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Microlocal Properties of Filtered Rings

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0. INTRODUCTION AND TERMINOLOGY

The process of microlocalization appeared first in a strongly analytic frame in relation to systems of linear differential equations with holomorphic coefficients (see, e.g., [13]). From the strictly algebraic point of view, the process of microlocalization gives an answer to the following universal problem: If R is a filtered ring and $S \subset R$ is a multiplicatively closed subset, does there exist a complete filtered ring, denoted by $Q_S^{\mu}(R)$, with a filtered ring homomorphism $\varphi: R \to Q_S^{\mu}(R)$ such that the two conditions below are satisfied?

(i) $\varphi(s)$ is a unit of $Q_S^{\mu}(R)$, for every $s \in S$.

(ii) If $f: R \to A$ is filtered ring homomorphism, where A is complete and $f(s)$ is a unit of A for each $s \in S$, then there exists a unique filtered ring homomorphism $g: Q_S^{\mu}(R) \to A$ such that $g \circ \varphi = f$.

The first purely algebraic approach to the matter was carried out by Springer [14], who solved the problem for the case when R is commutative. The general settlement of the solution was fulfilled by Van den Essen [16] for the case when $\sigma(S)$ is a left Ore set of the associated graded ring $G(R)$ (see ahead for the definition of $\sigma(S)$). However, a large number of analytic tools concerning the use of norms and pseudonorms were used by the author. This still very analytic sediment was completely avoided by Asensio, Van den Bergh, and Van Oystaeyen [1], who gave an utterly algebraic description of the process by considering the generalized Rees ring $\tilde{R}$ associated with the filtered ring R. Since then, microlocalization has been a subject of study in several papers (see, e.g., [5–8, 17]).

The aim of these notes is to show certain localization-like properties of the above process. We divide them into three sections, apart from this

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preliminary one. In Section 1 we see that the microlocalization with respect to $S \subset R$ actually depends on the ring of quotients $\sigma(S)^{-1} G(R)$ and not on the set S itself (Proposition 1.3). It is also proved in that section that, when $G(R)$ is left Noetherian and M is a filt-finitely generated left $\hat{R}$-module ($\hat{R}$ is the completion of R), a surjective order-preserving lattice homomorphism may be given between the lattice of $\hat{R}$-submodules of M and the lattice of $Q_S^\mu(R)$-submodules of $Q_S^\mu(M)$ (Theorem 1.5). As a corollary, in that situation, the dimension of $\hat{R}$ is an upper bound for the dimension of $Q_S^\mu(R)$, for several dimensions of rings (Corollary 1.6). In Section 2, we prove that if R is a left and right Zariski ring and N is a filt-finitely generated R–R-bimodule, then the right R-module $\operatorname{Ext}_R^n(M, N)$ can be given a good filtration for which $Q_S^\mu(\operatorname{Ext}_R^n(M, N)) \cong \operatorname{Ext}_Q^n(Q_S^\mu(M), Q_S^\mu(N))$, where $Q = Q_S^\mu(R)$ (Theorem 2.6). This result is used to give a microlocal estimation of the global and Krull dimensions of $\hat{R}$ (Theorem 2.9) and also to show that Auslander regularity is a microlocal property of $\hat{R}$ (Theorem 2.10). The last section is dedicated to apply the foregoing results to discrete and strongly filtered rings.

All rings considered in the sequel are assumed associative with 1 and, unless otherwise stated, module means left module. If A is a ring A-Mod stands for the (Grothendieck) category of A-modules. If M is an object of A-Mod, then $\mathfrak{L}({}_A M)$ is the lattice of submodules of M.

A *filtered ring* is a ring R together with a $\mathbb{Z}$-indexed ascending family $\{F_n R/n \in \mathbb{Z}\}$ of additive subgroups, which is called the *filtration*, such that $F_m R \cdot F_n R \subseteq F_{m+n} R$, for any integers m and n. A *filtered R-module* is a module M with a filtration $\{F_n M/n \in \mathbb{Z}\}$ of additive subgroups such that, for all integers m and n, $F_m R \cdot F_n M \subseteq F_{m+n} M$. All filtrations on modules are considered to be exhaustive (i.e., $\bigcup_{n \in \mathbb{Z}} F_n M = M$). Moreover, the filtration on the ring R is assumed to be separated (i.e., $\bigcap \{F_n R/n \in \mathbb{Z}\} = 0$). A filtration FM on an R-module M is said to be *complete* (or M is a *complete filtered R-module*) if the inverse system provided by the canonical projections $M/F_n M \to M/F_{n+1} M$ satisfies that $M = \varprojlim M/F_n M$. This is equivalent to saying that any Cauchy sequence has a unique limit in M. The *discrete filtrations* (i.e., there is a $p \in \mathbb{Z}$ such that $F_n M = 0$, for $n \leqslant p$) are a trivial example of complete filtrations. It is always possible to associate a complete R-module $\hat{M}$ with a given filtered R-module M by defining $\hat{M} = \varprojlim M/F_n M$ and filtration $F_p \hat{M} = \varprojlim_{n \leqslant p} F_p M/F_n M$. This filtered module is called the *completion* of M. When $M = {}_R R$ (i.e., R considered as left R-module), $\hat{R}$ is also a filtered ring and $\hat{M}$ becomes an $\hat{R}$-module, for any R-module M. An R-homomorphism f between filtered R-modules M and N is said to be a *filtered morphism of degree* $p \in \mathbb{Z}$ in case $f(F_n M) \subseteq F_{n+p} N$ for any $n \in \mathbb{Z}$. If we denote by $F_p \operatorname{HOM}_R(M, N)$ the additive subgroup of $\operatorname{Hom}_R(M, N)$ consisting of the filtered morphism of degree p, then we get a filtered $\mathbb{Z}$-module $\operatorname{HOM}_R(M, N) =$

