cdots X_{S}V_{TU\cdots W)_0}=0$ for ξ_A corresponding to any choice of scale ξ. Using that $X^A\xi_A=1$ and that the tractor section $\xi^P\xi^Q\cdots\xi^W$ is symmetric and trace-free it follows that $\xi^T\xi^U\cdots\xi^W V_{TU\cdots W}=0$ for ξ_A corresponding to any choice of scale ξ. From this we can conclude that $\chi^T\chi^U\cdots\chi^W V_{TU\cdots W}=0$ for any null tractor field χ^A such that $\chi^A X_A\neq 0$. But then by a continuity argument (c.f. the Proof of theorem 6.3) one may drop the requirement $\chi^A X_A\neq 0$ and conclude that the vanishing of $\xi^P\xi^Q\cdots\xi^W X_{(P}X_{Q}\cdots X_{S}V_{TU\cdots W)_0}$ for all scales ξ implies that the trace-free symmetric tractor field $V_{TU\cdots W}$ vanishes and so the isomorphism is established. Using this and the conformal invariance of the weak expression $E_{PQ\cdots W}$ it follows that a reduced weak expression $E'_{TU\cdots W}$ for E is conformally invariant. So $D^T D^U\cdots D^W E'_{TU\cdots W}$ is an invariant that we may associate with E.

At this point it is worthwhile observing that once we know the number x in (34) it is straightforward to write an explicit formula giving the mapping from E to E'. For example if

$$B_{PA} = X_{(P}B'_{A)_0}$$

then

$$B'_C = 2\xi^P\left(\delta^A_C - \frac{1}{2}X_C\xi^A + \frac{2}{n}\xi_C X^A\right)B_{PA}.$$

(Note that $2\xi^P(\delta^A_C-\frac{1}{2}X_C\xi^A+\frac{2}{n}\xi_C X^A)$ is not conformally invariant but it is conformally invariant as a mapping on the subspace of $\Gamma\mathcal{E}_{(PA)_0}[w]$ consisting of sections of the form $X_{(P}B'_{A)_0}$.) We will write $B'_C = S_C{}^{DE}B_{DE}$ to indicate this mapping. More generally we write $S_{A\cdots D}{}^{P\cdots W}$ for the section of $\mathcal{E}_{A\cdots D}{}^{P\cdots W}[m'-m]$ giving the 1-1 mapping

$$S_{\underbrace{AB\cdots D}_{m'}}{}^{\overbrace{PQ\cdots STU\cdots W}^{m}}: X_{(P}X_{Q}\cdots X_{S}E_{TU\cdots W)_0}\mapsto E_{AB\cdots D},$$

where the integers m, m' satisfy $0 \leqslant m' \leqslant m$. We will not concern ourselves here with the details of an explicit formula for this. It is an easy exercise for the reader to produce such a formula in terms of ξ^A, X^B and the tractor Kronecker delta δ_B^A in the case that $m' = m-1$ (which is of course sufficient). The existence of such a formula is useful for general discussions. In practice one need never use any formula for $S_{A\cdots D}{}^{P\cdots W}$ since one can use the standard rules of calculus (as applied in the proof of the proposition below) to manipulate a weak expression into the form on the right hand side of (34) at which point the weak expression can be simply read off.

Of course for metric invariants of a given principal degree d_0 and principal order k_0, the number x will in general depend on the particular metric invariant involved. However, given m as in the Definition 5.2 above, the next proposition puts a lower bound on the value of x. The proof of this is the subject of Section 7.

Proposition 5.4. *Suppose E is a metric invariant of principal degree d_0 and principal order k_0. Let $E_{PQ\cdots V}$ be a weak expression associated with E. Then*

$$E_{PQ\cdots W} = X_{(P} X_Q \cdots X_S E'_{\underbrace{TU\cdots W}_{m'})_0}$$

where, m' is given as follows:

if E is even, then

- *in dimensions $n \geqslant 4$, $m' \leqslant \min(k_0, (2d_0 + k_0)/2)$ and*
- *in dimension $n = 3$, $m' = \min(k_0 - d_0, [(d_0 + k_0)/2])$;*

if E is odd, then

- *in dimensions $n \geqslant 4$, $m' \leqslant 1 + \min(k_0, [(2d_0 + k_0)/2])$ and*
- *in dimension $n = 3$, $m' = 1 + \min(k_0 - d_0, [(d_0 + k_0)/2], (2d_0 + k_0 - 3)/2])$.*

Here $[a]$ is the integral part of a. This proposition leads to the following definition.

Definition 5.5 [Quasi-Weyl invariants]. For each normal expression for a metric invariant E of principal degree d_0 and total order k_0, we can construct a conformal invariant by simply forming

$$D^T D^U \cdots D^W (S_{\underbrace{TU\cdots W}_{m'}}{}^{AB\cdots H} E_{AB\cdots H}), \tag{35}$$

where $E_{AB\cdots H}$ is a weak expression for E and m' is given by the proposition above. We call any invariant which arises this way a quasi-Weyl invariant (or a q-Weyl invariant as an abbreviation).

Note that (35) gives an explicit formula for an invariant. Some other remarks are in order. It is not difficult to show that, up to the addition of invariants of degree d_0+1, two normal expressions for the same metric invariant E yield the same q-Weyl invariant. (This follows from an easily established generalisation of Theorem 6.8 which treats the difference between two normal expressions for E as a degree d_0 expression for the trivial invariant.) Thus we may speak loosely of the q-Weyl invariant E^q associated to a metric invariant E. Observe that if E and F are metric invariants, both sharing the same principal degree d_0 and principal total order k_0, then the formal sum of normal expressions for each of these is a normal expression for the sum (metric) invariant $E+F$. Furthermore it is clear from the Definition above that the q-Weyl invariant arising from this expression is the sum of the two q-Weyl invariants arising from the given normal expressions for E and F. Thus the set of q-Weyl invariants of a given principal degree and total order, d_0 and k_0, is closed under linear combinations. Since, as remarked above, the normal expression we choose for a given metric invariant E is not really important in terms of the q-Weyl invariant it determines, a finite linearly generating set of such invariants (at least modulo invariants of principal degree d_0+1) is given by the set of quasi-Weyl invariants $\{E^q\}$ corresponding to a (finite) linearly generating set $\{E\}$ for the metric invariants of degree d_0 and total order k_0. For each pair d_0, k_0 one can list the set $\{E\}$ very quickly using classical theory. Of course distinct metric invariants will in many cases yield the same q-Weyl invariant but the important point here is that the overall the task of listing a spanning set of q-Weyl invariants is at worst equivalent to the classical task of listing a spanning set of metric invariants. (This is similar to the situation in [2], for example.)

The following theorem asserts that almost all invariants are q-Weyl invariants and this is the main result of this paper.

THEOREM 5.5. *An invariant I, of principal degree d_0 and principal total order k_0 (and weight $u=-(2d_0+k_0)$), is quasi-Weyl if one of the following holds.*

- $n=3$
- *I is an even invariant and at least one of the following three conditions is satisfied:*

1. $\max(|u|, 4d_0) > 2n-2$,
2. *n is odd and either $k_0 = 0$ or $|u| < n$,* (36)
3. *n is even and either $k_0 = 0$ or $|u| < \frac{n+2}{2}$ or $|u| < n$ and $4d_0 > n$.*

or

- *I is odd and either $d_0 > \frac{n}{2}$ or $d_0 = \frac{n}{2}$ and $k_0 > n$.*

Proof of Theorem 5.5. Suppose that I is an invariant satisfying one of the conditions in the theorem. Let $I_{AB\cdots H}$ be a weak expression associated with I. From Theorem 6.8 and Proposition 5.4 it follows that there is a reduced weak expression

$$I'_{\underbrace{TU\cdots W}_{m'}} = S_{TU\cdots W}{}^{AB\cdots H} I_{AB\cdots H},$$

where

$$m' \leqslant \begin{cases} \min\left(k_0, \dfrac{2d_0+k_0}{2}\right) & \text{if } I \text{ is even and } n \geqslant 4 \\ 1+\min\left(k_0, \left[\dfrac{2d_0+k_0}{2}\right]\right) & \text{if } I \text{ is odd and } n \geqslant 4 \\ k_0 - d_0 & \text{if } I \text{ is even and } n = 3 \\ 1+k_0-d_0 & \text{if } I \text{ is odd and } n = 3, \end{cases}$$

and

$$I'_{TU\cdots W} = X_T X_U \cdots X_W I + J_{TU\cdots W},$$

where the conformally invariant tractor field $J_{TU\cdots W}$ consists of terms of degree $\geqslant d_0+1$).

