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Cells in any simple polygon formed by a planar point set

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Abstract

Let P be a finite point set in general position in the plane. We consider empty convex subsets of P such that the union of the subsets constitute a simple polygon S whose dual graph is a path, and every point in P is on the boundary of S. Denote the minimum number of the subsets in the simple polygons S's formed by P by $f_{\mathrm{p}}(P)$, and define the maximum value of $f_{\mathrm{p}}(P)$ by $F_{\mathrm{p}}(n)$ over all P with n points. We show that $\lceil (4n-17)/15 \rceil \leqslant F_{\mathrm{p}}(n) \leqslant \lfloor n/2 \rfloor$.

Keywords: Empty convex subsets; Simple polygons; The Erdős–Szekeres theorem

1. Introduction

Throughout the paper we consider only finite point sets in the plane, which are assumed to be in general position, that is, no three points on a line. For such a point set P, a subset of P that consists of the vertices of a convex polygon is called a *convex subset* of P and it is also said to be *in convex position*. We usually identify a convex subset with its convex hull. A convex subset is said to be *empty* if no point of P lies in the interior. More generally, a convex region in the plane is empty if its interior contains no points of P. An empty convex subset with size k is also called an empty convex k-gon in P.

In 1935, the historic paper of Erdős and Szekeres [3] asks for the value of the smallest integer $Y(k)$ such that any set of $Y(k)$ points contains a convex subset with size k. Subsequently, a similar question is asked by Erdős [2] for the smallest integer $Y_0(k)$ such that any set of $Y_0(k)$ points contains an empty convex subset with size k. It is proven that $Y_0(3)=4$ and $Y_0(4)=5$ by Klein in [3], and Harborth [5] shows that $Y_0(5)=10$. Horton gives a construction showing that $Y_0(7)$ is not finite in [6], that is, there are arbitrarily many points with no empty convex heptagons. For the remaining case of $k=6$, Overmars exhibits a set of 29 points, the largest known, with no empty convex hexagons in [9]. And recently, Gerken [4] shows that $Y_0(6)$ is finite; $Y_0(6) \leqslant 1717$. Namely, the current record is for $30 \leqslant Y_0(6) \leqslant 1717$. Some combinatorial results on partitioning a point set into disjoint empty convex subsets are presented in [8].

A polygon has its successive vertices and edges of line segments, called the *closed chain*. If the closed chain does not intersect itself, the polygon with its interior is said to be *simple*. We considered the variation on the convex partition theme in [7]: Given any planar point set P in general position, we consider empty convex subsets of P such that the union of the subsets form a single simple polygon S, and every point in P is on the boundary of S. We now call each

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such empty convex subset a *cell* in S. Let $f(P)$ represent the minimum number of cells in the simple polygons S's formed by P and let $F(n)$ be defined as the maximum value of $f(P)$ over all sets P with n points.

Then we showed the following results:

Theorem A.

$$\left\lceil \frac{n-1}{4} \right\rceil \leqslant F(n) \leqslant \left\lfloor \frac{3n-2}{5} \right\rfloor \quad \textit{for any integer } n \geqslant 3.$$

In other words, we investigate the minimum number of cells in any S formed by a given P. Note that *Horton sets* show $F(n) \geqslant n/4$ for an infinite sequence of n since they have no empty convex heptagons and that the trivial upper bound for $F(n)$ is $n-2$ if we triangulate any S.

The *dual graph* on S is defined as follows: The nodes of the graph correspond to the cells in S, and two nodes are adjacent if and only if the corresponding cells have a common side. Although it is natural that the dual graph of a simple polygon is a tree, we now deal with a simple polygon whose dual graph is a path. Let $f_{\mathrm{p}}(P)$ and $F_{\mathrm{p}}(n)$ be the same notations as $f(P)$ and $F(n)$, respectively, if the dual graphs of the simple polygons are restricted to paths.

Note that $f(P) \leqslant f_{\mathrm{p}}(P)$ holds for any set P of n points since a path is also a tree. Hence, $F(n) \leqslant F_{\mathrm{p}}(n)$ holds for any n. In addition, there always exists such a simple polygon S from an n point set P. In fact, let v be any vertex of the convex hull boundary of P. If we scan any other point of P by the half-line L with center v, L meets $p_0, p_1, \ldots, p_{n-2}$ with their order and we obtain an empty convex region Γ_i determined by $\{v, p_{i-1}, p_i\}$ for any i, $1 \leqslant i \leqslant n-2$. Since Γ_i contains exactly one empty triangle $\triangle_i = \triangle v p_{i-1} p_i$, we obtain $S = \triangle_1 \cup \triangle_2 \cup \cdots \cup \triangle_{n-2}$ with the dual graph a path.

In this paper, we present the following results where the upper bound of Theorem A is improved by Theorem 1:

Theorem 1.

$$F(n) \leqslant \left\lfloor \frac{n}{2} \right\rfloor \quad \textit{for any integer } n \geqslant 3.$$

Theorem 2.

$$\left\lceil \frac{4n-17}{15} \right\rceil \leqslant F_{\mathrm{p}}(n) \leqslant \left\lfloor \frac{n}{2} \right\rfloor \quad \textit{for any integer } n \geqslant 3.$$

In Section 2, we show that $F_{\mathrm{p}}(n) \leqslant \lfloor n/2 \rfloor$. Here, we can prove Theorem 1 by $F(n) \leqslant F_{\mathrm{p}}(n)$. For the lower bound for $F_{\mathrm{p}}(n)$, we find configurations in Section 3 with $F_{\mathrm{p}}(n) \geqslant \lceil (4n-17)/15 \rceil > \lceil (n-1)/4 \rceil$, where $\lceil (n-1)/4 \rceil$ is the lower bound of $F(n)$.

We begin with some notation used throughout the proofs. For any point set Q, we denote the convex hull of Q by $\mathrm{ch}(Q)$ and represent the boundary vertices of $\mathrm{ch}(Q)$ by $V_{\mathrm{ch}}(Q)$. We denote the vertices of the closed chain in any simple polygon T by $V_{\mathrm{sp}}(T)$.

We mainly use the following definitions in the next section: Let a, b and c be any three points in general position, not necessarily elements of P. We denote the *convex cone* by $\gamma(a; b, c)$ such that a is the center and b and c are on its boundary, i.e., $\gamma(a; b, c) = \{x \mid \overrightarrow{ax} = s\overrightarrow{ab} + t\overrightarrow{ac}$ for any scalars $s, t \geqslant 0\}$. For $\delta = b$ or c of the convex cone $\gamma(a; b, c)$, let δ' be a point collinear with a and δ, so that a lies on the line segment $\overline{\delta\delta'}$. For instance, we can consider the other convex cone $\gamma(a; b', c)$ for $\gamma(a; b, c)$ as shown in Fig. 1(i).

If $\gamma(a; b, c)$ is not empty, we define $\alpha(a; b, c)$ as the element of P in the interior of $\gamma(a; b, c)$ such that $\gamma(a; b, \alpha(a; b, c))$ is empty, called the *attack point* in $\gamma(a; b, c)$, from the half-line ab to ac. See Fig. 1(ii) where black points are elements of P. We let the *quasi-attack point* $\widetilde{\alpha}(a; b, c)$ in $\gamma(a; b, c)$ be the point c or the attack point $\alpha(a; b, c)$, respectively, if $\gamma(a; b, c)$ is empty or not.

Moreover, let R be a convex region in the plane and consider a convex cone $\gamma(a; b, c)$ such that $\{a, b, c\}$ is contained in R. Let $\gamma_R(a; b, c) = \gamma(a; b, c) \cap R$ denote the restriction of this convex cone to R. We similarly define $\alpha_R(a; b, c)$ as the point of P in the interior of $\gamma_R(a; b, c)$ so that $\gamma_R(a; b, \alpha_R(a; b, c))$ is empty. Finally, let $\widetilde{\alpha}_R(a; b, c)$ be c or $\alpha_R(a; b, c)$, respectively, if $\gamma_R(a; b, c)$ is empty or not.

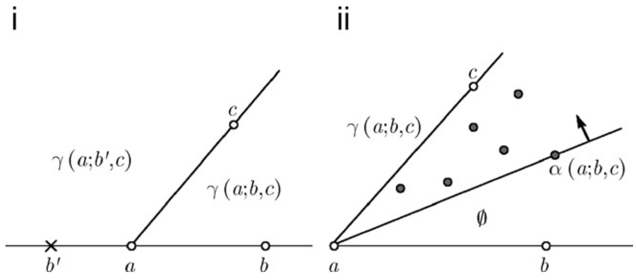

Fig. 1. (i) Two convex cones $\gamma(a; b, c)$ and $\gamma(a; b', c)$. (ii) Attack point $\alpha(a; b, c)$ from ab to ac.