$\bigcup_{p\in\mathbb{Z}} F_p \operatorname{HOM}_R(M, N)$. The pre-Abelian category whose objects are the filtered left (resp. right) R-modules and whose morphisms are the filtered morphisms of degree zero is denoted by *R-filt* (resp. *filt-R*). If $f \in F_0 \operatorname{HOM}_R(M, N) = \operatorname{Hom}_{R\text{-filt}}(M, N)$, it is called *strict* when $F_n N \cap \operatorname{Im} f = f(F_n M)$ for each $n \in \mathbb{Z}$. If $M \in R$-filt and $p \in \mathbb{Z}$, we can construct a new filtered R-module $M(p)$ as follows: $M(p) = M$ as R-modules and $F_n M(p) = F_{n+p} M$ for any $n \in \mathbb{Z}$. This is called the *p-shifted filtered module derived from M*. If $M, N \in R$-filt and $p, q \in \mathbb{Z}$, then $F_n \operatorname{HOM}_R(M(p), N(q)) = F_{n+q-p} \operatorname{HOM}_R(M, N)$ for every $n \in \mathbb{Z}$. A *(finitely generated) filt-free R-module* is the direct sum in R-filt of a (finite) family $\{R(p_i)/i \in I\}$ of filtered R-modules, where each $R(p_i)$ is the p_i-shifted R-module derived from R. A *filt-finitely generated* R-module is a filtered R-module M for which there is a surjective strict morphism $\pi: L \to M$, from a finitely generated filt-free R-module L onto M. In that case the filtration on M is said to be *good*. If $M \in R$-filt and $j: N \to M$ (resp. $\pi: M \to N$) is a monomorphism (resp. epimorphism) in R-Mod, then the filtration in N given by $F_n N = j^{-1}(F_n M)$ (resp. $F_n N = \pi(F_n M)$), for each $n \in \mathbb{Z}$, is called the *induced* (resp. *quotient*) *filtration* from FM. For a more detailed account of the topics related to filtered rings and modules, the reader is referred to [9, Chap. D].

A ring A is called a *$\mathbb{Z}$-graded ring* if there exists a $\mathbb{Z}$-indexed family $\{A_n/n \in \mathbb{Z}\}$ of additive subgroups of A such that $A = \bigoplus_{n\in\mathbb{Z}} A_n$ and $A_m \cdot A_n \subseteq A_{m+n}$, for all integers m and n. Although the notion of G-graded ring (G is a group) can be defined in a similar manner, here we are only concerned with $\mathbb{Z}$-graded rings so that, in the sequel, graded ring will mean $\mathbb{Z}$-graded ring. An A-module K for which there exists a family $\{K_n/n \in \mathbb{Z}\}$ of additive subgroups satisfying that $K = \bigoplus_{n\in\mathbb{Z}} K_n$ and $A_m \cdot K_n \subseteq K_{m+n}$, for all integers m and n, is said to be a *graded A-module*. An A-homomorphism f between two graded A-modules $K = \bigoplus_{n\in\mathbb{Z}} K_n$ and $H = \bigoplus_{n\in\mathbb{Z}} H_n$ is a *graded morphism of degree* $p \in \mathbb{Z}$ in the case where $f(K_n) \subseteq H_{n+p}$, for any $n \in \mathbb{Z}$. The category whose objects are the graded A-modules and whose morphisms are the graded morphisms of degree zero is denoted by A-gr and is a Grothendieck category. If $K = \bigoplus_{n\in\mathbb{Z}} K_n$ is an object of A-gr, the elements of $\bigcup_{n\in\mathbb{Z}} K_n$ are called *homogeneous* and, when Y is a subset of K, $h(Y)$ denotes the set of homogeneous elements of Y. $\mathscr{L}^{\mathrm{gr}}({}_A K)$ stands for the lattice of subobjects of K in A-gr (i.e., the *graded submodules* of K). [9] is also a valid reference for graded rings.