Suppose $m' = 0$. Then I is a linear combination of the quasi-Weyl invariant I' and the invariant J of principal degree d_0+1. Since J also has weight u it is easily verified that it satisfies one of the conditions (36). Now suppose $m' \geqslant 1$. Since one of the conditions (36) holds, then (see remark below)

$$I \prod_{i=1}^{m'} (n+2(u+i))(n+u+i-1) \neq 0$$

and so it follows from (28) that I is a linear combination of the quasi-Weyl invariant $D^T D^U \cdots D^W I'_{TU\cdots W}$ and the invariant $J = D^T D^U \cdots D^W J_{TU\cdots W}$ which is of principal degree d_0+1. Again J has weight u and it is easily verified that it satisfies one of the conditions (36). Since, in either case, J

also satisfies one of the conditions (36), by the same reasoning we can conclude that it in turn is a linear combination of a quasi-Weyl invariant and an invariant of degree d_0+2. Continuing this process (which, in view of (10), must stop after a finite number of steps) we obtain that I is a linear combination of quasi-Weyl invariants. ∎

Remark 5.7. We consider here, in the context of the above proof, the product, $\prod_{i=1}^{m'}(n+2(u+i))(n+u+i-1)$ for $m' \geqslant 1$. It is useful to regard this as the product of $2^{m'}$ with the two sub-products $\prod_{i=1}^{m'}(\frac{n}{2}+u+i)$ and $\prod_{i=1}^{m'}(n+u+i-1)$. Each of the latter two is a product of terms increasing by one. Observe also that, since $n \geqslant 3$,

$$\left(\frac{n}{2}+u+i\right) < (n+u+i-1) \text{ for all } i.$$

It follows, for instance, that in even dimensions if either $0 < (\frac{n}{2}+u+1)$ or $(n+u+m'-1) < 0$ then all terms in these products are non-zero. Observe that in odd dimensions $\prod_{i=1}^{m'}(\frac{n}{2}+u+i) \neq 0$ as u is integral for structure invariants. Thus in this case $\prod_{i=1}^{m'}(n+2(u+i))(n+u+i-1) \neq 0$ if either $0 < (n+u)$ or $(n+u+m'-1) < 0$. These observations plus the observation that $\prod_{i=1}^{m'}(n+2(u+i))(n+u+i-1) \neq 0$ if $(\frac{n}{2}+u+m') < 0 < (n+u)$ lead us to the conditions (36).

6. THEOREM 6.8 AND LINEARISED CURVATURE

The only result we will finally need from this section is Theorem 6.8 which was one of the two main ingredients for the proof of the main Theorem 5.6 above. Theorem 6.8 in turn follows from two key observations. The first is that any invariant determines a corresponding invariant of a "linearised curvature" field on conformally flat structures. This is the content of Theorem 6.7. The second is the powerful result that any such invariant turns up as a quotient as in Theorem 6.3. This is the linearised curvature analogue of Theorem 6.8.

6.1. *The Flat Case.* Here we will be concerned with invariants of certain special tensor fields on conformally flat geometries. On the conformally flat structures there is a natural preferred class of scales, namely those ξ such that $\mathsf{P}_{ab} = 0$. A metric corresponding to a choice of scale from this class is flat as a Riemannian or pseudo-Riemannian metric. In this section whenever we make a choice of scale it will be from this class and note that in this case

$$\tilde{D}_I \xi_B = -\xi_I \xi_B. \tag{37}$$

In each dimension, the sheaf of tensor fields we are interested in will be denoted $\mathscr{U}$ and its sections play the role of linearised Weyl curvature (or a linearised Cotton–York tensor if $n=3$). If $n \geqslant 4$ then $\mathscr{U}$ will mean the sheaf of local sections of trace-free tensor fields $U_{abc}{}^{d} \in \mathscr{E}_{abc}{}^{d}[0]$ with the following additional symmetries and properties:

$$\begin{aligned} U_{abcd} &= U_{[ab][cd]} \\ U_{[abc]\,d} &= 0 \\ (n-3)\,\nabla_{[a}U_{bc]\,de} &= g_{d[a}\nabla_{|s|}U_{bc]}{}^{s}{}_{e} - g_{e[a}\nabla_{|s|}U_{bc]}{}^{s}{}_{d}. \end{aligned} \tag{38}$$

(Note that the last of these is invariant under the transformations (1) and is a vacuous condition in dimension 4.) For M of dimension $n=3$ we take $\mathscr{U}$ to be local sections of trace-free tensor fields $U_{abc} \in \mathscr{E}_{[ab]\,c}[0]$ such that

$$\begin{aligned} U_{[abc]} &= 0 \\ \nabla_{[a}U_{bc]\,d} &= 0. \end{aligned}$$

In each case it is easy to check that there exist non-trivial tensor fields in $\mathscr{U}$. For example if $w_{ab} \in \mathscr{E}_{(ab)_0}(2)$ then, in each dimension $n \geqslant 4$,

$$\begin{aligned} U_{bc}{}^{de} = {} & 4(n-1)(n-2)\,\nabla_{[b}\nabla^{[d}w_{c]}{}^{e]} - 4(n-1)\,\delta^{[d}_{[b}\Delta w_{c]}{}^{e]} \\ & + 4(n-1)\,\delta^{[d}_{[b}\nabla^{e]}\nabla^{q}w_{c]\,q} + 4(n-1)\,\delta^{[d}_{[b}\nabla_{c]}\nabla_{q}w^{e]\,q} - 4\delta^{d}_{[b}\delta^{e}_{c]}\nabla^{p}\nabla^{q}w_{pq} \end{aligned}$$

provides a solution. For $n \geqslant 5$ all solutions are locally of this form. In dimension 3 solutions are locally of the form

$$U_{bc}{}^{d} = 2\Delta\nabla_{[b}w_{c]}{}^{d} - 2\nabla^{d}\nabla_{q}\nabla_{[b}w_{c]}{}^{q} - \delta^{d}_{[b}\nabla_{c]}\nabla^{p}\nabla^{q}w_{pq},$$

where, here also, $w_{ab} \in \mathscr{E}_{(ab)_0}(2)$. We will use the notation $U_{a\cdots d}$ to mean either U_{abcd} or U_{abd} according to the context and extend this ambiguous notation in an obvious way to the tractor expressions for these tensors.

Since we will discuss invariants of $\mathscr{U}$ we need some definitions.

DEFINITION 6.1. A metric invariant of $\mathscr{U}$ is a polynomial $E(\mathscr{U}, \mathbf{g})$ in the variables $\partial_i\partial_j\cdots\partial_k\mathbf{g}_{lm}$, $(\det \mathbf{g}_{ij})^{-1}$ and $\partial_i\partial_j\cdots\partial_k U_{lmp}{}^{q}$ (or $\partial_i\partial_j\cdots\partial_k U_{lmp}$ if $n=3$) which is independent of the coordinate system $\{x^i\}$ used. (Here $U_{lmp}{}^{q}$, U_{lmp} and $\mathbf{g}_{ij}$ indicate components in the coordinates $\{x^i\}$ and $\partial_i := \partial/\partial x^i$.)

Given this we have the notion of (conformal) invariants:

DEFINITION 6.2. A (conformal) invariant of $\mathscr{U}$ of weight u is a metric invariant of $\mathscr{U}$, $I(\mathscr{U}, \mathbf{g})$, such that $I(\mathscr{U}, \Omega^2\mathbf{g}) = \Omega^u I(\mathscr{U}, \mathbf{g})$ for any smooth positive function Ω.

The invariants $\mathcal{U}$ are just a special class of coupled invariants restricted to conformally flat structures. From classical invariant theory it follows that, in each choice of scale (from the preferred class so that the metric is flat), metric invariants of $\mathcal{U}$ can be expressed as linear combinations of complete contractions of the form

$$E = E_{\text{even}} + E_{\text{odd}},$$

where E_{even} is a linear combination of complete contractions of the form

$$\operatorname{contr}(g^{-1} \otimes \cdots \otimes g^{-1} \otimes g^{-1} \otimes U^{(k_1)} \otimes U^{(k_2)} \otimes \cdots \otimes U^{(k_d)}),$$

and E_{odd} is a linear combination of complete contractions of the form

$$\operatorname{contr}(\varepsilon \otimes g^{-1} \otimes \cdots \otimes g^{-1} \otimes g^{-1} \otimes U^{(k_1)} \otimes U^{(k_2)} \otimes \cdots \otimes U^{(k_d)}),$$

where, for $n \geqslant 4$, U^{ℓ} means the ℓth ∇-derivative of U_{abcd}, while, for $n = 3$, U^{ℓ} means the $(\ell - 1)$th ∇-derivative of U_{abc}. In each case we say that the expression is of *degree* d and refer to $k := \sum_1^d k_i$ as the *total order*. A metric invariant of $\mathcal{U}$, E is said to be *even* if $E = E_{\text{even}}$ while E is said to be *odd* if $E = E_{\text{odd}}$. A metric invariant of $\mathcal{U}$ expressed as a linear combination of complete contractions, as above, will be said to be in *standard form*.

Since in the conformally flat case the transformations (1) determine a linear transformation of the U^{ℓ}, any invariant of $\mathcal{U}$ is a linear combination of invariants of well defined degree d in the section of $\mathcal{U}$. For our purposes there will be no loss of generality in dealing only with invariants and metric invariants of $\mathcal{U}$ which are either even or odd and with well defined degree d in $\mathcal{U}$ and well defined total order k. In this case, for an invariant I of weight u, we have $u = -(2d + k)$ where k is the *total order* of I. If E is a metric invariant of $\mathcal{U}$ we can take $u := -(2d + k)$ as a definition of its weight.

For $n \geqslant 4$ we define the *lifted expression* for U_{bcde} to be

$$U_{ABCDEF} := 9X_{[A}U_{BC][DE}X_{F]}$$

and (for $n = 3$) we define the *lifted expression* for U_{bcd} to be

$$U_{ABCDE} := 6X_{[A}U_{BC][D}X_{E]}.$$

Now beginning with metric invariants of $\mathcal{U}$, *weak expressions* for such invariants can be constructed in the same way as for the curvature. That is given given a metric invariant E of $\mathcal{U}$, expressed in standard form and of total order k and degree d, one follows the prescription of Definition (5.2) except that in that Definition C_{abc}, C_{abcd}, C_{ABC}, C_{ABCD}, C_{ABCDE} and C_{ABCDEF} are replaced by U_{abc}, U_{abcd}, U_{ABC}, U_{ABCD}, U_{ABCDE} and U_{ABCDEF} respectively. In each case m is given by the same formulae except with k and d replacing k_0 and d_0.