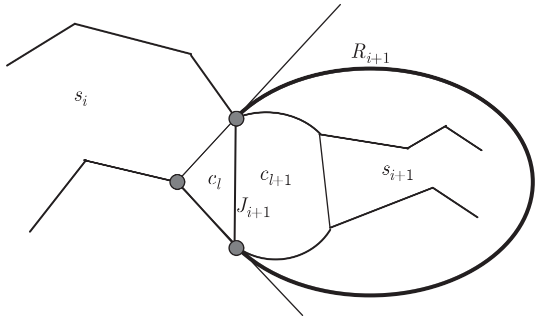

Fig. 2. The growing triangle c_l and the grown cell $c_l \cup c_{l+1}$.

2. Upper bound

We show that $F_{\mathrm{p}}(n) \leqslant \lfloor n/2 \rfloor$ for any $n \geqslant 3$, that is, we construct a simple polygon S from any n point set P whose dual graph is a path of length at most $\lfloor n/2 \rfloor$. That means, in particular, that the average size of all the cells in S is at least 4. Let $S = c_1 \cup c_2 \cup \cdots \cup c_N$ such that the cells c_i's are indexed in order of incidence, i.e., c_i has a common side with c_{i+1} for any i, $1 \leqslant i < N$. We call c_i the ith cell of S and we represent it by $c_i = (v_1 v_2 \ldots v_t)_t$ if it is a t-gon consisting of $\{v_1, v_2, \ldots, v_t\}$ with the counterclockwise order.

We present an iterative construction: At the first step we form a simple polygon s_1 whose dual graph is a path. At each ith step for $i \geqslant 2$, we form a simple polygon s_i for the union of simple polygons $S_{i-1} = s_1 \cup \cdots \cup s_{i-1}$ so that the dual graph of $S_i = S_{i-1} \cup s_i$ is a path. Then we obtain S as $S_L = s_1 \cup s_2 \cup \cdots \cup s_L$ at the last Lth step.

We call s_i the ith *subpolygon* of S, where we define that s_1 contains c_1. For any s_i, $i \geqslant 2$, we denote the line segment $S_{i-1} \cap s_i$ by J_i, called the *starting joint* of s_i, and we particularly define the starting joint J_1 of s_1 by any edge on the boundary of $\mathrm{ch}(P)$. Then s_i is said to *grow* from J_i for every i. The construction must proceed so that s_1 grows in the region $R_1 = \mathrm{ch}(P)$ and s_i grows in $R_i = \mathrm{ch}((P \setminus V_{\mathrm{sp}}(S_{i-1})) \cup J_i)$, satisfying $S_{i-1} \cap R_i = J_i$ for $i \geqslant 2$. We call R_i the *growing region* of s_i where $R_1 \supsetneq R_2 \supsetneq \cdots \supsetneq R_L$.

Naturally, a subpolygon consists of cells. If a cell in s_i has a common side with s_{i-1} or s_{i+1}, the cell is called the *first* or *last cell* of s_i, respectively, where we define the first cell of s_1 and the last cell of s_L as c_1 and c_N, respectively. We now introduce the special cells. Consider the last cell of s_i, say c_l and the growing region of s_{i+1} and suppose that c_l is a triangle and $c_l \cup R_{i+1}$ is a convex region. After the construction, we can moreover join c_l to the first cell c_{l+1} of s_{i+1} to form a single bigger cell. We call c_l and $c_l \cup c_{l+1}$ the *growing triangle* and the *grown cell*, respectively, as shown in Fig. 2.

We now consider the possible subpolygons in S which are classified into five *types* A, B, C, D and L as follows:

Type A: The first cell and the last cell are a triangle and a growing triangle, respectively, and the other cells are quadrilaterals.

Type B: The first and the last cell are a triangle and a pentagon, respectively, and the others are quadrilaterals.

Type C: All the cells are quadrilaterals.

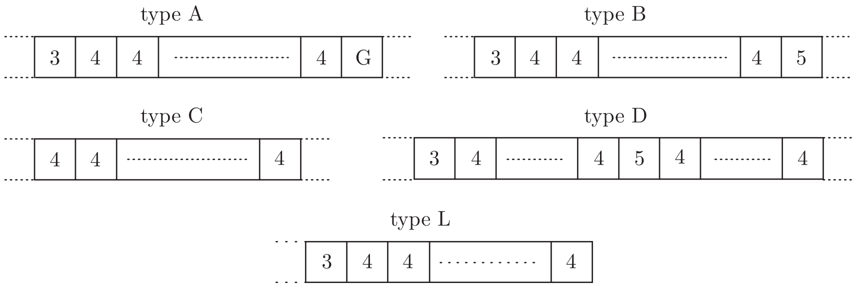

Fig. 3. All the types of subpolygons.

Type D: The first and the last cell are a triangle and a quadrilateral, respectively, and the others consist of a single pentagon and quadrilaterals.

We remark that the intermediate quadrilaterals between the first cell and the last cell may not exist in types except type C and we also define type A for the last subpolygon s_L though the triangle c_N is no longer growing. Note that type D is needed, though it is the union of types B and C, since it will occur that R_{i+1} is not defined if $s_i \cup s_{i+1} =$ type B $\cup$ type C.

These types are also candidates of s_L. In particular, we define type L as s_L such that the first cell is a triangle and any other cell is a quadrilateral which may not exist.

All the types are illustrated in Fig. 3 where 3, 4 or 5 stands for a triangle, quadrilateral or pentagon, respectively, and G is a growing triangle. The leftmost cell is the first in each type.

The following proposition holds where we rewrite $|T| = |V_{\rm sp}(T)|$ for simplicity for any simple polygon T.

Proposition 2.1. *If we construct a simple polygon* $S = s_1 \cup \cdots \cup s_L$ *from* P *so that every subpolygon belongs to some type*, *then* $F_{\rm p}(n) \leqslant \lfloor n/2 \rfloor$.

Proof. We observe that if s_L is in type L, $|s_L|$ is odd and if s_i belongs to any other type for each i, $1 \leqslant i \leqslant L$, $|s_i|$ is even. Any simple polygon T is said to be *good* if it consists of at most $\lfloor |T|/2 \rfloor$ cells. We claim that a subpolygon s_i consists of exactly $|s_i|/2$, $(|s_i|/2) - 1$ or $\lfloor |s_i|/2 \rfloor$ cells, respectively, if it is in type A, type j or type L for $j =$ B, C, D, from which it follows that every subpolygon is good.

We show that S is good by induction. Since the last subpolygon s_L is good, we suppose that $S^{i+1} = s_{i+1} \cup \cdots \cup s_{L-1} \cup s_L$ is good and show that $S^i = s_i \cup S^{i+1}$ is good for any i, $1 \leqslant i < L$. If s_i is in type A, we join the first cell of s_{i+1} to the last growing triangle of s_i to form one grown cell of a quadrilateral or pentagon. Therefore, S^i consists of at most $|s_i|/2 + \lfloor |S^{i+1}|/2 \rfloor - 1 = \lfloor |S^i|/2 \rfloor$ cells with $|S^i| = |s_i| + |S^{i+1}| - 2$. For otherwise, since s_i is in type j, S^i also consists of at most $\{(|s_i|/2) - 1\} + \lfloor |S^{i+1}|/2 \rfloor$ cells. $\square$

We now prove the upper bound for $F_{\rm p}(n)$.

Proof of the upper bound. We form a subpolygon s_i which belongs to some type for each i by Proposition 2.1. Consider any ith step for $i \geqslant 1$. We form such an s_i for a given J_i and a given R_i and determine the next J_{i+1} and R_{i+1}. We denote the first cell of s_i by c_k and let p and q be the endpoints of J_i. We first consider an element r of P in R_i such that the convex cone $\gamma_{R_i}(p; q, r)$ is empty, where $\{p, q, r\}$ is in the counterclockwise order.

If $\gamma_{R_i}(r; p, q')$ is not empty, i.e., the attack point $a_1 = \alpha_{R_i}(r; p, q')$ exists, we obtain the quadrilateral $pqra_1$ as c_k. Then since we can think of c_k itself as the ith subpolygon in type C, we proceed to the next step, that is, consider the first cell of s_{i+1} such that J_{i+1} is the line segment $\overline{a_1 r}$ and $R_2 = {\rm ch}(P \backslash J_1)$ with $J_1 = \overline{pq}$ and $R_{i+1} = {\rm ch}(P \backslash V_{\rm sp}(S_{i-1}))$ for $i \geqslant 2$. We remark that if $V_{\rm sp}(S_i) = P$, our construction ends in the ith step.