When R is a filtered ring, the additive group $G(R) = \bigoplus_{n\in\mathbb{Z}} (F_n R/F_{n-1} R)$ has a canonical structure of ring. It is called the *associated graded ring* of R. Analogously if M is a filtered R-module, then $G(M) = \bigoplus_{n\in\mathbb{Z}} (F_n M/F_{n-1} M)$ has an obvious structure of graded $G(R)$-module. These constructions give rise to a functor $G(-)$: R-filt $\to G(R)$-gr, whose main properties may be found, e.g., in [9, Chap. D]. On the other hand, the additive subgroup $\tilde{R} = \bigoplus_{n\in\mathbb{Z}} F_n R \cdot X^n$ of $R[X, X^{-1}]$ has a canonical structure of graded ring and, for each $M \in R$-filt, $\tilde{M} = \bigoplus_{n\in\mathbb{Z}} F_n M \cdot X^n$ is a graded $\tilde{R}$-module in a natural way. If $I = \tilde{R}X$ is the (graded) ideal of $\tilde{R}$ generated by the centralizing regular homogeneous element X, there is a unique torsion theory in $\tilde{R}$-Mod (see [15, Chap. VI] for the definition) whose torsion-free objects are those $V \in \tilde{R}$-Mod such that $Xv \neq 0$, for any nonzero element $v \in V$. Those torsion-free $\tilde{R}$-modules are referred to as *X-torsion-free $\tilde{R}$-modules* in the sequel. The graded $\tilde{R}$-modules which are X-torsion-free give a full subcategory $\mathfrak{F}_X$ of $\tilde{R}$-gr. The assignment $M \to \tilde{M}$ yields a functor from R-filt to $\tilde{R}$-gr that identifies R-filt with $\mathfrak{F}_X$. We refer the reader to [1, 12] for a more detailed study of the properties and relations between R, $G(R)$, and $\tilde{R}$.

The *principal symbol map* $\sigma\colon R \to h(G(R))$ is defined as follows: (i) if $r = 0$, then $\sigma(r) = 0$; (ii) if $r \neq 0$, then there is a unique $n \in \mathbb{Z}$ such that $r \in F_n R$ and $r \notin F_{n-1} R$ and $\sigma(r)$ is the class of r in $F_n R / F_{n-1} R = G(R)_n$. Analogously, a map $\sim\colon R \to h(\tilde{R})$ is given by writing $\tilde{0} = 0X^0$ and $\tilde{r} = rX^n$, when $r \in (F_n R \setminus F_{n-1} R)$. If $S \subset R$ is a multiplicatively closed subset, then $\sigma(S)$ and $\tilde{S}$ are multiplicatively closed subsets of $G(R)$ and $\tilde{R}$, respectively, consisting of homogeneous elements. In fact, when $G(R)$ is identified with $\tilde{R}/I$ (see [1, Lemma 2.1]), $\sigma(S)$ is identified with $\tilde{S}_1 = \{\tilde{s} + I / \tilde{s} \in \tilde{S}\}$. For each natural number n, we denote by $\tilde{S}_n$ the multiplicatively closed subset of $\tilde{R}/I^n$ given by $\{\tilde{s} + I^n / \tilde{s} \in \tilde{S}\}$. If $\sigma(S)$ is a left Ore set of $G(R)$, then $\tilde{S}_n$ is a left Ore set of $\tilde{R}/I^n$, for each $n \in \mathbb{N}$. If $M \in R$-filt and we write $Q_{\tilde{S}_n}(\tilde{M}/I^n\tilde{M})$ for the module of quotients of the $\tilde{R}/I^n$-module $\tilde{M}/I^n\tilde{M}$ with respect to $\tilde{S}_n$, then we get an inverse system in $\tilde{R}$-gr, when we take the canonical homomorphisms $Q_{\tilde{S}_{n+1}}(\tilde{M}/I^{n+1}\tilde{M}) \to Q_{\tilde{S}_n}(\tilde{M}/I^n\tilde{M})$. Its inverse limit in $\tilde{R}$-gr is denoted by $Q^{\mu}_{\tilde{S}}(\tilde{M})$ and turns out to be an X-torsion-free graded $\tilde{R}$-module. By the equivalence of categories R-filt $\cong \mathfrak{F}_X$, we get a unique (complete) filtered R-module $Q^{\mu}_S(M)$ such that $\widetilde{Q^{\mu}_S(M)} = Q^{\mu}_{\tilde{S}}(\tilde{M})$, which is called the *microlocalization of M with respect to S*. The (graded) $\tilde{R}$-homomorphisms $\tilde{M} \to \tilde{M}/I^n\tilde{M} \to Q_{\tilde{S}_n}(\tilde{M}/I^n\tilde{M})$ are compatible with the inverse system given above, thus yielding a unique morphism in $\tilde{R}$-gr, $\tilde{\varphi}_M\colon \tilde{M} \to Q^{\mu}_{\tilde{S}}(\tilde{M})$, which comes again from a unique morphism in R-filt, $\varphi_M\colon M \to Q^{\mu}_S(M)$. When $M = {}_R R$, $Q^{\mu}_S(R)$ is a complete filtered ring that, together with the (ring) homomorphism $\varphi = \varphi_R\colon R \to Q^{\mu}_S(R)$, solves the universal problem mentioned at the beginning. Moreover, $Q^{\mu}_S(M)$ is a filtered $Q^{\mu}_S(R)$-module, for every $M \in R$-filt. The assignment $M \to Q^{\mu}_S(M)$ gives rise to a functor from R-filt to $Q^{\mu}_S(R)$-filt. The reader wishing to have a more detailed knowledge of the microlocalization process and, in particular, of what is written in this paragraph can look up [1].