The following theorem describes the surprising form of the weak expression in the case that one starts with an invariant.

THEOREM 6.3. *Let I be an invariant of $\mathcal{U}$. If $I_{PQ\cdots V}$ is a weak expression for I then*

$$I_{PQ\cdots V} = X_P X_Q \cdots X_V I.$$

Remark 6.4. Note that this theorem shows that the weak expression for the trivial invariant is trivial no matter what standard expression (for the trivial invariant) we start with.

Toward proving Theorem 6.3 we need the following results.

LEMMA 6.5. *Fix a choice of ξ. Let E be a metric invariant of $\mathcal{U}$. In a standard formula for E replace each $U_{a\cdots d}$ with its tractor expression $U_{A\cdots D}$, replace each ∇_a by $\tilde{D}_A$, replace each g^{ab} with h^{AB} and finally replace each $\varepsilon^{bc\cdots e}$ with $\varepsilon^{BC\cdots E}$. The result is another expression for E.*

Proof. Certainly another expression for E is obtained by replacing, in the standard formula, each $U_{a\cdots d}$ with its tractor expression $U_{A\cdots D}$, each g^{ab} with g^{AB}, each $\varepsilon^{bc\cdots e}$ with $\varepsilon^{BC\cdots E}$ and each ∇_a with its tractor expression as given by (25) (with $\mathsf{P}_{AB} = 0$). The result now follows by substituting for g^{AB} using (23) and then employing the identities (37), $\varepsilon^{BC\cdots E}\xi_B = 0$ and

$$\xi^E \tilde{D}_A \tilde{D}_B \cdots \tilde{D}_E \cdots \tilde{D}_F U_{GH\cdots J} = 0 = \xi^H \tilde{D}_A \tilde{D}_B \cdots \tilde{D}_D U_{E\cdots H\cdots J}, \tag{39}$$

where, on the right hand side, the "$_H$" indicates any index of the tractor field $U_{E\cdots J}$. The displayed identities are straightforward consequences of (37), (19) and that $U_{A\cdots D}$ is ξ^C-saturated. ∎

LEMMA 6.6. *Fix a choice of ξ. Let $E_{PQ\cdots V}$ be a weak expression for a metric invariant E of $\mathcal{U}$. Then*

$$\xi^P \xi^Q \cdots \xi^V E_{PQ\cdots V} = E. \tag{40}$$

Proof. Substituting for D_{AP} using (17), expand out the left hand side of (40) in terms of ξ^P, X_Q and $\tilde{D}_A$ derivatives of $U_{A\cdots D}$. Eliminate occurrences of X_P using (16), (21) and $\xi^P X_P = 1$. Of the terms present, all have ξ^P contracted into one of the indices of some $\tilde{D}_A$ derivative of $U_{A\cdots D}$ except for one. Now using the Eqs. (39) above, we see that all terms vanish apart from an exceptional one. This remaining term, if we eliminate occurrences of X_P using that $\xi^P X_P = 1$, is given by an expression which is the standard formula for E only with each $U_{a\cdots d}$ replaced by $U_{A\cdots D}$, ∇_a replaced by $\tilde{D}_A$, g^{ab} replaced by h^{AB} and $\varepsilon^{ab\cdots e}$ replaced by $\varepsilon^{AB\cdots E}$. The result thus follows from Lemma 6.5. ∎

Proof of Theorem 6.3. Suppose that $I_{PQ\cdots V}$ has valence m. From Lemma 6.6

$$\xi^P\xi^Q\cdots\xi^V I_{PQ\cdots V}=I, \tag{41}$$

and this is independent of choice of scale ξ as both I and the weak expression $I_{PQ\cdots V}$ are conformally invariant. So, for each $p\in\mathscr{M}$, regarding χ^P as homogeneous coordinates on $\mathbb{P}^{n+1}$,

$$\chi^P\chi^Q\cdots\chi^V(I_{PQ\cdots V}-X_PX_Q\cdots X_VI) \tag{42}$$

is a continuous weighted function (of weight m) on $\mathbb{P}^n$ which vanishes on χ_P such that $\chi_PX^P\neq 0$ and $\chi_P\chi^P=0$. By continuity it must in fact vanish on all χ_P describing the quadric $\chi_P\chi^P=0$ and so it follows that, at each $p\in\mathscr{M}$,

$$I_{PQ\cdots T}-X_PX_Q\cdots X_TI=h_{(PQ}B_{RS\cdots T)}$$

for some tractor B. However the left-hand-side here is trace-free and so the theorem follows. ∎

This proof is due to Robin Graham [19]. In fact there are many alternative proofs. For example, one can also easily deduce the result from the Theorem C.1 in [17]. This states that if S is an H-submodule of an H-module V, for any group H, such that S has trivial intersection with the first composition factor of V, then $S=0$. Here we would use the corresponding result for induced bundles which follows immediately (see the discussion introducing appendix C in [17]).

6.2. *Linearising Invariants and the Theorem.* We now return to the setting of invariants for general conformal manifolds. We first observe that such invariants determine invariants of $\mathscr{U}$ on flat structures.

THEOREM 6.7. *Let I denote an invariant of principal degree d_0. Then I determines, on the flat structures, an invariant $\tilde{I}$ of $\mathscr{U}$. This is of degree $d=d_0$ in $\mathscr{U}$. When $n\geqslant 4$, a formula for $\tilde{I}$ is given by replacing each C_{abcd}, in the principal part $I_{(d_0)}$ of a normal expression for I, by a U_{abcd}. Similarly, when $n=3$, a formula for $\tilde{I}$ is given by replacing each C_{abc}, in the principal part $I_{(d_0)}$ of a normal expression for I, by a U_{abc}.*

Proof. Consider the $n\geqslant 4$ case first and let us examine the behaviour of the principal part $I_{(d_0)}$ under a change of conformal scale. $I_{(d_0)}$ is some complete contraction expression involving various covariant derivatives of the Weyl tensor C_{abcd}. If we transform to a new connection ∇ (no longer necessarily satisfying (11)) Then the expression $I_{(d_0)}$ transforms, by (1), to $\hat{I}_{(d_0)} = I_{(d_0)} + T$ where T indicates a linear combination of terms each of which is a contraction of juxtapositions of Υ_a, C_{abcd} and various covariant derivatives of each of these objects. Since $I - I_{(d_0)}$ is of degree d_0+1 and I is conformally invariant it follows immediately that T may be expressed as a complete contraction of principal degree d_0+1 in the curvature. That is $I_{(d_0)}$ is invariant if we calculate modulo terms of order d_0+1.

Let us then imagine formally verifying that $I_{(d_0)}$ is invariant in this sense by using the transformations (1), (5). It follows from Proposition 2.1, that any identities between the covariant derivatives of C_{abcd} that are used in this calculation, follow from (2), (3), the definition of C_{abcd} as the trace-free part of R_{abcd} and the covariant derivatives of these. Since $I_{(d_0)}$ is expressed in terms of the Weyl curvature we should understand the identities (3) in terms of this. Using the Definition of C_{abcd} as the trace-free part of R_{abcd} one finds that the identities satisfied by the Weyl tensor that one may deduce from (3) are:

$$\begin{aligned} C_{abcd} &= C_{[ab][cd]} \\ C_{[abc]\,d} &= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (43) \\ (n-3)\,\nabla_{[a}C_{bc]\,de} &= g_{d[a}\nabla_{|s|}C_{bc]}{}^{s}{}_{e} - g_{e[a}\nabla_{|s|}C_{bc]}{}^{s}{}_{d}. \end{aligned}$$

In fact (in dimensions $n \geqslant 4$) the system of identities (3) is equivalent to the combination of the system (43) plus the contracted Bianchi identity $\nabla_d C_{ab}{}^{d}{}_{c} = 2(n-3)\,\nabla_{[a}\mathsf{P}_{b]\,c}$. It follows that with the trace-free property, the identities (43) generate all identities between the covariant derivatives of the Weyl curvature.

Since we are calculating modulo terms of degree d_0+1 it is clear that, for the purposes of this calculation, the identity (2) may be replaced by the assumption that the ∇-operators commute, $[\nabla_a, \nabla_b] = 0$. For the same reason, for the purposes of this calculation, we may also replace (5) with the identity

$$\nabla_a \Upsilon_b = \Upsilon_a \Upsilon_b - \tfrac{1}{2}\Upsilon_k \Upsilon^k g_{ab}. \tag{44}$$

Since the Weyl curvature is invariant under change of conformal scale it follows that the relationship, at any point, between the jets of Υ_a and the change of curvature is fully described by (5) and its covariant derivatives. In view of this and that the function Ω we use to make a change of conformal scale is a completely arbitrary positive function it follows that, at any point, the 0-jet of Υ_a can be freely altered without affecting the curvature. Thus for the purposes of our calculation all identities between the covariant derivatives of Υ_a follow from (44) and its covariant derivatives.