For otherwise, since $\gamma_{R_i}(q; p, r)$ is also empty, we cannot help choosing $\triangle pqr$ as c_k, that is, we hereafter form s_i in types except type C, starting with this triangle. We consider the set $P' = P \backslash (V_{\rm sp}(S_{i-1}) \cup \{r\})$ for $i \geqslant 2$ and $P' = P \backslash \{p, q, r\}$ for $i = 1$. Let $V_{\rm ch}(P') = \{1, 2, \ldots, m, \ldots\}$ with the order counterclockwise such that ${\rm ch}(P')$ is included in $\gamma(r; 1, m)$ and $\triangle 1rm$ contains $Q = \{1, 2, \ldots, m\}$ as shown in Fig. 4. We remark that $\gamma_{R_i}(1; p', r')$ may not be empty. We now denote $\gamma_i(a; b, c) = \gamma_{R_i}(a; b, c)$ and $\alpha_i = \alpha_{R_i}$ for simplicity.

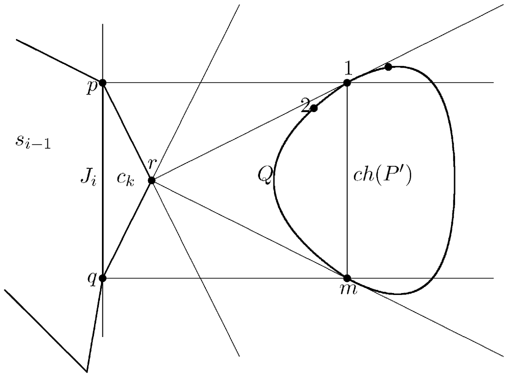

Fig. 4. s_i grows in R_i with the first triangular cell.

If $P' = \emptyset$, the construction ends in this step since c_k itself is the last subpolygon in type L. If $|P'| = 1$, $c_k \cup c_{k+1}$ is the last subpolygon in type A since we obtain $\triangle 1pr$ as c_{k+1}.

For $m \geqslant 2$, we present the first assumption.

Assumption 2.1. $\{1, m, p, r\}$ is in convex position, that is, m is in $\gamma_i(p; 1, r)$ for $m \geqslant 2$. In particular, $\{1, 2, p, r\}$ forms an empty convex quadrilateral.

In fact, we show that $\{1, m, p, r\}$ or $\{1, m, q, r\}$ is in convex position. If $\{1, m, q, r\}$ is not in convex position, the point m is contained in $\triangle 1rq$, implying that $\{1, m, p, r\}$ is in convex position. We obtain this assumption by symmetry.

For $m = 2$, $\triangle 2pr$ is empty by Assumption 2.1. Since P' is contained in $\gamma_i(r; 2, p)$, $\triangle 2pr \cup \mathrm{ch}(P' \cup \{p\})$ is convex. Thus we can obtain $s_i = c_k \cup c_{k+1}$ in type A such that $c_{k+1} = (2pr)_3$ is the growing triangle for $R_{i+1} = \mathrm{ch}(P' \cup \{p\})$. We proceed to the next $(i + 1)$st step by taking $J_{i+1} = \overline{2p}$ and R_{i+1}.

We hereafter consider the case $m \geqslant 3$. We propose the next assumption.

Assumption 2.2. Both the points 2 and 3 are in $\triangle 1rq$ for $m \geqslant 3$.

In fact, if the point 2 is not in $\triangle 1rq$, $\triangle 1rq$ is empty and we obtain $s_i = c_k \cup c_{k+1}$ in type A such that $c_{k+1} = (1rq)_3$ is the growing triangle where $J_{i+1} = \overline{1q}$ and $R_{i+1} = \mathrm{ch}(P' \cup \{q\})$. Suppose that 2 is in $\triangle 1rq$ and 3 is not in $\triangle 1rq$. We remark that 3 is not in $\gamma_i(1; p', r')$ since, otherwise, m is also in $\gamma_i(1; p', r')$ since m is in $\gamma(2; 1, 3)$, contradicting Assumption 2.1 and that 3 is in $\gamma(r; 2, p')$ by the configuration of $Q \cup \{r\}$, i.e., $\{2, 3, q, r\}$ is in convex position. We suppose that $\triangle 123$ is not empty since, if it is empty, we obtain $s_i = c_k \cup c_{k+1} \cup c_{k+2}$ in type A such that $c_{k+1} = (21pr)_4$ and $c_{k+2} = (123)_3$ is growing with $J_{i+1} = \overline{13}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2\})$.

If 3 is in $\gamma_i(2; p', q)$, we consider $a_1 = \alpha_i(3; 2, 1)$ since $\triangle 123$ is not empty. If a_1 is in $\gamma(2; 3, r')$, we obtain $s_i = c_k \cup c_{k+1}$ in type B by $c_{k+1} = (2rq3a_1)_5$ with $J_{i+1} = \overline{3a_1}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2\})$. If a_1 is in $\gamma(2; 1, r')$, we use the quasi-attack point $\widetilde{\alpha}_i = \widetilde{\alpha}_i(a_1; 3', 1)$ as shown in Fig. 5. Since $\{2, p, r, a_1, \widetilde{\alpha}_i\}$ forms an empty convex pentagon by adding $\triangle pa_1\widetilde{\alpha}_i$ to the convex quadrilateral $pr2a_1$, we also obtain s_i in type B by $c_{k+1} = (pr2a_1\widetilde{\alpha}_i)_5$ with $J_{i+1} = \overline{a_1\widetilde{\alpha}_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2\})$.

If 3 is in $\gamma_i(2; p', r')$, we assume that $\triangle 23p$ is empty since, if $a_2 = \alpha_i(p; 2, 3)$ is in $\gamma(2; 1, r')$ or not, we obtain s_i in type B by $c_{k+1} = (pr2a_2\widetilde{\alpha}_i(a_2; p, 1))_5$ or $(a_2 2rq\widetilde{\alpha}_i(a_2; p', 3))_5$ with $J_{i+1} = \overline{a_2\widetilde{\alpha}_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2\})$ for $\widetilde{\alpha}_i = \widetilde{\alpha}_i(a_2; p, 1)$ or $\widetilde{\alpha}_i(a_2; p', 3)$, respectively. Finally, if $a_3 = \alpha_i(3; p, 1)$ is in $\gamma(2; 1, r')$ or not, we obtain s_i in type B by $c_{k+1} = (pr2a_3\widetilde{\alpha}_i(a_3; 3', 1))_5$ or $(2rq3a_3)_5$ with $J_{i+1} = \overline{a_3\widetilde{\alpha}_i}$ or $\overline{3a_3}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2\})$, respectively.

Under Assumption 2.2, if $\{2, 3, q, r\}$ is not in convex position, $\{2, 3, p, r\}$ is in convex position since the point 3 is in $\triangle 2rq$. Therefore, we have the following three cases I, II and III for $m \geqslant 3$.

(I) Both $\{2, 3, p, r\}$ and $\{2, 3, q, r\}$ are in convex position: The point 3 is in $\gamma(p; 2, r) \cap \triangle 12q$.

If $m = 3$, i.e., $\gamma_i(r; 3, p')$ is empty, we obtain s_i in type A by the growing triangle $c_{k+1} = (3pr)_3$ with $J_{i+1} = \overline{3p}$ and $R_{i+1} = \mathrm{ch}(P' \cup \{p\})$.

For $m \geqslant 4$, the point 4 is in $\gamma_i(3; 2', r')$. If $\triangle 234$ is empty, we obtain s_i in type A such that $c_{k+1} = (32pr)_4$ and $c_{k+2} = (234)_3$ is growing with $J_{i+1} = \overline{24}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{3\})$. If not so, we consider $a_1 = \alpha_i(2; 3, 4)$. Then if a_1 is

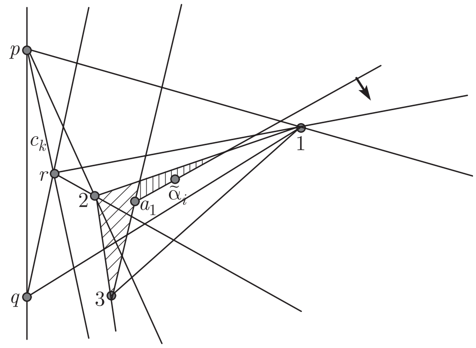

Fig. 5. Pentagonal cell formed by the quasi-attack point.