One of the most suitable situations to study microlocalizations appears when $\tilde{R}$ is left Noetherian and, in particular, when $F_{-1}R$ is included in the

Jacobson radical $J(F_0R)$ of F_0R. The filtered rings satisfying these two conditions are called *left Zariski rings* in [4]. Many characterizations of these rings have been given in the recent literature (see, e.g., [4, Theorem 3.3; 8, Theorem 3.12]. We add here another one. Recall that the filtration FR on R is said to be *faithful* in the case where if M is any filt-finitely generated R-module for which $G(M)=0$, then $M=0$. Two filtrations FM and $F'M$ on the same R-module M are said to be *algebraically equivalent* if there exists a natural number w such that $F_{n-w}M \subseteq F'_nM \subseteq F_{n+w}M$, for every $n \in \mathbb{Z}$.

THEOREM 0.1. *The following assertions are equivalent for the filtered ring R:*

(1) *R is a left Zariski ring.*

(2) *$\tilde{R}$ is left Noetherian and, for every left ideal $\mathfrak{a}$ of R, we have $\mathfrak{a}=\bigcap_{m\in\mathbb{Z}}(\mathfrak{a}+F_mR)$.*

Proof. The condition

$$\mathfrak{a}=\bigcap_{m\in\mathbb{Z}}(\mathfrak{a}+F_mR), \tag{\&}$$

for every left ideal $\mathfrak{a}$ of R, is equivalent to saying that any quotient filtration of FR is separated.

(1)$\Rightarrow$(2) Clear since any quotient filtration of FR is good and good filtrations are separated (see [4, Theorem 3.3]).

(2)$\Rightarrow$(1) We prove condition 5 of Theorem 3.3 in [4] (i.e., $G(R)$ is left Noetherian, FR is faithful, and, for every left ideal $\mathfrak{a}$ of R with good filtration $F\mathfrak{a}$, $F_n\mathfrak{a}=\bigcap_{m\in\mathbb{Z}}(F_n\mathfrak{a}+F_mR)$ for all $n\in\mathbb{Z}$). By the isomorphism $\tilde{R}/I\cong G(R)$, the latter ring is left Noetherian. We prove now that FR is faithful. Let M be a filt-finitely generated R-module such that $G(M)=0$ and let $x_1, ..., x_k\in M$ and $n_1, ..., n_k\in\mathbb{Z}$ be elements so that

$$F_nM=\sum_{1\leqslant i\leqslant k}F_{n-n_i}R\cdot x_i, \tag{\&\&}$$

for all integers $n\in\mathbb{Z}$ (this is the same as giving the strict epimorphism $\bigoplus_{1\leqslant i\leqslant k}R(-n_i)\to M$, which maps the ith vector of the canonical basis onto x_i, for every $i=1, ..., k$!). We proceed by induction on k. The case $k=1$ reduces to the case $M=R/\mathfrak{a}$ with the quotient filtration from $R(p)$, for some integer p. It can be trivially checked, by using condition (&). If FM is as (&&), then we get a strict exact sequence in R-filt: $0\to Rx_1\to M\to M'=M/Rx_1\to 0$, when the induced and the quotient filtrations are taken, respectively, in Rx_1 and M'. By [9, Theorem D.III.3], the corresponding sequence $0\to G(Rx_1)\to G(M)\to G(M')\to 0$ is exact in $G(R)$-gr.

Since $G(M)=0$, it follows that $G(Rx_1)=0=G(M')$. The induction hypothesis can be applied to the filtration on M' so that we get $M'=0$. Hence the problem is reduced to the case when FM is a good filtration in a cyclic module M. But the result is true in this situation due to the fact that any two good filtrations on M are algebraically equivalent and the result holds for one of them if, and only if, it also holds for the other one (a cyclic R-module can be always given a good "cyclic" filtration!). Let now $\mathfrak{a}$ be a left ideal of R and consider the induced filtration on it. Since $\tilde{R}$ is left Noetherian, $\tilde{\mathfrak{a}}$ is finitely generated and, by [1, Lemma 2.1], that filtration is good. Since any good filtration on $\mathfrak{a}$ is equivalent to the latter one, it is very easy to see that proving the last condition reduces to the case of the induced filtration. But, in this case, $F_n\mathfrak{a}+F_mR=(\mathfrak{a}\cap F_nR)+F_mR$. When $m\leqslant n$, this equals $(\mathfrak{a}+F_mR)\cap F_nR$. For an arbitrary $n\in\mathbb{Z}$, $\bigcap_{m\in\mathbb{Z}}(F_n\mathfrak{a}+F_mR)=\bigcap_{m\leqslant n}[(\mathfrak{a}+F_mR)\cap F_nR]$. From the hypothesis we deduce that the latter member of the equality is just $\mathfrak{a}\cap F_nR=F_n\mathfrak{a}$ and we are done.

A particular case of left Zariski ring is provided by a complete filtered ring R whose associated graded ring $G(R)$ is left Noetherian (see [4, Theorem 3.3; 16, Corollary 6.7, Definition 6.10]).