In summary in our formal calculation to show that $\hat{I}_{(d_0)} = I_{(d_0)}$ modulo terms of degree d_0+1 we can assume that covariant derivatives commute, that (44) holds, we can use that $C_{abcd} \in \mathcal{E}_{abcd}[2]$ is completely trace-free and satisfies the identities (43) and all identities these generate amongst covariant derivatives of C_{abcd} and Υ_a. There are no other identities that we can use non-trivially. These identities are formally identical to the identities satisfied by U_{abcd} (see (38)). Thus the formal calculation we have described corresponds precisely to a calculation to verify that $\tilde{I}$, as described in the theorem, is an invariant in the flat case.

For the $n=3$ case the argument is the same except that now we observe that C_{abc} has the same symmetries as U_{abc} and we deduce from Proposition (2.1) that all identities amongst the covariant derivatives of $\nabla_{[b}\mathsf{P}_{c]a}$ arise from its symmetries (viz it is skew in the first two indices and $\nabla_{[b}\mathsf{P}_{ca]}=0$) and (2). It follows that in the calculation to show that $\hat{I}_{(d_0)} = I_{(d_0)}$ (modulo terms of degree (d_0+1) in the curvature and its covariant derivatives) we are again restricted to use identities which hold in the linearised case and so the calculation can be identified with a calculation to verify that $\tilde{I}$ is invariant.

Now let I and J be two normal *expressions* for the same invariant. Then, by similar reasoning to that just above and if again we ignore terms of degree $\geqslant (d_0+1)$ in the curvature, a calculation to establish that I and J determine the same invariant corresponds precisely to a calculation verifying that $\tilde{I}$ and $\tilde{J}$ determine the same invariant of $\mathcal{U}$. ▮

THEOREM 6.8. *For an invariant I of principal degree d_0, let $I_{PQ\cdots T}$ be a weak expression associated with I. Then $I_{PQ\cdots T}$ can be expressed in the form*

$$I_{PQ\cdots T} = X_P X_Q \cdots X_T I + (\textit{terms of principal degree} \geqslant d_0+1).$$

Proof. Let $\tilde{I}$ be the expression for the invariant of $\mathcal{U}$ determined by $I_{(d)}$ as described in Theorem 6.7. Note that the formula for the corresponding weak expression $\tilde{I}_{PQ\cdots T}$ in terms of X^P, h^{AB}, U_{ABCD}, $\tilde{D}_A$, $\eta^I{}_Q{}^{A\cdots E}$ may be obtained from the formula for $I_{PQ\cdots T}$, in terms of X^P, h^{AB}, C_{ABCD}, $\tilde{D}_A$, $\eta^I{}_Q{}^{A\cdots E}$, by formally replacing each C_{ABCD} with U_{ABCD} (or, in the case $n=3$, by formally replacing each C_{ABC} with U_{ABC}). Consider now explicitly calculating the components of $I_{PQ\cdots T}$ in terms of ∇-derivatives of C_{abcd} (or C_{abc} in the $n=3$ case). As mentioned above C_{abcd} satisfies the same (formal) symmetries and identities as U_{abcd} (as described in (38)) and similarly C_{abc} has the same symmetries as U_{abc}. Also $\tilde{C}_{ABCD}$ is the tractor expression for C_{abcd} just as U_{ABCD} is the tractor expression for U_{abcd}. A similar comment applies for the relationship between $\tilde{C}_{ABC}$ and C_{abc} as compared with the relationship between U_{ABC} and U_{abc}. If we are allowed to ignore terms of degree $\geqslant (d_0+1)$, then this calculation is essentially the same as *verifying* Theorem 6.3, for the invariant $\tilde{I}$ of $\mathcal{U}$, by explicitly calculating the components of $\tilde{I}_{PQ\cdots T}$ in terms of ∇-derivatives of U_{abcd} (or U_{abc} in the $n=3$ case).

With C_{abcd} replacing U_{abcd}, C_{abc} replacing U_{abc} etc., the only possible differences in the calculation here arise from the fact that now $\tilde{D}_A \xi_B + \xi_A \xi_B = \mathsf{P}_{AB}$ rather than zero and now the $\tilde{D}_A$-derivatives and the ∇-derivatives do not commute with themselves. However these differences are all by curvature terms and so only affect the terms of degree $\geqslant (d_0 + 1)$. ▮

Note that the sum of the "terms of principal degree $\geqslant d_0 + 1$" in the theorem is some conformally invariant tractor field ($J_{PQ \cdots T}$ say) since both $I_{PQ \cdots T}$ and I are invariant.

7. THE PROOF OF PROPOSITION 5.4

The remaining major task is to establish Proposition 5.4. We do this using some results of the tractor calculus and consideration of what terms could turn up in a weak expression for a metric invariant.

Proof of Proposition 5.4, *Part* I. In fact here we will just discuss the cases $n \geqslant 4$ and show, in these cases, the weaker result that, for even invariants,

$$m' \leqslant k_0.$$

The point here is to set up the approach and illustrate that one can get quite close to the results claimed in the proposition using only very elementary observations. The second part of the proof, which follows, is then essentially a fine tuning of the ideas used here but is necessarily technical.

Consider the cases $n \geqslant 4$. Let C_{ABDEFG} be the lifted expression for the Weyl curvature C_{bdef}. For some d consider forming a degree d juxtaposition of various orders of D_{AP} derivatives of the C_{ABDEFG} tractors. Suppose that the total number of D_{AP}'s appearing in this juxtaposition is k. The terms in the weak expression associated with an invariant of principal degree $d_0 \leqslant d$ and principal order $k_0 \geqslant k$ (with $2d_0 + k_0 = 2d + k$ even) arise from a contraction of some number of h^{AB}'s, and possibly a $X_P \eta^{P\,AB \cdots E}_{\;Q}$, into such a juxtaposition so let us consider its form before we do this contraction. Using the Definition $D_{AP} := 2X_{[P} \tilde{D}_{A]}$ and similarly the description of the lifted expression C_{ABDEFG} in terms of the tractor expression $\tilde{C}_{BDEF}$ and X_P's one can expand out the juxtaposition. If, as we carry out this expansion, we move the X_M's in each term to the left of the operators $\tilde{D}_A$ by repeatedly using (22) then it is straightforward to verify that our juxtaposition can be rewritten as a linear combinations of terms, a typical term of which is of the form

$$\underbrace{h_{GH}h_{IJ}\cdots h_{KL}}_{a}\,\underbrace{X_M X_N\cdots X_P}_{b}\,(\underbrace{\tilde{D}_Q\tilde{D}_R\cdots\tilde{D}_S}_{l_1}\,\tilde{C}_{A\cdots B})$$
$$\times(\underbrace{\tilde{D}_T\tilde{D}_U\cdots\tilde{D}_V}_{l_2}\,\tilde{C}_{C\cdots D})\cdots(\underbrace{\tilde{D}_W\tilde{D}_X\cdots\tilde{D}_Y}_{l_d}\,\tilde{C}_{E\cdots F}),\tag{45}$$

where $a=k-\sum_1^d l_i$ and $b=2d+k-a$.

The next step toward building a term in the weak expression associated to an <u>even</u> invariant is to contract all indices of $\frac{4d+k}{2}$ h^{AB}'s into this. However if the result is then purely trace on any pair of free indices it will contribute zero to $E_{APBQ\cdots DS}$ as the latter is constructed from the trace-free part of such terms. It follows that we may as well assume that $a\leqslant\frac{4d+k}{2}$ and dedicate a number a of the $\frac{4d+k}{2}$ h^{AB}'s for contracting into the a h_{GH}'s in (45). That is at least one index of each of the h^{AB}'s should be contracted with an index on one of the a h_{GH}'s. Although there are a number of different possibilities as to where the second index of each h^{AB} is contracted in order to produce a non-zero result, the effect on (45) of contracting in the a h^{AB}'s in any of these ways is always the same up to scale and index relabelling, with result

$$\underbrace{X_M X_N\cdots X_P}_{b}\,(\underbrace{\tilde{D}_Q\tilde{D}_R\cdots\tilde{D}_S}_{l_1}\,\tilde{C}_{A\cdots B})$$
$$\times(\underbrace{\tilde{D}_T\tilde{D}_U\cdots\tilde{D}_V}_{l_2}\,\tilde{C}_{C\cdots D})\cdots(\underbrace{\tilde{D}_W\tilde{D}_X\cdots\tilde{D}_Y}_{l_d}\,\tilde{C}_{E\cdots F}).\tag{46}$$

Now we wish to contract the remaining $\frac{4d+k-2a}{2}$ h^{AB}'s into this in such a way as to leave as few free (i.e., uncontracted) X_M's as possible. Using (21) and that each $\tilde{C}_{ABCD}$ is X-saturated it follows that one can contract at most ℓ X^A's into

$$\underbrace{\tilde{D}_I\tilde{D}_J\ldots\tilde{D}_L}_{\ell}\,\tilde{C}_{ABCD}\tag{47}$$

without killing it. Thus if $k\geqslant 4d$ then we may contract at most $\frac{4d+k-2a}{2}$ of the b X_M's into the

$$(\underbrace{\tilde{D}_Q\tilde{D}_R\cdots\tilde{D}_S}_{\ell_1}\,\tilde{C}_{A\cdots B})(\underbrace{\tilde{D}_T\tilde{D}_U\cdots\tilde{D}_V}_{\ell_2}\,\tilde{C}_{C\cdots D})\cdots(\underbrace{\tilde{D}_W\tilde{D}_X\cdots\tilde{D}_Y}_{\ell_d}\,\tilde{C}_{E\cdots F})\tag{48}$$

part of (46). This leaves $\frac{k}{2}$ free X_M's whereas any other contraction leaves either a greater number or annihilates the expression. If on the other hand $k<4d$ then $l_1+l_2+\cdots+l_d=k-a$ of the $b=2d+k-a$ X_P's may be contracted into (48) and this leaves $2d$ free X_P's. Again any other contraction results in zero or a greater number of free X_P's remaining. So for such a term in the weak expression for the invariant we have $m'=\min(k, \frac{4d+k}{2})$.