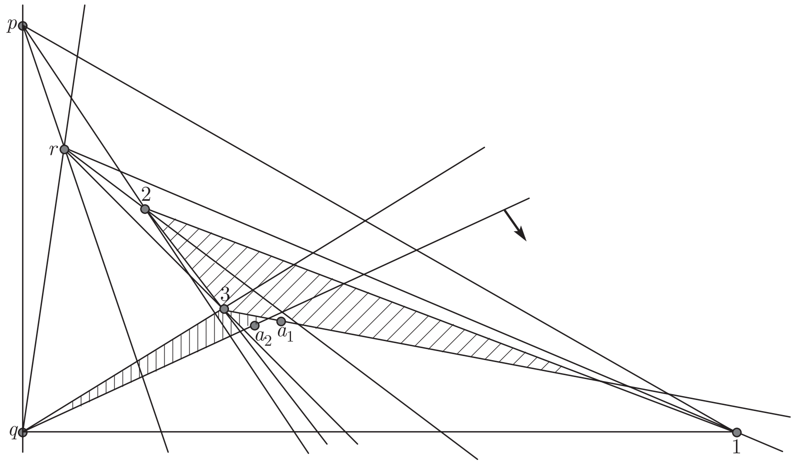

Fig. 6. Forming three cells in type B.

in $\gamma(r; 3, 2)$ or not, we obtain s_i in type B by $c_{k+1} = (2pr3a_1)_5$ or $(a_1 3rq\widetilde{\alpha}_i(a_1; 2', 4))_5$ with $J_{i+1} = \overline{2a_1}$ or $\overline{a_1\widetilde{\alpha}_i}$ and $R_{i+1} = \text{ch}(P'\backslash\{3\})$, respectively.

(II) $\{2, 3, p, r\}$ is not in convex position and $\{2, 3, q, r\}$ is in convex position: The point 3 is in $\gamma(2; p', r') \cap \triangle 12q$.

We consider $a_1 = \alpha_i(3; 2, 1)$ since, if $\triangle 123$ is empty, we obtain s_i in type A such that $c_{k+1} = (32rq)_4$ and $c_{k+2} = (123)_3$ is growing with $J_{i+1} = \overline{13}$ and $R_{i+1} = \text{ch}(P'\backslash\{2\})$. If a_1 is in $\gamma(2; 1, r')$, we obtain s_i in type B by $c_{k+1} = (pr2a_1\widetilde{\alpha}_i(a_1; 3', 1))_5$ with $J_{i+1} = \overline{a_1\widetilde{\alpha}_i}$ and $R_{i+1} = \text{ch}(P'\backslash\{2\})$.

Suppose that a_1 is in $\gamma(2; 3, r')$. If a_1 is moreover in $\gamma(3; 2, q')$, we obtain s_i in type B by $c_{k+1} = (2rq3a_1)_5$ with $J_{i+1} = \overline{3a_1}$ and $R_{i+1} = \text{ch}(P'\backslash\{2\})$. For $a_1 \in \gamma(3; 1, q')$, we consider $a_2 = \alpha_i(q; 3, a_1)$ since, if $\triangle 3qa_1$ is empty, we obtain s_i in type A such that $c_{k+1} = (32rq)_4$ and $c_{k+2} = (3qa_1)_3$ is growing with $J_{i+1} = \overline{a_1q}$ and $R_{i+1} = \text{ch}((P'\backslash\{2, 3\}) \cup \{q\})$. If a_2 is in $\gamma(3; a_1, r')$ as shown in Fig. 6, we obtain s_i in type B by $c_{k+1} = (21pr)_4$ and $c_{k+2} = (2r3a_2\widetilde{\alpha}_i(a_2; q', a_1))_5$ with $J_{i+1} = \overline{a_2\widetilde{\alpha}_i}$ and $R_{i+1} = \text{ch}(P'\backslash\{1, 2, 3\})$.

Suppose that a_2 is in $\gamma(3; 2', r')$. If $\gamma_i(3; 2', a_2)$ is empty, i.e., $a_2 = 4$, we obtain s_i in type A such that $c_{k+1} = (32rq)_4$ and $c_{k+2} = (234)_3$ is growing with $J_{i+1} = \overline{24}$ and $R_{i+1} = \text{ch}(P'\backslash\{3\})$. If not so, we consider $a_3 = \alpha_i(a_2; q, 3')$ and form $c_{k+1} = (a_2 3rqa_3)_5$. Then if we choose $c_{k+2} = (23a_2a_4)_4$ for $a_4 = \widetilde{\alpha}_i(a_2; q', a_1)$ as shown in Fig. 7, we obtain $s_i = c_k \cup c_{k+1} \cup c_{k+2}$ in type D with $J_{i+1} = \overline{a_2a_4}$ and $R_{i+1} = \text{ch}(P'\backslash\{2, 3, a_3\})$. We remark that $c_k \cup c_{k+1}$ is not adopted as a subpolygon in type B since we cannot determine the growing region of s_{i+1} then.

(III) $\{2, 3, p, r\}$ is in convex position and $\{2, 3, q, r\}$ is not in convex position: The point 3 is in $\gamma(2; 1', q)$.

If $m = 3$, we obtain s_i in type A by the growing triangle $c_{k+1} = (3pr)_3$ with $J_{i+1} = \overline{3p}$ and $R_{i+1} = \text{ch}(P' \cup \{p\})$.

Suppose that $m \geqslant 4$. We have the two subcases (a) and (b) by the position of the point 4.

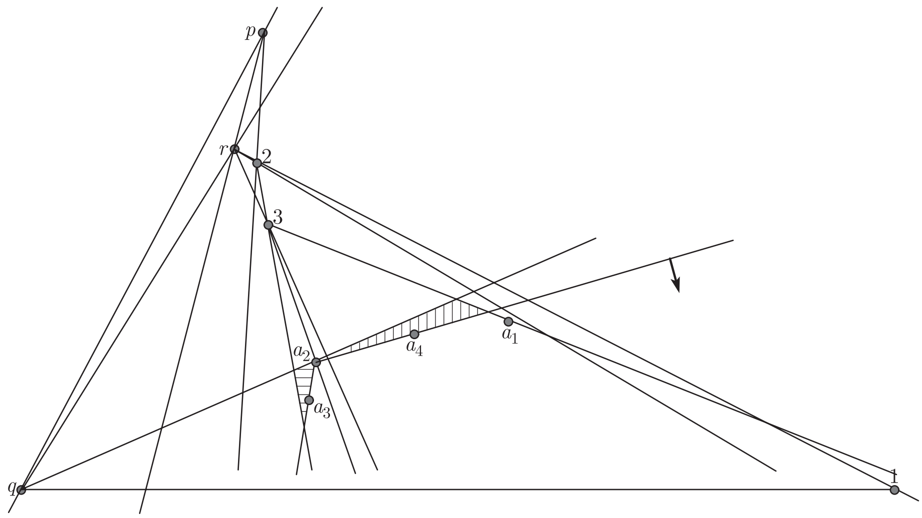

Fig. 7. Forming a subpolygon in type D.

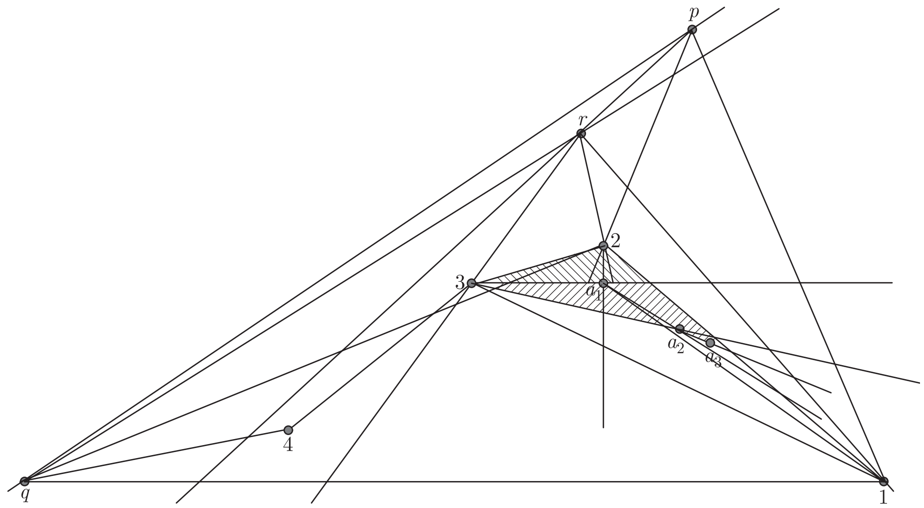

Fig. 8. Pentagonal cell formed by a_3.