1. Some Properties of Microlocalizations

In the sequel, a multiplicatively closed subset of the filtered ring R will be referred as one *in the standard situation* where $\sigma(S)$ is a left Ore set of $G(R)$. The following lemma was announced without proof in [4, p. 6]. Here we include a proof to be used subsequently.

LEMMA 1.1. *Let* $S\subset R$ *be a subset in the standard situation. If* $T=\sigma^{-1}(\sigma(S))=\{t\in R/\sigma(t)\in\sigma(S)\}$, *then* $Q_S^{\mu}(M)=Q_T^{\mu}(M)$ *for every* $M\in R$-filt.

Proof. We prove that $\tilde{S}_n$ and $\tilde{T}_n$ define the same 1-topology (see [15, VI.6.1]) in $\tilde{R}/I^n$, for each $n\geqslant 1$. This is the same as proving that, for any $\tilde{t}\in\tilde{T}$, there exist $\tilde{r}_n\in h(\tilde{R})$ and $\tilde{s}_n\in\tilde{S}$ such that $\tilde{r}_n\tilde{t}-\tilde{s}_n$ is an homogeneous element of $\tilde{R}X^n=I^n$. We proceed by induction on n. The result is clear for $n=1$, because $\tilde{S}_1=\tilde{T}_1$ is identified with $\sigma(S)$ by the isomorphism $\tilde{R}/I\cong G(R)$. If the result is true for $k\leqslant n-1$, then we can choose $\tilde{r}_{n-1}\in h(\tilde{R})$ and $\tilde{s}_{n-1}\in\tilde{S}$ such that

$$\tilde{r}_{n-1}\tilde{t}-\tilde{s}_{n-1}=\tilde{a}_{n-1}X^{n-1}, \tag{\&}$$

for certain $\tilde{a}_{n-1}\in h(\tilde{R})$. By applying the second left Ore condition in the graded sense (see [9, Lemma I.6.1]) to the subset $\tilde{S}_1 = \tilde{T}_1$ of $\tilde{R}/I$, we get a pair $(\tilde{s}'_{n-1}, \tilde{a}'_{n-1}) \in \tilde{S} \times h(\tilde{R})$ such that $\tilde{s}'_{n-1}\tilde{a}_{n-1} - \tilde{a}'_{n-1}\tilde{t} \in h(I)$. Then $\tilde{s}'_{n-1}\tilde{a}_{n-1} = \tilde{a}'_{n-1}\tilde{t} + \tilde{b}X$, for some $\tilde{b} \in h(\tilde{R})$. Now we can multiply on the left by $\tilde{s}'_{n-1}$ in (&) and, after using the fact that X is a centralizing element, we obtain that $(\tilde{s}'_{n-1}\tilde{r}_{n-1} - \tilde{a}'_{n-1}X^{n-1})\tilde{t} - \tilde{s}'_{n-1}\tilde{s}_{n-1} = \tilde{b}X^n$. By taking $\tilde{r}_n = \tilde{s}'_{n-1}\tilde{r}_{n-1} - \tilde{a}'_{n-1}X^{n-1}$ and $\tilde{s}_n = \tilde{s}'_{n-1}\tilde{s}_{n-1}$, the induction is finished. Since, consequently, $Q_{\tilde{S}_n}(\tilde{M}/I^n\tilde{M}) = Q_{\tilde{T}_n}(\tilde{M}/I^n\tilde{M})$ for each $n \geqslant 1$, the result follows from the algebraic construction of microlocalizations given before.

COROLLARY 1.2. *If $S, T \subset R$ are in the standard situation and $\sigma(S) = \sigma(T)$, then $Q^{\mu}_S(M) = Q^{\mu}_T(M)$ for every $M \in R$-filt.*

Note. Bearing in mind the equivalence of categories R-filt $\cong \mathfrak{F}_X$, the above lemma shows that $Q^{\mu}_S(M) = Q^{\mu}_T(M)$ is an equality in R-filt. In other words, the equality of the lemma preserves the filtrations. Unless otherwise remarked, equalities between microlocalizations of modules (resp. rings) mean equalities in this filtered sense.

The following is an interesting result overlooked in previous papers.

PROPOSITION 1.3. *Let $S \subset T \subset R$ be subsets in the standard situation such that $\sigma(S)^{-1}G(R) = \sigma(T)^{-1}G(R)$. Then $Q^{\mu}_S(M) = Q^{\mu}_T(M)$ for every $M \in R$-filt.*