With a view to eventually improving and generalising the previous argument it is worthwhile repeating it in the more mechanical setting of linear programming. Thus we return to equation (46) and the problem of contracting $\frac{4d+k-2a}{2}$ h^{AB}'s into this in such a way as to leave as few free X_M's as possible. Suppose we use $\tilde{\beta}$ of these h^{AB}'s each to contract one of the X_M's of (46) into an index which is not on another X. (Note that since X is a null tractor we may as well assume that none of the h^{AB}'s are used to contract two X's.) Suppose that we use the remaining $\tilde{\delta}$ h^{AB}'s to contract amongst pairs of indices such that neither index of each pair is on an X. Then

$$\tilde{\beta}+\tilde{\delta}=\frac{4d+k-2a}{2}.$$

We have the constraint

$$\tilde{\beta}\leqslant k-a$$

arising from (47). There is also the constraint $\tilde{\beta}+2\tilde{\delta}\leqslant 4d+k-a$ given by the total number of indices not on X's, but this follows from the previous constraint and equality. The number of free X's is $b-\tilde{\beta}=2d+k-a-\tilde{\beta}$ and so minimising this is equivalent to maximising z subject to the following system,

$$\begin{aligned} z-a-\tilde{\beta}&=0\\ a+\tilde{\beta}+\tilde{\delta}&=\frac{4d+k}{2}\\ a+\tilde{\beta}+s&=k. \end{aligned}\tag{49}$$

All variables on the left hand sides of these equations are non-negative and s is a "slack variable". By adding the first two equations we see that $z=\frac{4d+k}{2}-\tilde{\delta}$ and so $z\leqslant\frac{4d+k}{2}$. Similarly adding the first and third equations reveals that $z\leqslant k$. Thus

$$z\leqslant\min\left(k, \frac{4d+k}{2}\right).\tag{50}$$

For all $d\geqslant 2$ and $k\leqslant 4d$ there are solutions with $z=k$ and similarly for all such d and $k\leqslant 4d$ there are solutions with $z=\frac{4d+k}{2}$ and so the estimate is sharp for the system (49).

Given our metric invariant with fixed d_0 and k_0 and weight $u = -(2d_0+k_0)$, d and k will in general vary from term to term in the weak expression. But all non-trivial terms have $k<k_0$, so this introduces no problems. On the other hand $\frac{4d+k}{2} \geqslant (4d_0+k_0)/2$ so for this part of the estimate things "get worse" (i.e., z possibly increases) as we move away from the terms in the principal part of the invariant to other terms of the invariant. Using (10) we can put an estimate on how much worse. This yields $\frac{4d+k}{2} \leqslant 2d_0+k_0$ so finally we have just $z \leqslant k_0$ as claimed. ▮

As the proposition asserts, we can do better than the result just above. The key is that the only property of the $\tilde{D}_I \tilde{D}_J \cdots \tilde{D}_L \tilde{C}_{A \cdots D}$ that we have used is that contracting $\ell+1$ X^A's into (47) results in annihilation. By using the symmetries and properties of the Weyl tensor (Cotton–York when $n=3$) we can improve our estimate. To exploit these symmetries we need to consider contracting one or two X's into the indices on the $\tilde{C}$ part of $\tilde{D}_I \tilde{D}_J \cdots \tilde{D}_L \tilde{C}_{A \cdots D}$.

Let us treat an example first. Consider

$$X^P \tilde{D}_A \tilde{D}_B \tilde{D}_C \tilde{D}_E \tilde{C}_{PQRS}$$

and first observe that

$$\begin{aligned} X^P \tilde{D}_A \tilde{D}_B \tilde{D}_C \tilde{D}_E \tilde{C}_{PQRS} &\\ = -[\tilde{D}_A, X^P]\, \tilde{D}_B \tilde{D}_C \tilde{D}_E \tilde{C}_{PQRS} &- \tilde{D}_A [\tilde{D}_B, X^P]\, \tilde{D}_C \tilde{D}_E \tilde{C}_{PQRS} \\ - \tilde{D}_A \tilde{D}_B [\tilde{D}_C, X^P]\, \tilde{D}_E \tilde{C}_{PQRS} &\\ - \tilde{D}_A \tilde{D}_B \tilde{D}_C [\tilde{D}_E, X^P]\, \tilde{C}_{PQRS}, & \end{aligned} \tag{51}$$

as $\tilde{C}_{PQRS}$ is X-saturated. Now recall (21) gives $[\tilde{D}_C, X^P] = \delta_C^P - X_C \xi^P$ and so the third term on the right hand side of above may be replaced by

$$-\tilde{D}_A \tilde{D}_B \tilde{D}_E \tilde{C}_{CQRS} + \tilde{D}_A \tilde{D}_B X_C \xi^P \tilde{D}_E \tilde{C}_{PQRS}.$$

Now we leave the first term in this last expression, as it is. However the other term is to be replaced by the result of commuting the X to the left of all $\tilde{D}$ operators:

$$\begin{aligned} \tilde{D}_A \tilde{D}_B X_C \xi^P \tilde{D}_E \tilde{C}_{PQRS} = \tilde{D}_A [\tilde{D}_B, X_C]\, \xi^P \tilde{D}_E \tilde{C}_{PQRS} &+ [\tilde{D}_A, X_C]\, \tilde{D}_B \xi^P \tilde{D}_E \tilde{C}_{PQRS} \\ &+ X_C \tilde{D}_A \tilde{D}_B \xi^P \tilde{D}_E \tilde{C}_{PQRS}. \end{aligned}$$

Once again using (21) we have

$$\begin{aligned} \tilde{D}_A \tilde{D}_B X_C \xi^P \tilde{D}_E \tilde{C}_{PQRS} = {} & h_{BC} \tilde{D}_A \xi^P \tilde{D}_E \tilde{C}_{PQRS} - \tilde{D}_A X_B \xi_C \xi^P \tilde{D}_E \tilde{C}_{PQRS} \\ & + h_{AC} \tilde{D}_B \xi^P \tilde{D}_E \tilde{C}_{PQRS} - X_A \xi_C \tilde{D}_B \xi^P \tilde{D}_E \tilde{C}_{PQRS} \\ & + X_C \tilde{D}_A \tilde{D}_B \xi^P \tilde{D}_E \tilde{C}_{PQRS}. \end{aligned}$$

Finally

$$\begin{aligned}\tilde{D}_A\tilde{D}_B X_C\xi^P\tilde{D}_E\tilde{C}_{PQRS} &= h_{BC}\tilde{D}_A\xi^P\tilde{D}_E\tilde{C}_{PQRS} - h_{AB}\xi_C\xi^P\tilde{D}_E\tilde{C}_{PQRS}\\ &\quad + X_A\xi_B\xi_C\xi^P\tilde{D}_E\tilde{C}_{PQRS} + h_{AC}\tilde{D}_B\xi^P\tilde{D}_E\tilde{C}_{PQRS}\\ &\quad - X_A\xi_C\tilde{D}_B\xi^P\tilde{D}_E\tilde{C}_{PQRS} + X_C\tilde{D}_A\tilde{D}_B\xi^P\tilde{D}_E\tilde{C}_{PQRS}.\end{aligned}$$

Thus we see that third term in the expansion (51) can be replaced by this final right hand side plus the $-\tilde{D}_A\tilde{D}_B\tilde{D}_E\tilde{C}_{CQRS}$ term which arose above. Of course all the terms on the right hand side of (51) can be similarly replaced and this is the way we wish to expand any occurrences of $X^P\tilde{D}_A\tilde{D}_B\cdots\tilde{D}_F\tilde{C}_{P\cdots S}$.