(a) $\{3, 4, q, r\}$ is in convex position where the point 4 is in $\gamma_i(3; q, r')$: We consider $a_1 = \alpha_i(3; 2, 1)$ since, if $\triangle 123$ is empty, we obtain s_i in type A such that $c_{k+1} = (21pr)_4$ and $c_{k+2} = (123)_3$ is growing with $J_{i+1} = \overline{13}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2\})$. If a_1 is in $\gamma(2; 3, p')$ or $\gamma(2; r', 1)$, we obtain s_i in type B by $c_{k+1} = (2pr3a_1)_5$ or $(pr2a_1\widetilde{\alpha}_i(a_1; 3', 1))_5$ with $J_{i+1} = \overline{3a_1}$ or $\overline{a_1\widetilde{\alpha}_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2\})$, respectively.

Suppose that a_1 is in $\gamma(2; p', r')$. If $\gamma_i(3; a_1, 1)$ is empty, we obtain s_i in type A such that $c_{k+1} = (43rq)_4$, $c_{k+2} = (2r3a_1)_4$ and $c_{k+3} = (31a_1)_3$ is growing with $J_{i+1} = \overline{13}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 4, a_1\})$. Thus we consider $a_2 = \alpha_i(3; a_1, 1)$. If a_2 is in $\gamma(a_1; 3, 2')$ or $\gamma(a_1; 2', 1)$, we obtain s_i in type B by $c_{k+1} = (21pr)_4$ or $(32pr)_4$ and $c_{k+2} = (2r3a_2a_1)_5$ or $(p2a_1a_2\widetilde{\alpha}_i(a_2; 3', 1))_5$ with $J_{i+1} = \overline{3a_2}$ or $\overline{a_2\widetilde{\alpha}_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{1, 2, a_1\})$ or $\mathrm{ch}(P' \backslash \{2, 3, a_1\})$, respectively.

We suppose that a_2 is in $\gamma(a_1; 1, 3')$. If $a_3 = \alpha_i(a_2; 3', a_1')$ exists as shown in Fig. 8, we obtain s_i in type B by $c_{k+1} = (32pr)_4$ and $c_{k+2} = (p2a_1a_2a_3)_5$ with $J_{i+1} = \overline{a_2a_3}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 3, a_1\})$. If $\gamma_i(a_2; 3', a_1')$ is empty, we obtain s_i in type A such that $c_{k+1} = (21pr)_4$ and $c_{k+2} = (2r3a_1)_4$ and $c_{k+3} = (3a_2a_1)_3$ is growing with $J_{i+1} = \overline{3a_2}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{1, 2, a_1\})$.

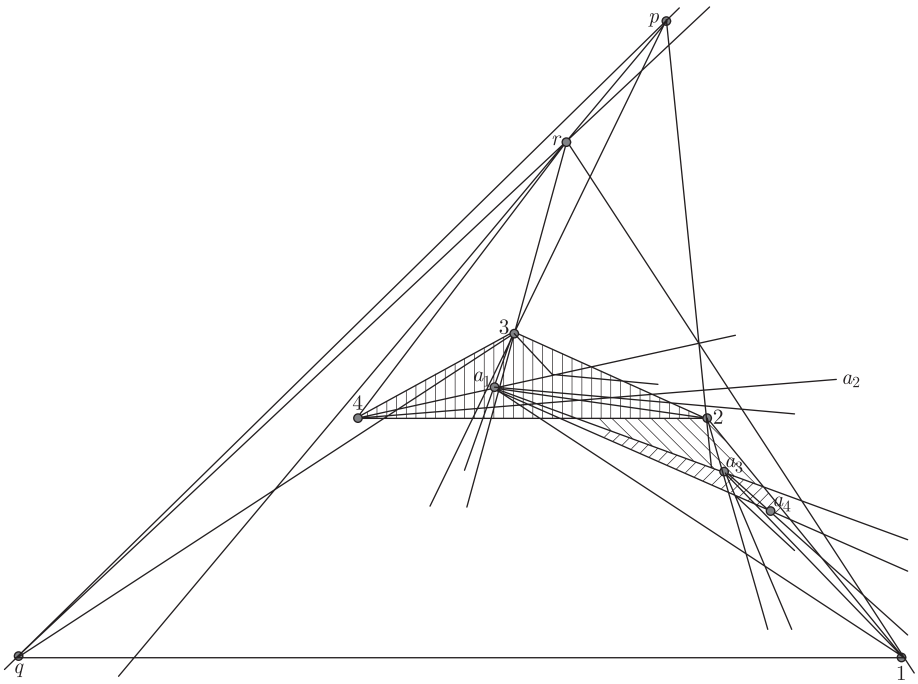

Fig. 9. Argument in III-b.

(b) $\{3, 4, p, r\}$ is in convex position and $\{3, 4, q, r\}$ is not in convex position where the point 4 is in $\gamma_i(3; 2', q)$: We first consider $a_1 = \alpha_i(4; 3, 2)$ since, if $\triangle 234$ is empty, we obtain s_i in type A such that $c_{k+1} = (32pr)_4$ and $c_{k+2} = (234)_3$ is growing with $J_{i+1} = \overline{24}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{3\})$. If a_1 is in $\gamma(3; 4, p')$ or $\gamma(3; r', 2)$, we obtain s_i in type B by $c_{k+1} = (3pr4a_1)_5$ or $(pr3a_1\widetilde{\alpha}_i(a_1; 4', 2))_5$ with $J_{i+1} = \overline{4a_1}$ or $\overline{a_1\widetilde{\alpha}_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{3\})$, respectively.

If a_1 is in $\gamma(3; p', r')$, we can assume that a_1 is the only element of P in the interior of $\triangle 234$. In fact, we consider $a_2 = \alpha_i(4; a_1, 2)$. If a_2 is in $\gamma(a_1; 3', 4)$ or $\gamma(a_1; 3', 2)$, we obtain s_i in type B by $c_{k+1} = (32pr)_4$ or $(43pr)_4$ and $c_{k+2} = (3r4a_2a_1)_5$ or $(p3a_1a_2\widetilde{\alpha}_i(a_2; 4', 2))_5$ with $J_{i+1} = \overline{4a_2}$ or $\overline{a_2\widetilde{\alpha}_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 3, a_1\})$ or $\mathrm{ch}(P' \backslash \{3, 4, a_1\})$, respectively. For $a_2 \in \gamma(a_1; 2, 4')$, if $\gamma_i(a_2; 4', a_1')$ is empty, we obtain s_i in type A such that $c_{k+1} = (32pr)_4$, $c_{k+2} = (3r4a_1)_4$ and $c_{k+3} = (4a_2a_1)_3$ is growing with $J_{i+1} = \overline{4a_2}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 3, a_1\})$. If not so, we obtain s_i in type B by $c_{k+1} = (43pr)_4$ and $c_{k+2} = (p3a_1a_2\alpha_i(a_2; 4', a_1'))_5$ with $J_{i+1} = \overline{a_2\alpha_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{3, 4, a_1\})$.

Now, if $\triangle 12a_1$ is empty, then $\triangle 123$ is also empty and we obtain s_i in type A such that $c_{k+1} = (21pr)_4$ and $c_{k+2} = (123)_3$ is growing with $J_{i+1} = \overline{13}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2\})$. If not so, we consider $a_3 = \alpha_i(a_1; 2, 1)$. If a_3 is in $\gamma(2; p', 4)$, we obtain s_i in type B by $c_{k+1} = (43pr)_4$ and $c_{k+2} = (2p3a_1a_3)_5$ with $J_{i+1} = \overline{a_1a_3}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 3, 4\})$.

Suppose that a_3 is in $\gamma(2; p', 1)$. If $\gamma_i(a_1; a_3, 1)$ is empty, we obtain s_i in type A such that $c_{k+1} = (32pr)_4$, $c_{k+2} = (23a_1a_3)_4$ and $c_{k+3} = (1a_3a_1)_3$ is growing with $J_{i+1} = \overline{1a_1}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 3, a_3\})$. We finally consider $a_4 = \alpha_i(a_1; a_3, 1)$. If a_4 is in $\gamma(2; 4, a_3)$ or $\gamma(a_3; 2', 1)$, we obtain s_i in type B by $c_{k+1} = (32pr)_4$ and $c_{k+2} = (23a_1a_4a_3)_5$ or $(p2a_3a_4\widetilde{\alpha}_i(a_4; a_1', 1))_5$ with $J_{i+1} = \overline{a_1a_4}$ or $\overline{a_4\widetilde{\alpha}_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 3, a_3\})$, respectively. For $a_4 \in \gamma(a_3; 1, a_1')$ as shown in Fig. 9, if $\gamma_i(a_4; a_1', a_3')$ is empty, we obtain s_i in type A such that $c_{k+1} = (32pr)_4$, $c_{k+2} = (23a_1a_3)_4$ and $c_{k+3} = (a_1a_4a_3)_3$ is growing with $J_{i+1} = \overline{a_1a_4}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 3, a_3\})$. If not so, we obtain s_i in type B by $c_{k+1} = (32pr)_4$ and $c_{k+2} = (p2a_3a_4\alpha_i(a_4; a_1', a_3'))_5$ with $J_{i+1} = \overline{a_4\alpha_i}$ and $R_{i+1} = \mathrm{ch}(P' \backslash \{2, 3, a_3\})$.