Proof. As in Lemma 1.1, it is enough to prove that $\tilde{S}_n$ and $\tilde{T}_n$ define the same 1-topology in $\tilde{R}/I^n$, for each $n \geqslant 1$. We proceed by induction on n to see that, for any $\tilde{t} \in \tilde{T}$, there is a pair $(\tilde{s}_n, \tilde{r}_n) \in \tilde{S} \times h(\tilde{R})$ such that $\tilde{r}_n\tilde{t} - \tilde{s}_n \in I^n$. Due to the identifications of $\tilde{R}/I$ with $G(R)$ and $\tilde{S}_1$ (resp. $\tilde{T}_1$) with $\sigma(S)$ (resp. $\sigma(T)$), the case $n = 1$ is just the hypothesis. The truth of the assertion for $n-1$ implies the existence of $\tilde{s}_{n-1} \in \tilde{S}$ and $\tilde{r}_{n-1}$, $\tilde{a}_{n-1} \in h(\tilde{R})$ such that $\tilde{r}_{n-1}\tilde{t} = \tilde{s}_{n-1} + \tilde{a}_{n-1}X^{n-1}$. This is an element of $h(\tilde{S} + \tilde{R}X) = \sigma^{-1}(\sigma(S))$. Now the proof of Lemma 1.1 gives a pair $(\tilde{s}_n, \tilde{a}_n) \in \tilde{S} \times h(\tilde{R})$ such that $\tilde{a}_n\tilde{r}_{n-1}\tilde{t} - \tilde{s}_n \in I^n$, and we are done.

Remark 1.4. The hypothesis $S \subset T$ can be avoided in the above theorem because, in any case, we can take $\sigma(S)_{\mathrm{sat}} = \sigma(T)_{\mathrm{sat}} = \{x \in h(G(R))/G(R)x \cap \sigma(S) \neq \varnothing\}$. This is not necessarily a left Ore set of $G(R)$ but defines the same 1-topology in $G(R)$ as $\sigma(S)$ and $\sigma(T)$ and, as sketched in [1, p. 12], the microlocalization of M with respect to the linear topology in R generated by $\{Ru/\sigma(u) \in \sigma(S)_{\mathrm{sat}} = \sigma(T)_{\mathrm{sat}}\}$ exists and equals $Q^{\mu}_S(M)$ or $Q^{\mu}_T(M)$.

The separability of the filtration in R implies that the canonical ring homomorphism $R \to \hat{R}$ is injective. Thus we can identify R with its image and, from now on, assume that R is included in $\hat{R}$. If $S \subset R$ is a subset in the standard situation, then so is it as a subset of $\hat{R}$, because $G(R) = G(\hat{R})$

(see [9, Proposition D.III.1]). Just from the universal properties satisfied by microlocalizations (see [16, Theorems I and II]), one easily deduces that $Q_S^{\mu}(M)=Q_S^{\mu}(\hat{M})$, for any $M\in R$-filt, where the last filtered module can be indifferently viewed as an object of R-filt, $\hat{R}$-filt, or $Q_S^{\mu}(R)$-filt. This fact will be profusely used in the sequel. The following result relates the lattices of submodules of ${}_{\hat{R}}M$ and ${}_{Q_S^{\mu}(R)}Q_S^{\mu}(M)$ in a particular situation.

THEOREM 1.5. *Let R be a filtered ring such that $G(R)$ is left Noetherain, $S\subset R$ a subset in the standard situation, and $Q=Q_S^{\mu}(R)$. If M is a complete filt-finitely generated left $\hat{R}$-module, then the map $\Phi\colon \mathfrak{L}({}_{\hat{R}}M)\to \mathfrak{L}({}_{Q}Q_S^{\mu}(M))$, given by $L\to Q\varphi_M(L)$, is a surjective order-preserving map.*

Proof. By the above considerations, it is not restrictive to assume that $R=\hat{R}$ so that it is a complete left Zariski ring. Let U be a Q-submodule of $Q_S^{\mu}(M)$ and consider the induced filtration in it. Since $G(Q)=\sigma(S)^{-1}(G(R))$ is left Noetherian and Q is a complete ring, Q is a Zariski ring and we can assert that $Q_S^{\mu}(M)$ and U are filt-finitely generated Q-modules. Let now $u_{n,n-1}\colon Q_{\tilde{S}_n}(\tilde{M}/I^n\tilde{M})\to Q_{\tilde{S}_{n-1}}(\tilde{M}/I^{n-1}\tilde{M})$ be the canonical map and $p_n\colon Q_S^{\mu}(\tilde{M})\to Q_{\tilde{S}_n}(\tilde{M}/I^n\tilde{M})$ the map from the inverse limit onto the nth component of the inverse system. Then $p_n(\tilde{U})$ is a graded $Q_{\tilde{S}_n}(\tilde{R}/I^n)$-submodule of $Q_{\tilde{S}_n}(\tilde{M}/I^n\tilde{M})$, for each $n\geqslant 1$. By well-known properties of the modules of quotients, there exists a unique graded $\tilde{R}/I^n$-submodule $\tilde{U}_n/I^n\tilde{M}$ of $\tilde{M}/I^n\tilde{M}$ satisfying that $p_n(\tilde{U})=Q_{\tilde{S}_n}(\tilde{U}_n/I^n\tilde{M})$ and $(\tilde{M}/I^n\tilde{M})/(\tilde{U}_n/I^n\tilde{M})\cong \tilde{M}/\tilde{U}_n$ is $\tilde{S}_n$-torsion-free (i.e., no nonzero element of it is annihilated by an element of $\tilde{S}_n$). This last property and the identity $u_{n,n-1}\circ p_n=p_{n-1}$ entail that