Let us introduce some notation to incorporate a range of terms that could arise. Suppose we have a word of length a made of the two symbols ξ and $\tilde{D}$ (e.g., an $a=6$ word is $\underbrace{\xi\xi\tilde{D}\xi\tilde{D}\xi}_{a}$) and we adorn this with a distinct indices (e.g., for our example $\xi_A\xi_B\tilde{D}_C\xi_D\tilde{D}_E\xi_F$). We may view this as an expression for an operator. Suppose we (formally) act with this on the tractor field $\xi^P\underbrace{\tilde{D}_J\cdots\tilde{D}_K}_{b}\tilde{C}_{P\cdots S}$. Then we will use the notation

$$\{\tilde{D},\xi\}_{\underbrace{AB\cdots K}_{\ell}}\tilde{C}_{Q\cdots S}\qquad\text{where}\quad \ell=a+b \tag{52}$$

to indicate the resulting term. (For example

$$\xi_A\xi_B\tilde{D}_C\xi_D\tilde{D}_E\xi_F\xi^P\underbrace{\tilde{D}_J\cdots\tilde{D}_K}_{b}\tilde{C}_{P\cdots S}$$

is a term $\{\tilde{D},\xi\}_{\underbrace{AB\cdots K}_{6+b}}\tilde{C}_{Q\cdots S}$.) We will also use the notation (52) to indicate terms of the form

$$\underbrace{\tilde{D}_A\tilde{D}_B\cdots\tilde{D}_K}_{\ell}\tilde{C}_{Q\cdots S}.$$

Using this notation it is easily verified that

$$X^P\underbrace{\tilde{D}_A\tilde{D}_B\cdots\tilde{D}_F}_{\ell}\tilde{C}_{P\cdots S}$$

is a linear combination of terms of the form

$$\{\tilde{D},\xi\}_{\underbrace{AB\cdots K}_{\ell-1}}\tilde{C}_{Q\cdots S},\quad X_T\{\tilde{D},\xi\}_{\underbrace{AB\cdots K}_{\ell-1}}\tilde{C}_{Q\cdots S}\quad\text{and}\quad h_{TU}\{\tilde{D},\xi\}_{\underbrace{AB\cdots K}_{\ell-2}}\tilde{C}_{Q\cdots S}.$$

In fact we must also deal with expressions $X^R X^P \tilde{D}_A \tilde{D}_B \cdots \tilde{D}_F \tilde{C}_{P \dots RS}$. The X^P can be eliminated as described and then on the result we eliminate the X^R to obtain that

$$X^R X^P \underbrace{\tilde{D}_A \tilde{D}_B \cdots \tilde{D}_F}_{\ell} \tilde{C}_{P \dots RS}$$

may be expressed as a linear combination of terms of the form,

$$\{\tilde{D}, \xi\}_{\underbrace{AB \cdots K}_{\ell-2}} \tilde{C}_{Q \cdots S}, \quad X_T \{\tilde{D}, \xi\}_{\underbrace{AB \cdots K}_{\ell-2}} \tilde{C}_{Q \cdots S} \quad \text{and} \quad h_{TU} \{\tilde{D}, \xi\}_{\underbrace{AB \cdots K}_{\ell-3}} \tilde{C}_{Q \cdots S}.$$

and also

$$X_T X_U (\tilde{D}, \xi)_{\underbrace{AB \cdots K}_{\ell-2}} \tilde{C}_{Q \cdots S}, \qquad X_T h_{UV} (\tilde{D}, \xi)_{\underbrace{AB \cdots K}_{\ell-3}} \tilde{C}_{Q \cdots S} \qquad \text{and}$$

$$h_{TU} h_{VW} (\tilde{D}, \xi)_{\underbrace{AB \cdots K}_{\ell-4}} \tilde{C}_{Q \cdots S},$$

where,

$$(\tilde{D}, \xi)_{\underbrace{AB \cdots K}_{\ell}} \tilde{C}_{Q \cdots S} \tag{53}$$

indicates a contraction of a term of the form (52),

$$h^{ER} \{\tilde{D}, \xi\}_{\underbrace{AB \cdots E \cdots K}_{\ell+1}} \tilde{C}_{Q \cdots RS}$$

and the index "$_E$" is attached to a ξ. Henceforth, to simplify our discussion, we use the notation (52) to include the contracted terms of the form (53). (The reader is warned that total number of free indices on the object (52) may be either $\ell+3$ or $\ell+4$ while the total number of free indices on the object (53) is $\ell+2$. While in this sense the notation for these objects is a little misleading it is nevertheless convenient for our discussions.)

We are now in a position to improve and complete the proof of the proposition.

Proof of Proposition 5.3, *Part* II. Here, as above, we consider certain characteristics of the form of a typical term in an expansion of the weak expression for a metric invariant. In particular we seek to find what is smallest number of free X's that could be on a non-trivial term. Recall the metric invariant has principal degree d_0 and principal order k_0.

Even invariants, $n \geqslant 4$: We again consider the problem of contracting $\frac{4d+k}{2}$ h^{AB}'s into the expression (45) where $d \geqslant d_0$ and $2d+k=2d_0+k_0$. We will do this as follows. We suppose that first we contract a number β of these h^{AB}'s in such a way that each h^{AB} pairs an X_A with one of the indices of one these $\tilde{C}_{ABCD}$. Now we re-express the result by eliminating these contracted X's as described above. Using our observations just above we find that the new result is then a linear combination of expressions of the form

$$\underbrace{h_{GH}h_{IJ}\cdots h_{KL}}_{a+\beta_2}\,\underbrace{X_M X_N\cdots X_P}_{b-\beta_0-\beta_2}(\{\tilde{D},\xi\}_{\underbrace{QR\cdots S}_{l'_1}}\tilde{C}_{A\cdots B})$$
$$\times(\{\tilde{D},\xi\}_{\underbrace{TU\cdots V}_{l'_2}}\tilde{C}_{C\cdots D})\cdots(\{\tilde{D},\xi\}_{\underbrace{QR\cdots S}_{l'_d}}\tilde{C}_{E\cdots F}), \tag{54}$$

where the notation $\{\tilde{D},\xi\}_{\underbrace{QR\cdots S}_{l}}\tilde{C}_{A\cdots B}$ is as described above. From the observations above it is clear that $\beta \geqslant \beta_2 \geqslant 0$. By counting indices, for example, it is clear that there can be at most $b-\beta_2$ free indices on X's in (54). Thus $\beta_0 \geqslant 0$. On the other hand in any non-trivial term (54) there must be a minimum of $b-\beta$ free indices on X's. That is $b-\beta_0-\beta_2 \geqslant b-\beta$ and so $\beta_1 := \beta-\beta_0-\beta_2 \geqslant 0$. Then it follows from our observations above that $\sum_1^d l'_i = k-a-\beta_0-\beta_1-2\beta_2$, where, as in (45), $a=k-\sum_1^d l_i$ and, $b=2d+k-a$.

From this point the argument is very straightforward. First we observe that since the final result will be trace free we can assume that $a+\beta_2$ of the remaining $((4d+k-2\beta_0-2\beta_1-2\beta_2)/2)$ h^{AB}'s are used to eliminate the $a+\beta_2$ h_{GH}'s in (54). This leaves $((4d+k-2a-2\beta_0-2\beta_1-4\beta_2)/2)$ h^{AB}'s and these are used as follows: Suppose that α of these h^{AB}'s pair an X with a $\tilde{D}$ or a ξ, that γ of these h^{AB}'s pair a $\tilde{D}$ or a ξ with a $\tilde{C}$, that δ of these h^{AB}'s pair a $\tilde{C}$ with a $\tilde{C}$, and that ε of these h^{AB}'s pair a $\tilde{D}$ or a ξ with a $\tilde{D}$ or a ξ. (As before since X is null we may assume that none of the h^{AB}'s is used to pair an X with an X. Similarly since $\tilde{C}$ is trace free we may assume that none of the h^{AB}'s is used to contract two indices of the same $\tilde{C}$ tractor.) Thus we have

$$\alpha+\gamma+\delta+\varepsilon=\frac{4d+k-2a-2\beta_0-2\beta_1-4\beta_2}{2}.$$

Using that $X^A\xi_A=1$ and (21) it follows that we can contract at most ℓ X's into

$$\{\tilde{D},\xi\}_{\underbrace{QR\cdots S}_{l}}\tilde{C}_{A\cdots D}$$

(cf. (47) etc.). It follows that

$$\alpha \leqslant k-a-\beta_0-\beta_1-2\beta_2.$$

However this follows from the condition imposed by the total number of free indices on $\tilde{D}'s$ and ξ's which is

$$\alpha+\gamma+2\varepsilon \leqslant k-a-\beta_0-\beta_1-2\beta_2.$$

The total number of available indices on $\tilde{C}$'s imposes the constraint,

$$\beta_1+\beta_2+\gamma+2\delta \leqslant 4d.$$

Finally we observe that, because of the symmetry $\tilde{C}_{ABCD}=\tilde{C}_{[AB][CD]}$, it is clear that

$$\beta \leqslant 2d$$

and also that at most $2d$ of the indices on the $\tilde{C}$'s can remain free if the term is to possibly contribute non-trivially to the weak expression $E_{PQ\cdots W}$ (which is symmetric) and so

$$\beta_1+\beta_2+\gamma+2\delta \geqslant 2d.$$

Minimising the number of remaining free X's

$$2d+k-a-\alpha-\beta_0-\beta_2$$

is equivalent to maximising z subject to the system

$$\begin{aligned}
z-a-\alpha-\beta_0-\beta_2 &= 0\\
a+\alpha+\beta_0+\beta_1+2\beta_2+\gamma+\delta+\varepsilon &= \frac{4d+k}{2}\\
\beta_1+\beta_2+\gamma+2\delta-e &= 2d\\
a+\alpha+\beta_0+\beta_1+2\beta_2+\gamma+2\varepsilon+s_1 &= k\\
\beta_0+\beta_1+\beta_2+s_2 &= 2d\\
\beta_1+\beta_2+\gamma+2\delta+s_3 &= 4d
\end{aligned} \tag{55}$$

where, as usual, all variables on the left hand side are to be non-negative integers.

Adding the first two equations of this system and subtracting half of the third gives

$$z+\frac{\beta_1}{2}+\frac{\beta_2}{2}+\frac{\gamma}{2}+\varepsilon+\frac{e}{2}=\frac{2d+k}{2},$$

and so $z \leqslant \frac{2d+k}{2} = (2d_0 + k_0)/2$. On the other hand adding the first and forth equations of the system yields

$$z + \beta_1 + \beta_2 + \gamma + 2\varepsilon + s_1 = k$$

and so $z \leqslant k \leqslant k_0$. Thus we get

$$m' = \max(z) \leqslant \min\left(k, \frac{2d+k}{2}\right) \leqslant \min\left(k_0, \frac{2d_0+k_0}{2}\right).$$

Recall that in this case k is even and so this upper bound for z is integral.