The proof of the upper bound is complete since we have considered all the possible cases. □

3. Lower bound

We show that $F_{\mathrm{p}}(n) \geqslant \lceil (4n - 17)/15 \rceil$, that is, there exists a configuration P of an n point set such that any simple polygon S formed by P with the dual graph a path needs $\lceil (4n - 17)/15 \rceil$ cells. For $3 \leqslant n \leqslant 8$, the lower bound trivially holds since $F_{\mathrm{p}}(n) \geqslant 1$ for any $n \geqslant 3$.

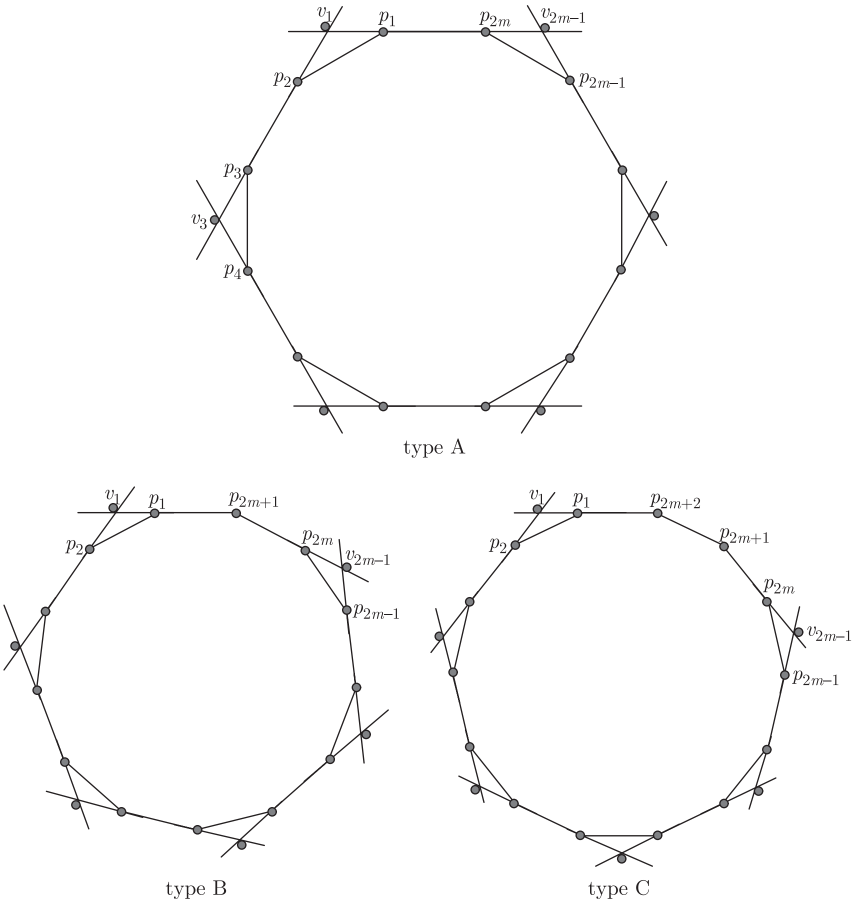

Fig. 10. Configuration of each type for $m = 6$.

For any $n \geqslant 9$, we give a configuration $P = I \cup V$, satisfied with (a) and (b):

(a) $I = \{p_i\}_{i \geqslant 1}$ construct a regular polygon, located in the order of their indices. Let x_i be the point of intersection of the lines $p_{i-1}p_i$ and $p_{i+2}p_{i+1}$.

(b) Each element v_i of $V = \{v_{2j-1}\}_{j \geqslant 1}$ is very near to x_i and in $\gamma(x_i;\, p'_i,\, p'_{i+1})$.

We deal with such a configuration by the following three types where $1 \leqslant j \leqslant m$ for any $m \geqslant 3$:
Type A: $|I| = 2m$ and $|V| = m$.
Type B: $|I| = 2m + 1$ and $|V| = m$.
Type C: $|I| = 2m + 2$ and $|V| = m$.
Fig. 10 gives configurations of all the types.

Note that in the example of type A, we cannot construct any simple polygons with 4 cells but with 5 cells as shown in Fig. 11(a). On the other hand, if we admit the dual graph to a tree, we obtain a simple polygon with 4 cells as in Fig. 11(b).

Observe that each element of V is on the boundary of $\mathrm{ch}(P)$, i.e., $V_{\mathrm{ch}}(P) = V$, $V \cup \{p_{2m+1}\}$ or $V \cup \{p_{2m+1},\, p_{2m+2}\}$ for type A, B or C, respectively. A pair of elements in V are called *friends* if they constitute an edge on the boundary of $\mathrm{ch}(P)$.

Let $S = c_1 \cup c_2 \cup \cdots \cup c_N$ for any simple polygon S from P where the cells c_i's are indexed in order of incidence. We first present the basic property of a cell in any S.

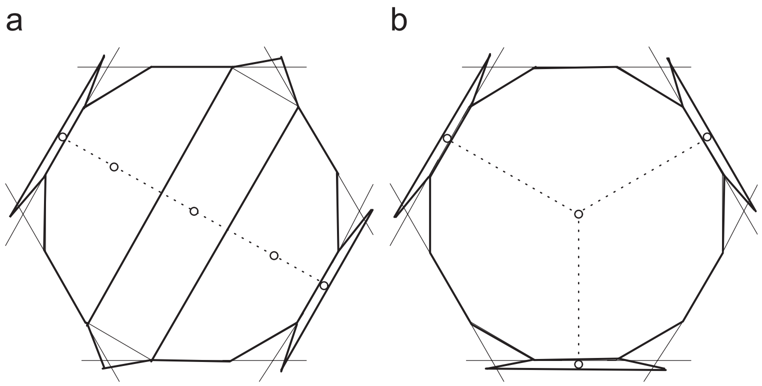

Fig. 11. Simple polygons from P and their dual graphs.

Proposition 3.1. *No cell in S contains more than two elements of V.*

Proof. Suppose that there exists a cell c_s with at least three elements of V. We consider any triangle in c_s determined by elements of V and denote it by $\triangle v_i v_j v_k$ with $i < j < k$. See Fig. 10. We consider three disjoint regions R_t's such that $\mathrm{ch}(P) \backslash \triangle v_i v_j v_k = R_1 \cup R_2 \cup R_3$ where the boundary of R_1, R_2 or R_3 has $\{v_i, v_j\}$, $\{v_j, v_k\}$ or $\{v_k, v_i\}$, respectively. We show that each R_t contains an element of P which does not belong to c_s. Then since the node corresponding to c_s has degree at least three in the dual graph, contradicting that our dual graph is a path.

In fact, we first consider the case for R_1. If $\{v_i, v_j\}$ are friends, i.e., $v_j = v_{i+2}$, c_s contains both p_{i+1} and p_{i+2}, a contradiction. If not so, R_1 contains an element of P and v_{i+2} or v_{j-2} is necessarily in R_1. Then c_s does not have, say v_{i+2} since, otherwise, c_s would have friends $\{v_i, v_{i+2}\}$ again. By the same reason, we have only to consider R_3 in types B and C as $v_i = v_1$ and $v_k = v_{2m-1}$. Then c_s would contain p_{2m} if c_s had p_{2m+1}. $\square$

We enumerate the cells in S by assigning elements of V by Proposition 3.1. A *subpolygon* is a simple polygon contained in S. Let $V(S') = V \cap V_{\mathrm{sp}}(S')$ for any subpolygon S'. If $|V(c_i)| = 0$, 1 or 2 for any i, we call c_i a 0-, 1- or 2-*cell*, respectively. A 2-cell is said to be *small* or *big*, respectively, if it contains friends or not. For any cell c_i, let j_i^+ and j_i^- be the line segments of $c_i \cap c_{i+1}$ and $c_i \cap c_{i-1}$, called the *right joint* and the *left joint* of c_i, respectively, where each of c_1 and c_N has the only joint. A joint is said to be *free*, *single* or *double*, respectively, if it contains 0, 1 or 2 elements of V.

We give the following properties about 2-cells, where (P-1) are by the configurations and (P-2) is derived since a big cell divides $\mathrm{ch}(P)$ into two disjoint regions, each of which contains an element of P as mentioned in the proof of Proposition 3.1:

(P-1) No small cell contains more than one free joint.
No pair of small cells have any common joint.
(P-2) A big cell c_i has both c_{i-1} and c_{i+1} for $S = c_1 \cup \cdots \cup c_N$.