$$\tilde{U}_n+I^{n-1}\tilde{M}=\tilde{U}_{n-1}, \tag{\&}$$

for each $n\geqslant 2$. In such way, we get a descending chain $\{\tilde{U}_n\}_{n\geqslant 1}$ of graded $\tilde{R}$-submodules of $\tilde{M}$. Take $\tilde{N}=\bigcap_{n\geqslant 1}\tilde{U}_n$. We prove that $\tilde{N}+I^n\tilde{M}=\tilde{U}_n$, for each $n\geqslant 1$. We only have to check the inclusion $\supseteq$. Let us fix $n\geqslant 1$ and take a homogeneous element $\tilde{u}=\tilde{u}_0$ of $\tilde{U}_n$. By repeated use of the formula (&), we can obtain a sequence $\{\tilde{u}_p\}_{p\geqslant 0}$ of homogeneous elements of $\tilde{M}$ with the same degree as $\tilde{u}$ and with the property that $\tilde{u}_p\in\tilde{U}_{n+p}$ and $\tilde{u}_p-\tilde{u}_{p+1}\in I^{n+p}\tilde{M}$, for each $p\geqslant 0$. It can be seen in a straightforward way that both $\{\tilde{u}_p\}$ and $\{\tilde{u}-\tilde{u}_p\}$ are Cauchy sequences with respect to the I-adic topology in $\tilde{M}$. By [1, Proposition 2.5] and the completeness of M, they have limits which are denoted by $\tilde{w}$ and $\tilde{v}$, respectively. Bearing in mind that $\tilde{u}-\tilde{u}_p=\sum_{0\leqslant k\leqslant p}(\tilde{u}_k-\tilde{u}_{k+1})\in I^n\tilde{M}$, it is clear that its limit $\tilde{v}$ belongs to $I^n\tilde{M}$. On the other hand, the definition of limit says that, for an arbitrary $p\geqslant 1$, $\tilde{w}-\tilde{u}_k\in I^p\tilde{M}$ for k large enough. Hence $\tilde{w}\in\tilde{U}_{n+k}+I^p\tilde{M}$. When $k\geqslant p$ the latter $\tilde{R}$-module is included in $\tilde{U}_{p+1}+I^p\tilde{M}=\tilde{U}_p$ due to the repeated use of (&). The arbitrariness of p implies that $\tilde{w} \in \bigcap_{p \geq 1} \tilde{U}_p = \tilde{N}$ and thereby $\tilde{u} = \tilde{w} + \tilde{v}$ belongs to $\tilde{N} + I^n\tilde{M}$, as desired.

We claim that $\tilde{M}/\tilde{N}$ is an X-torsion-free graded $\tilde{R}$-module. Let $\tilde{m} + \tilde{N} \in \tilde{M}/\tilde{N}$ be an element annihilated by X (i.e., $X\tilde{m} \in \tilde{N}$) and $\tilde{\varphi}_M : \tilde{M} \to Q^{\mu}_{\tilde{S}}(\tilde{M})$ the canonical $\tilde{R}$-homomorphism. Since $p_n \circ \tilde{\varphi}_M$ is the composition $\tilde{M} \to \tilde{M}/I^n\tilde{M} \to Q_{\tilde{S}_n}(\tilde{M}/I^n\tilde{M})$, $p_n(X\tilde{\varphi}_M(\tilde{m})) = (p_n \circ \tilde{\varphi}_M)(X\tilde{m})$ is an element of $Q_{\tilde{S}_n}((\tilde{N} + I^n\tilde{M})/I^n\tilde{M})$, for each $n \geq 1$. The last module is just $Q_{\tilde{S}_n}(\tilde{U}_n/I^n\tilde{M}) = p_n(\tilde{U})$, by the above paragraph. Then $X\tilde{\varphi}_M(\tilde{m}) \in p_n^{-1}(p_n(\tilde{U})) = \tilde{U} + I^n Q^{\mu}_{\tilde{S}}(\tilde{M})$, for each $n \geq 1$, and thus $X\tilde{\varphi}_M(\tilde{m})$ belongs to $\bigcap_{n \geq 1} (\tilde{U} + I^n Q^{\mu}_{\tilde{S}}(\tilde{M}))$. But the Zariskian condition of Q and the fact that $Q^{\mu}_S(M)$ is a filt-finitely generated $Q^{\mu}_S(R)$-module tell us that the quotient filtration on $Q^{\mu}_S(M)/U$ is separated, which is equivalent (see [1, Lemma 2.1]) to saying that $Q^{\mu}_{\tilde{S}}(\tilde{M})/\tilde{U}$ is I-adically separated. So $\tilde{U} = \bigcap_{n \geq 1} (\tilde{U} + I^n Q^{\mu}_{\tilde{S}}(\tilde{M}))$ and, consequently, $X\tilde{\varphi}_M(\tilde{m}) \in \tilde{U}$. But $Q^{\mu}_{\tilde{S}}(\tilde{M})/\tilde{U}$ is X-torsion-free (see, e.g., [12, Theorem 1.2]) so that $\tilde{\varphi}_M(\tilde{m}) \in \tilde{U}$, whence $(p_n \circ \tilde{\varphi}_M)(\tilde{m}) = \tilde{m} + I^n\tilde{M}$ is an element of $p_n(\tilde{U}) = Q_{\tilde{S}_n}(\tilde{U}_n/I^n\tilde{M})$. The $\tilde{S}_n$-torsion freedom of $\tilde{M}/\tilde{U}_n$ entails that $\tilde{m} \in \tilde{U}_n$. Since this has been done for an arbitrary $n \geq 1$, it is valid for all of them and thereby $\tilde{m} \in \bigcap_{n \geq 1} \tilde{U}_n = \tilde{N}$, showing the claim. Now $0 \to \tilde{N} \to \tilde{M} \to \tilde{M}/\tilde{N} \to 0$ is an exact sequence of X-torsion-free graded $\tilde{R}$-modules. This means that $\tilde{N}$ is the X-torsion-free graded $\tilde{R}$-module assigned to a submodule N of M, considered with its induced filtration, by means of the equivalence of categories $R\text{-filt} \simeq \mathfrak{F}_X$ (see, e.g., [12, Theorem 1.2]).