Note that it is the third equation of (55) that is key to our improvement on the estimate (50).

Odd invariants, $n \geqslant 4$: In this case we consider contracting $X_P \eta^{P\,AB\cdots E}_{\;Q}$ and $\frac{4d+k-n}{2}$ h^{AB}'s into (45). One should note that to build a possible term of the weak expression $E_{APBQ\cdots DS}$ in line with Definition 5.2 then precisely one index (say the index Q) of $X_P \eta^{P\,AB\cdots E}_{\;Q}$ must remain free. Observe that $X_P \eta^{P\,AB\cdots E}_{\;Q}$ is annihilated upon contraction with X_M. Note also that if we contract $X_P \eta^{P\,AB\cdots E}_{\;Q}$ into one of h_{MN}'s of (45) then this lowers an index of $X_P \eta^{P\,AB\cdots E}_{\;Q}$. This lowered index cannot remain free as Q is to be a free index (and $X_P \eta^{P\,AB\cdots E}_{\;Q}$ is completely skew on free indices). It follows that there is no loss of generality in assuming that n upstairs indices of $X_P \eta^{P\,AB\cdots E}_{\;Q}$ are to be contracted into the part of (45) displayed in (47) (and we postpone performing these contractions for now). Thus we consider the problem of contracting all indices of $\frac{4d+k-n}{2}$ h^{AB}'s into (45) in such a way as to leave as few free X_M's as possible. As for the even invariants case above, we will first use β of these h^{AB}'s in such a way that each h^{AB} pairs an X_A with one of the indices of one the $\tilde{C}_{ABCD}$. Now we re-express the result by eliminating these contracted X's as described above. Again the result is a linear combination of expressions of the form (54) with $a = k - \sum_1^d l_i$, $b = 2d + k - a$, $\beta = \beta_0 + \beta_1 + \beta_2$, $\sum_1^d l_i' = k - a - \beta_0 - \beta_1 - 2\beta_2$. Without loss of generality we can now use $a + \beta_2$ of the remaining h^{AB}'s to remove that number of h_{GH}'s in (54). This leaves $((4d + k - n - 2a - 2\beta_0 - 2\beta_1 - 4\beta_2)/2)$ h^{AB}'s and these are used as follows: Suppose that α of these h^{AB}'s pair an X with a $\tilde{D}$ or a ξ, that γ of these h^{AB}'s pair a $\tilde{D}$ or a ξ with a $\tilde{C}$, that δ of these h^{AB}'s pair a $\tilde{C}$ with a $\tilde{C}$, and that ε of these h^{AB}'s pair a $\tilde{D}$ or a ξ with a $\tilde{D}$ or a ξ. (As above other pairings need not be considered without loss of generality.) Finally we assume that μ of the indices of $X_P \eta^{P\,AB\cdots E}_{\;Q}$ are contracted into the $\tilde{C}$ tractors (and $n - \mu$ of the remaining free indices are contracted into $\tilde{D}$'s and ξ's). Of course $\mu \leqslant n$. Also note that, since to build a weak expression we begin with a normal form expression for a metric invariant, we may assume that

$$d \geqslant n - \mu.$$

Via a almost identical arguments to the even case above we get

$$\alpha+\gamma+\delta+\varepsilon=\frac{4d+k-n-2a-2\beta_0-2\beta_1-4\beta_2}{2},$$

and

$$\beta \leqslant 2d.$$

The condition imposed by the total number of free $\tilde{D}'s$ and ξ's now becomes

$$\alpha+\gamma+2\varepsilon+(n-\mu)\leqslant k-a-\beta_0-\beta_1-2\beta_2.$$

The total number of available indices on $\tilde{C}$'s imposes the constraint,

$$\beta_1+\beta_2+\gamma+2\delta+\mu\leqslant 4d.$$

As observed above, because of the symmetries possessed by the $\tilde{C}_{ABCD}$'s, we may assume that at most $2d$ of the indices on the $\tilde{C}$'s remain free. So

$$\beta_1+\beta_2+\gamma+2\delta+\mu\geqslant 2d.$$

Furthermore note that $X_P\eta^{P\,AB\cdots E}_{Q}\tilde{C}_{ABST}$ is skew on the pair "$_{ST}$" and so at least one index of each $\tilde{C}$ tractor must end up contracted into something other than $X_P\eta^{P\,AB\cdots E}_{Q}$. So

$$\beta_1+\beta_2+\gamma+2\delta\geqslant d.$$

Minimising the number of remaining free X's

$$2d+k-a-\alpha-\beta_0-\beta_2$$

is equivalent to maximising z subject to the system

$$\begin{aligned}
z-a-\alpha-\beta_0-\beta_2&=0\\
a+\alpha+\beta_0+\beta_1+2\beta_2+\gamma+\delta+\varepsilon&=\frac{4d+k-n}{2}\\
\beta_1+\beta_2+\gamma+2\delta+\mu-e_1&=2d\\
\mu-e_2&=n-d\\
a+\alpha+\beta_0+\beta_1+2\beta_2+\gamma+2\varepsilon-\mu+s_1&=k-n
\end{aligned}$$

$$\begin{aligned}\beta_1+\beta_2+\gamma+2\delta+\mu+s_2&=4d\\ \mu+s_3&=n\\ \beta_1+\beta_2+\gamma+2\delta-e_3&=d\\ \beta_0+\beta_1+\beta_2+s_4&=2d,\end{aligned}\tag{56}$$

where all variables on the left hand side are to be non-negative integers.

Adding the first two equations to half of the seventh and subtracting half of the third equation gives

$$z+\frac{\beta_1}{2}+\frac{\beta_2}{2}+\frac{\gamma}{2}+\varepsilon+\frac{e_1}{2}+\frac{s_3}{2}=\frac{2d+k}{2}.$$

Thus $z \leqslant \frac{2d+k}{2}=(2d_0+k_0)/2$. On the other hand adding the first and the fifth equations of the system yields

$$z+\beta_1+\beta_2+\gamma+2\varepsilon+(n-\mu)+s_1=k.$$

So $z \leqslant k \leqslant k_0$. Thus we again have

$$z \leqslant \min\left(k_0, \frac{2d_0+k_0}{2}\right).$$

Finally note that subtracting half the eighth equation from the sum of the first two equations gives

$$z+\frac{\beta_1}{2}+\frac{\beta_2}{2}+\frac{\gamma}{2}+\varepsilon+\frac{e_3}{2}=\frac{3d-n+k}{2}$$

and $\frac{3d-n+k}{2}$ may be less than $\min(k_0, (2d_0+k_0)/2)$. Now $3d-n+k$ will vary for different terms in an expression for an invariant and so, in general the best we can get from this (using that, in any non-trivial term, $d \leqslant [d_0+k_0/2]$) is that

$$z \leqslant \frac{3d_0-n+\frac{3}{2}k_0}{2}.$$

However this latter estimate follows from the above and so we have

$$m'=1+\max(z) \leqslant 1+\min\left(k_0, \left[\frac{2d_0+k_0}{2}\right]\right)$$

as claimed. Again it is easy to find solutions of the above system which achieve these upper bounds.

We now treat the cases of even and odd invariants for $n=3$. The details have been condensed here as the arguments are similar to above. The key point is that one again considers the problem of contracting h^{AB}'s into (45) except now $\tilde{C}$ means the tractor expression for the Cotton–York tensor.

Even invariants, $n=3$: Arguing as above we are led to the system

$$z-a-\alpha-\beta_0-\beta_2=0$$

$$a+\alpha+\beta_0+\beta_1+2\beta_2+\gamma+\delta+\varepsilon=\frac{2d+k}{2}$$

$$\beta_1+\beta_2+\gamma+2\delta-e=d$$

$$a+\alpha+\beta_0+\beta_1+2\beta_2+\gamma+2\varepsilon+s_1=k-d$$

$$\beta_0+\beta_1+\beta_2+s_2=2d$$

$$\beta_1+\beta_2+\gamma+2\delta+s_3=3d.$$

As before we have $\beta\leqslant 2d$ and this leads to the fifth equation. Otherwise the right-hand-sides of equations 2–6 here differ from those in the system (55) for the $n\geqslant 4$. In the case of the second equation this is because when $n=3$ there are initially $\frac{2d+k}{2}$ h^{AB}'s to contract into (45) (cf. $\frac{4d+k}{2}$ h^{AB}'s in the $n\geqslant 4$ cases). For the third equation we observe that since in the three dimensional case $\tilde{C}$ has valence 3 the condition that at most two indices on each $\tilde{C}$ should remain free now implies that at least *one* (cf. *two* indices for the cases $n\geqslant 4$) index of each $\tilde{C}$ must be contracted away. The forth equation has $k-d$ on the right-hand-side since here there are initially $k-d$ $\tilde{D}$'s (cf. k as in the cases $n\geqslant 4$). Finally we observe the fifth equation has $3d$ on the right-hand-side as this is the total number of indices on the $\tilde{C}$ tractors for this case.

Taking linear combinations of these equations as in the $n\geqslant 4$ case we obtain

$$z=\frac{d+k}{2}-\frac{\beta_1}{2}-\frac{\beta_2}{2}-\frac{\gamma}{2}-\varepsilon-\frac{e}{2}$$

and

$$z=k-d-\beta_1-\beta_2-\gamma-2\varepsilon-s_1.$$

Thus using these we get

$$z\leqslant \min\left(k-d,\frac{d+k}{2}\right).$$

Since $k-d \leqslant k_0-d_0$, $d+k \leqslant d_0+k_0$ and z must be integral the result follows.