Each small cell trivially has no double joint, and even if a big cell c_i has a double joint, say j_i^+, we can obtain a new cell $c_i \cup c_{i+1}$ without a double joint. Thus we present the following assumption without loss of generality.

Assumption 3.1. S contains no cells with double joints.

Any joint of a cell is free or single by the assumption. We prepare for the following lemmas to show the result where we choose c_{i+1} as any next cell to c_i without loss of generality.

Lemma 3.1. *If c_i is small and j_i^+ is free, then c_{i+1} is not a* 2*-cell. In particular, if c_1 or c_N is small with the joint free,* c_2 or c_{N-1} *is a* 0*-cell, respectively.*

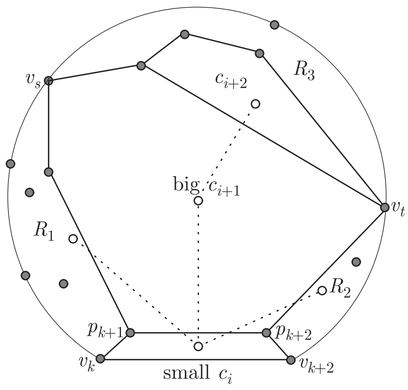

Fig. 12. c_i has degree three in the dual graph.

Proof. Let c_i be with $V(c_i) = \{v_k, v_{k+2}\}$. We remark that c_i is a quadrilateral by the free joint $\overline{p_{k+1}p_{k+2}}$ and any other possible joint is single, $\overline{v_k p_{k+1}}$ or $\overline{v_{k+2} p_{k+2}}$ by the configurations.

Since c_{i+1} is not small by (P-1), we suppose that c_{i+1} is big with $V(c_{i+1}) = \{v_s, v_t\}$ such that $\{v_s, v_k, v_{k+2}, v_t\}$ are located by their order on the boundary of $\mathrm{ch}(P)$. We consider three disjoint regions by $\mathrm{ch}(P)\backslash(c_i \cup c_{i+1}) = R_1 \cup R_2 \cup R_3$ where v_k and v_{k+2} are on the boundary of R_1 and R_2, respectively. If R_1 contains no elements of P, $\{v_s, v_k\}$ are friends in any type, or $v_k = v_1$ and $v_s = v_{2m-1}$ in types B and C since, otherwise, c_{i+1} would have more elements of V. Then c_{i+1} would contain p_{s+1} for each case. Suppose that each of R_1 and R_2 contains an element of P by symmetry. Since c_{i+2} lies in the opposite side of c_i with respect to c_{i+1} by (P-2), the node corresponding to c_i has degree exactly three, a contradiction. See Fig. 12 where a white point is the node in the dual graph.

Suppose that c_2 is a 1-cell for small c_1 with the common joint free. Let $V(c_2) = \{v_s\}$ and $V(c_1) = \{v_k, v_{k+2}\}$. We consider two disjoint regions by $\mathrm{ch}(P)\backslash(c_1 \cup c_2) = R_1 \cup R_2$ where v_k is on the boundary of R_1. Since the node to c_1 has degree one, if each of R_1 and R_2 contained an element of P, the node to c_2 would have degree at least three. If R_1 contained no elements of P by symmetry, c_2 would not be empty by the same way as the above. □

Lemma 3.2. *If c_i is big and j_i^+ is free, then c_{i+1} is a* 0*-cell.*

Proof. We suppose that c_{i+1} is a 1-cell or big since it is not small by Lemma 3.1. Let $V(c_i) = \{v_k, v_l\}$ and $V(c_{i+1}) = \{v_s, v_t\}$ such that $\{v_s, v_k, v_l, v_t\}$ are located by their order on the boundary of $\mathrm{ch}(P)$, where let $t = s$ if c_{i+1} is a 1-cell. Though $\mathrm{ch}(P)\backslash(c_i \cup c_{i+1})$ has three or four disjoint regions, we consider two regions R_1 and R_2 for both cases such that R_1 and R_2 have $\{v_k, v_s\}$ and $\{v_l, v_t\}$ on the boundary, respectively. If each of R_1 and R_2 contains an element of P, the node to c_{i+1} has degree at least three since c_{i-1} is in the opposite side to c_{i+1} with respect to c_i by (P-2), a contradiction.

Suppose that R_1 contains no elements of P by symmetry. We have the two cases (i) and (ii) since, otherwise, $|V(c_i)|$ or $|V(c_{i+1})|$ would increase. We consider the location of the common free joint $j_i^+ = \overline{p_x p_y}$.

(i) $\{v_s, v_k\}$ are friends in any type.

We have only to consider type A since we are similarly done for any other type. We suppose that $l < s < k$ without loss of generality and set $v_s = v_{k-2}$. Let $I = \{l+1, l+2, \ldots, k-3, k-2\}$ where $\{x, y\} \subset I \cup \{k-1, k\}$.

For $j_i^+ = \overline{p_{k-1}p_k}$, c_i would contain p_k. If $x = k-1$ or k for any $y \in I$, c_i or c_{i+1} would contain p_k or p_{k-1}, respectively. Suppose that $\{x, y\} \subseteq I$. Then if $j_i^+ \neq \overline{p_{k-3}p_{k-2}}$, $\triangle p_x p_y v_k$ overlaps $\triangle p_x p_y v_{k-2}$, i.e., c_i and c_{i+1} would overlap as shown in Fig. 13. For $j_i^+ = \overline{p_{k-3}p_{k-2}}$, $c_{i+1} = \triangle p_{k-2}p_{k-3}v_{k-2}$ and c_i would contain p_k.

(ii) $v_k = v_1$ and $v_s = v_{2m-1}$ in types B and C.

We can argue by the same way as (i). Let $I = \{l+1, \ldots, 2m-1\}$ and let $I' = \{1, 2m, 2m+1\}$ or $\{1, 2m, 2m+1, 2m+2\}$ for type B or C, respectively. If $\{x, y\} \subseteq I$ or $\{x, y\} \subset I'$, then c_i would contain p_1 or $\triangle p_x p_y v_1$ would overlap $\triangle p_x p_y v_{2m-1}$. For $x \in I$ and $y \in I'$, c_i or c_{i+1} would contain p_1 or p_{2m}, respectively. □

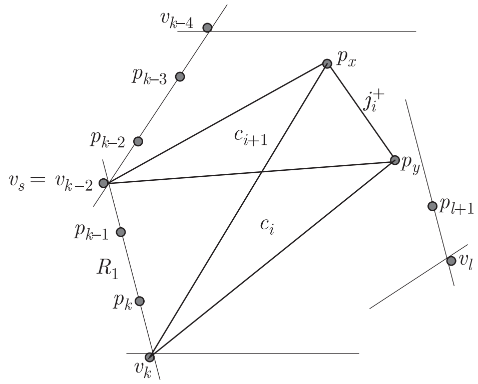

Fig. 13. $\triangle p_x p_y v_k$ would overlap $\triangle p_x p_y v_{k-2}$.

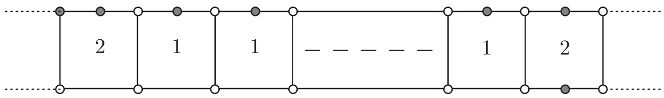

Fig. 14. The monster m_k.

We present the following propositions from which we derive the lower bound. We denote the number of cells in a subpolygon S' of S by $N(S')$ and we use the same notation $V(j_i^{\pm}) = V \cap j_i^{\pm}$ for any joint $j_i^{\pm}$ in S.

We consider a subpolygon $c_{i+1} \cup c_{i+2} \cup \cdots \cup c_{i+k} \subseteq S$ for $k \geqslant 3$ such that every joint j_j^+ is free for $i+1 \leqslant j \leqslant i+(k-1)$ and represent it by σ_k. We call σ_k the *monster* with size k, denoted by m_k if both c_{i+1} and c_{i+k} are 2-cells and any other cell is a 1-cell. Fig. 14 illustrates the monster where a black or white point is in V or I, respectively, and 1 or 2 stands for a 1- or 2-cell, respectively. We remark that $|V(m_k)| = k + 2$.

Proposition 3.2. *A subpolygon σ_k is only the monster m_k in S for $k \geqslant 3$ if $|V(\sigma_k)| \geqslant k + 2$.*

Proof. We claim that no c_j is small in σ_k for $i + 2 \leqslant j \leqslant i + (k - 1)$ by (P-1). Let $t_j = c_{i+1} \cup c_{i+2} \cup \cdots \cup c_{i+j}$ for $1 \leqslant j \leqslant k - 1$ and we first show that $|V(t_j)| \leqslant j + 1$ for any j by induction on j. Suppose that $|V(t_l)| \leqslant l + 1$ for any $l \leqslant j - 1$ by $|V(t_1)| \leqslant 2$. Then c_{i+j} is a 2-cell for $j \geqslant 2$ since, otherwise, $|V(t_j)| = |V(t_{j-1})| + |V(c_{i+j})| - |V(j_{i+j}^-)| \leqslant j + 1 - 0 = j + 1$ and we are done. Since c_{i+j} is big and $c_{i+(j-1)}$ is a 0-cell by Lemma 3.2, it holds that $|V(t_j)| = |V(t_{j-2})| + |V(c_{i+(j-1)})| + |V(c_{i+j})| \leqslant (j - 1) + 0 + 2 = j + 1$ for any $j \geqslant 3$ and $|V(t_2)| = 2$.

Since $|V(t_{k-1})| \leqslant k$, c_{i+k} is a 2-cell if $|V(\sigma_k)| \geqslant k + 2$, following by symmetry that c_{i+1} is also a 2-cell. If there exists a big cell c_{i+b} for $b \neq 1, k$, both $c_{(i+b)-1}$ and $c_{(i+b)+1}$ are 0-cells by Lemma 3.2 where $3 \leqslant b \leqslant k - 2$. Let $t^{k-b+1} = c_{i+k} \cup c_{i+(k-1)} \cup \cdots \cup c_{i+b}$ with $N(t^{k-b+1}) = k - b + 1$. Since $|V(t^{k-b+1})| \leqslant (k - b + 1) + 1$ by symmetry, it holds that $|V(\sigma_k)| = |V(t_{b-2})| + |V(c_{i+(b-1)})| + |V(t^{k-b+1})| \leqslant \{(b - 2) + 1\} + 0 + (k - b + 2) = k + 1$. For otherwise, every cell in σ' is a 1-cell for $\sigma' = c_{i+2} \cup \cdots \cup c_{i+(k-1)}$ since $|V(\sigma')| \geqslant (k + 2) - 4 = k - 2 = N(\sigma')$ if $|V(\sigma_k)| \geqslant k + 2$. $\square$

Proposition 3.3. *If a subpolygon $s_k = c_{i+1} \cup c_{i+2} \cup \cdots \cup c_{i+k} \subseteq S$ does not contain any size of monsters for $k \geqslant 1$, then $|V(s_k)| \leqslant k + 1$.*

Proof. Let $J_{s_k} = \{j_{i+1}^+, j_{i+2}^+, \ldots, j_{i+(k-1)}^+\}$ for $k \geqslant 2$ and $J_{s_1} = \emptyset$. Recall Assumption 3.1 and we show by induction on the number of single joints in J_{s_k}, denoted by $n(s_k)$. If $n(s_k) = 0$, i.e., every joint in J_{s_k} is free, we are done. In fact, $|V(s_1)| \leqslant 2$ and $|V(s_2)| \leqslant 3$ holds since, otherwise, both c_{i+1} and c_{i+2} are 2-cells with the free common joint, contradicting Lemma 3.1 or 3.2. Since $s_k = \sigma_k$ for $k \geqslant 3$, $|V(s_k)| \leqslant k + 1$ holds by the contraposition of Proposition 3.2.

For $n(s_k) \geqslant 1$, there is a single joint, say j_{i+m}^+ in J_{s_k}. Let $s_k = t_m \cup t^{k-m}$ for $t_m = c_{i+1} \cup \cdots \cup c_{i+m}$ and $t^{k-m} = c_{i+(m+1)} \cup \cdots \cup c_{i+k}$. Since $n(t_m) < n(s_k)$ and $n(t^{k-m}) < n(s_k)$, it holds that $|V(s_k)| = |V(t_m)| + |V(t^{k-m})| - |V(j_{i+m}^+)| \leqslant (m + 1) + (k - m + 1) - 1 = k + 1$. $\square$

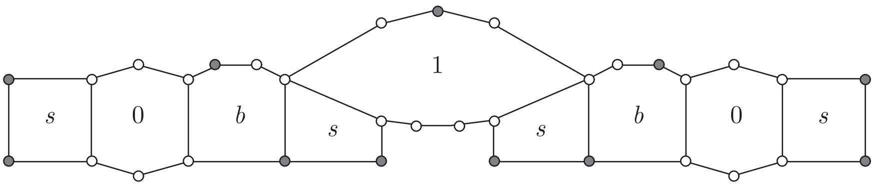

Fig. 15. S contains the monster m_3 with $n = 33$.

We finally show that $N \geqslant l(n) = (4n - 17)/15$ for any $n \geqslant 9$ with any simple polygon $S = c_1 \cup \cdots \cup c_N$, following that $F_{\rm p}(n) \geqslant \lceil (4n - 17)/15 \rceil$ for any $n \geqslant 3$.

Proof of the lower bound. If S does not contain any monsters, $|V(S)| \leqslant N(S) + 1$ holds by Proposition 3.3. Then $N \geqslant |V| - 1 \geqslant (n-2)/3 - 1 \gneqq l(n)$ since $|V| \geqslant (n-2)/3$.

Suppose that S contains a monster $m_k = c_{i+1} \cup c_{i+2} \cup \cdots \cup c_{i+k}$. We claim that both c_{i+1} and c_{i+k} are small since, if big, c_{i+2} and c_{i+k-1} would be 0-cells by Lemma 3.2. Thus, no pair of monsters are consecutive in S by (P-1) where they may have a common point of V. Moreover, $c_{i+1} \neq c_1$ and $c_{i+k} \neq c_N$ since, otherwise, c_{i+2} and c_{i+k-1} would be also 0-cells by Lemma 3.1, following by (P-1) and Assumption 3.1 that both j^-_{i+1} and j^+_{i+k} are single.

We now think of S as the union of odd subpolygons by $S = s_1 \cup s_2 \cup \cdots \cup s_{2L+1}$, indexed in order of incidence such that s_{2j} is a monster for $1 \leqslant j \leqslant L$ and s_{2j-1} contains no monsters for $1 \leqslant j \leqslant L+1$, where $|V(s_{2j})| = N(s_{2j}) + 2$ and $|V(s_{2j-1})| \leqslant N(s_{2j-1}) + 1$ and each of $s_{2j-1} \cap s_{2j}$ and $s_{2j} \cap s_{2j+1}$ has the single joint for any monster s_{2j}.

Let $S_j = s_{2j-1} \cup s_{2j}$ for $1 \leqslant j \leqslant L$. Since $|V(S_j)| = |V(s_{2j-1})| + |V(s_{2j})| - |V(s_{2j-1} \cap s_{2j})| \leqslant (N(s_{2j-1}) + 1) + (N(s_{2j}) + 2) - 1 = N(S_j) + 2$ for any j, the next inequality holds:

$$
\begin{aligned}
|V(S)| &= |V(S_1 \cup \cdots \cup S_L \cup s_{2L+1})| \\
&= |V(S_1)| + \cdots + |V(S_L)| + |V(s_{2L+1})| - L \\
&\leqslant (N(S_1) + 2) + \cdots + (N(S_L) + 2) + (N(s_{2L+1}) + 1) - L = N(S) + 2L + 1 - L.
\end{aligned}
$$

Hence, we have $L \geqslant |V| - N - 1(1)$. On the other hand, since S_j contains a monster m_k for $k \geqslant 3$, $N(S_j) \geqslant 1 + 3 = 4$ for every j and $N(s_{2L+1}) \geqslant 1$. We obtain $N = N(S_1) + \cdots + N(S_L) + N(s_{2L+1}) \geqslant 4L + 1(2)$. It follows from (1) and (2) that $N \geqslant 4L + 1 \geqslant 4(|V| - N - 1) + 1$, that is, $N \geqslant (4|V| - 3)/5 \geqslant l(n)$ holds for $|V| \geqslant (n-2)/3$. □

We remark that there certainly exists such a monster in S. Fig. 15 illustrates the example $S = c_1 \cup \cdots \cup c_9$ from type A with $|P| = 33$ such that $m_3 = c_4 \cup c_5 \cup c_6$ and S is symmetric with respect to c_5 where s or b stands for a small or big cell, respectively.

4. Discussion

Although it was conjectured in [7] that $F(n) = n/2$, we can now expect that $F(n) < n/2$ by $F_{\rm p}(n) \leqslant n/2$. On the other hand, they presented the open problem in [1] whether it is true that $F(n) \geqslant (n-2)/3$, where the negative answer of $F(n) < (n-2)/3$ also implies the finiteness of $Y_0(6)$. We finally present the following conjecture:

Conjecture. $F(n) = n/3$ and $F_{\rm p}(n) = n/2$ for any $n \geqslant 3$.

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