Odd invariants, $n=3$: Arguing as for the $n \geqslant 4$ case above we are led to the system,

$$
\begin{aligned}
z-a-\alpha-\beta_0-\beta_2 &= 0 \\
a+\alpha+\beta_0+\beta_1+2\beta_2+\gamma+\delta+\varepsilon &= \frac{2d+k-3}{2} \\
\beta_1+\beta_2+\gamma+2\delta+\mu-e_1 &= d \\
\mu-e_2 &= 3-d \\
a+\alpha+\beta_0+\beta_1+2\beta_2+\gamma+2\varepsilon-\mu+s_1 &= k-d-3 \\
\beta_1+\beta_2+\gamma+2\delta+\mu+s_2 &= 3d \\
\mu+s_3 &= 3 \\
\beta_0+\beta_1+\beta_2+s_4 &= 2d.
\end{aligned}
$$

Comparing this system with the system (56) for $n \geqslant 4$, first observe that the eighth equation of the latter system does not arise in this case as now each $\tilde{C}$ tractor has just three indices. Note that the conditions $\beta \leqslant 2d$, $\mu \leqslant n$, $d \geqslant n-\mu$ are as before (with $n=3$), as is the equation defining z. Otherwise the equations above correspond to the other four equations of the system (56) with the right-hand sides differing for the following reasons: In the case of the second equation this is because when $n=3$ there are initially $\frac{2d+k-3}{2}$ h^{AB}'s to contract into (54) (c.f. $\frac{4d+k-n}{2}$ h^{AB}'s in the $n \geqslant 4$ cases). For the third equation we observe that, since in the three dimensional case $\tilde{C}$ has valence 3, the condition that at most two indices on each $\tilde{C}$ should remain free now implies that at least *one* (cf. *two* indices for the cases $n \geqslant 4$) index of each $\tilde{C}$ must be contracted away. The fifth equation here has $k-d-3$ on the right-hand-side (cf. $k-n$ for the $n \geqslant 4$ cases) since here there are initially $k-d$ $\tilde{D}$'s (cf. k in the cases $n \geqslant 4$). Finally we note that the sixth equation now has $3d$ on the right-hand-side as this is the total number of indices on the $\tilde{C}$ tractors for this case.

Taking linear combinations of these equations as in the $n \geqslant 4$ case we obtain

$$
z \leqslant \frac{d+k}{2} \qquad \text{and} \qquad z \leqslant k-d.
$$

Similarly adding the first two equations of the system we obtain

$$
z = \frac{2d+k-3}{2} - \beta_1-\beta_2-\gamma-\delta-\varepsilon
$$

and so $z \leqslant \frac{2d+k-3}{2}$. Thus, using these and that z must be integral, we get

$$z \leqslant \min\left(k-d, \left[\frac{d+k}{2}\right], \frac{2d+k-3}{2}\right)$$
$$\leqslant \min\left(k_0-d_0, \left[\frac{d_0+k_0}{2}\right], \frac{2d_0+k_0-3}{2}\right) = m'. \quad \blacksquare$$

Remark 7.1. In the proof, for each of the four cases we maximised z for some system with a given fixed (but unspecified) d and k. In the end we have considered the implications of allowing d and k to vary subject to $d \geqslant d_0$ and $2d+k = 2d_0+k_0 = |u|$ for fixed u and d_0. These last two constraints could of course be added to each of the four systems in the first place. The results given in each case are sharp for the full systems.

For example consider the case of even invariants in dimensions $n \geqslant 4$. If $d+\frac{k}{2} = \min(k, d+\frac{k}{2})$ then $k \geqslant 2d$ and for any such k and d and a solution to (55), with all variables non-negative, integral and with $z = d+\frac{k}{2}$, is given by $0 = \alpha = \beta = \gamma = \varepsilon = e$, $\delta = d$, $z = a = d+\frac{k}{2}$, $s_1 = \frac{k}{2}-d$ and $s_2 = s_3 = 2d$. On the other hand if $k = \min(k, d+\frac{k}{2})$ then $k \leqslant 2d$ and a solution for any such k and d with all variables non-negative, integral and with $z = k$ is given by $0 = \alpha = \beta = \gamma = \varepsilon = s_1$ and $\delta = 2d-\frac{k}{2}$, $e = 2d-k$, $s_2 = 2d$ and $z = a = s_3 = k$. Now allowing d and k to vary subject to $d \geqslant d_0$ and $2d+k = 2d_0+k_0 = |u|$ we achieve the bound claimed in the proposition by replacing d with d_0 and k with k_0 in the solutions above.

Matters are slightly more subtle in the case of odd invariants in dimensions $n \geqslant 4$ but in this case and the n=3 cases one can also find solutions achieving the bound for all $d_0 \geqslant 2$ and $k_0 \geqslant 0$.

8. WEYL INVARIANTS

Weyl invariants, as defined above, are easier than q-Weyl invariants to expand into standard expressions (i.e., expressions in terms of the Riemannian curvature and the Levi–Civita connection). This can help, for example, in determining if two invariants are distinct. For this reason it is helpful to establish which invariants arise as Weyl invariants. The following proposition is a step toward this.

PROPOSITION 8.1. *In odd dimensions the invariants established in Theorem 5.5 to be quasi-Weyl are also Weyl invariants.*

In even dimensions invariants of weight $|u| < \frac{n+2}{2}$ are Weyl.

Proof. Let us first treat the case of odd dimensions $n \geqslant 5$. In this case if w is <u>integral</u> then $(n+2w-2) \neq 0$ and so for $f \in \Gamma\mathcal{E}[w]$, $2X_{[P}D_{A]}f$ is a non-vanishing multiple of $D_{AP}f$. Thus, up to scale, the weak expressions are unaltered if, in their construction, we replace C_{ABCDEF} with

$9X_{[A}W_{BC][DE}X_{F]}$ and replace $D_{AP}f$ with $2X_{[P}D_{A]}f$. Now in each term of such a weak expression we move the X's to the left of the D_A's, using now just (29) to do this, and the result follows provided we establish in this setting that the estimates of Proposition 5.4 still hold. In fact this is readily verified to be the case. If one follows the programme for proving Proposition 5.4 then the argument goes through, essentially as before except with D_A replacing $\tilde{D}_A$ and W_{ABCD} replacing $\tilde{C}_{ABCD}$. Again contracted X's are eliminated. This time this involves using (27), that W_{ABCD} is X-saturated and (29) to move any X's past the tractor-D operators. The key point is that one is once again led to (54) only with each

$$\{\tilde{D}, \xi\}_{\underbrace{AB\cdots K}_{l}} \tilde{C}_{Q\cdots S}$$

now replaced by $D_A D_B \cdots D_K W_{Q\cdots S}$, or

$$\underbrace{D_A D_B \cdots D_C D^P D_E \cdots D_K}_{\ell+1} W_{Q\cdots P\cdots S}$$

(where P could be any index of W) or a contraction of such a term such as

$$h^{FR} \underbrace{D_A D_B \cdots D_C D^P D_E \cdots D_F \cdots D_K}_{\ell+2} W_{Q\cdots P\cdots R\cdots S}.$$

Using this and that $W_{ABCD} = W_{[AB][CD]}$ we again arrive at the set of Eqs. (56) and also to the estimate as in the proposition. This deals with odd $n \geqslant 5$ cases.

Recall that for all metric invariants we assume that $d_0 \geqslant 2$. It follows that metric invariants with $|u| < \frac{n+2}{2}$ have $k_0 < \frac{n-6}{2}$. In particular the order of the metric invariant is less than $\frac{n-6}{2}$ (and so clearly $n > 6$ if there are any non-trivial such metric invariants). For invariants satisfying this, the argument as above for odd dimensions carries over to the even dimensional case since we need only consider dimensions $\geqslant 8$ and, acting on

$$\underbrace{D_A \cdots D_B}_{\ell} W_{CDEF}, \quad \text{where} \quad \ell \leqslant \frac{n-8}{2},$$

$2X_{[P}D_{A]}$ is proportional to a non-zero multiple of D_{AP}.

Finally for the $n = 3$ case, in the construction of the weak expressions, we should replace $2X_{[A}C_{BC][D}X_{E]}$ with $X_{[A}W_{BC]DE}$ and, as above, replace $D_{AP}f$ with $2X_{[P}D_{A]}f$. Now moving the $X's$ to the left of the D's and so forth as for the $n \geqslant 4$ cases we quickly establish that the new reduced expression (which is expressed in terms of D operators acting on W_{ABCD} rather than $\tilde{D}$ operators on C_{ABC}) m' satisfies $m' \leqslant \min(k_0, (2d_0+k_0)/2)$ for even invariants and $m' \leqslant 1+\min(k_0, (2d_0+k_0)/2)$ for odd invariants. (That is we now get the same formula for the estimate as for invariants in higher dimensions.) Acting with an m' power of D^A it is easily verified that all invariants are recovered. ∎

Remark 8.2. In fact one can extend the above result to show that the set of Weyl invariants includes all invariants in odd dimensions and, in even dimensions, all invariants of weight u with $|u| < n$. This recovers the invariants that are constructible by the methods of [2]. The method for doing this is simply an adaption of the arguments used by [2] and so is not suitable for discussion here.

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