Our goal is to prove that $Q\varphi_M(N) = U$, which ends the proof. What we have done above with $X\tilde{\varphi}_M(\tilde{m})$ may be done with any element of $\tilde{\varphi}_M(\tilde{N})$, showing that $\tilde{\varphi}_M(\tilde{N}) \subseteq \tilde{U}$ and hence $\varphi_M(N) \subseteq U$. Since U is a Q-submodule of $Q^{\mu}_S(M)$, the inclusion $Q\varphi_M(N) \subseteq U$ is clear. On the other hand, the canonical projection $p_n : \tilde{U} \to p_n(\tilde{U}) = Q_{\tilde{S}_n}(\tilde{N} + I^n\tilde{M}/I^n\tilde{M})$ yields a homomorphism $\check{p}_n : \tilde{U} \to Q_{\tilde{S}_n}(\tilde{N}/I^n\tilde{N})$, taking into account that $(\tilde{N} + I^n\tilde{M})/I^n\tilde{M} = \tilde{N}/(\tilde{N} \cap I^n\tilde{M}) = \tilde{N}/I^n\tilde{N})$ due to the X-torsion freedom of $\tilde{M}/\tilde{N}$. It is very easy to see that $\{\check{p}_n/n \geq 1\}$ is a family of homomorphisms compatible with the inverse system $Q_{\tilde{S}_n}(\tilde{N}/I^n\tilde{N})$, which gives rise to $Q^{\mu}_{\tilde{S}}(\tilde{N})$. Then we get a homomorphism $\tilde{\gamma}: \tilde{U} \to Q^{\mu}_{\tilde{S}}(\tilde{N})$ in the category $\tilde{R}$-gr. It is left as an exercise to prove that $\tilde{\gamma}$ is injective. When $Q^{\mu}_{\tilde{S}}(\tilde{N})$ is viewed as a graded submodule of $Q^{\mu}_{\tilde{S}}(\tilde{M})$, $\tilde{\gamma}$ may be seen as an inclusion and hence, by the equivalence of categories $R\text{-filt} \simeq \mathfrak{F}_X$, $U \subseteq Q^{\mu}_S(N)$. Thus

$$Q\varphi_M(N) \subseteq U \subseteq Q^{\mu}_S(N). \qquad (\&\&)$$

From the Zariskian condition of R we deduce that N is a filt-finitely generated R-module and, by [16, Proposition 6.22], the canonical map $Q \otimes_R N \to Q^{\mu}_S(N)$ bijective. Its image is just $Q\varphi_M(N)$, thus showing that we have equality in (&&).

COROLLARY 1.6. *Let R be a filtered ring such that $G(R)$ is left Noetherian and S a subset in the standard situation. For any of the dimensions Krull, Gabriel, global, and weak global, the following inequality holds:* l-dim $Q_S^\mu(R) \leqslant$ l-dim $\hat{R}$, *where the prefix* l *indicates that they are taken on the left.*

Proof. Since $\hat{R}$ and $Q_S^\mu(R)$ are left Noetherian (see [9, Corollary D.IV.4]), the equalities G-dim $\hat{R} =$ K-dim $\hat{R}+1$, gl-dim $\hat{R} =$ w-gl-dim $\hat{R}$ and the corresponding ones for $Q = Q_S^\mu(R)$ hold (here, and in the rest of the paper, G-dim, K-dim, gl-dim, and w-gl-dim denote, respectively, the Gabriel, Krull, global, and weak global dimensions on the left). Consequently, we only have to prove the result for, say, the Krull and the global dimensions. The Krull case is clear since the Krull dimension of Q is that of the lattice $\mathfrak{L}({}_Q Q)$ (see [10, Chaps. 3 and 4]) and this is an epimorphic image of $\mathfrak{L}({}_{\hat{R}}\hat{R})$ by the previous theorem. For the global dimension case, recall that gl-dim $Q = \operatorname{Sup}\{p\text{-dim}(Q/J)/J$ is a left ideal of $Q\}$, where p-dim denotes the projective dimension. By the foregoing theorem, any left ideal of Q is of the type $Q_S^\mu(\mathfrak{a}) = Q\varphi(\mathfrak{a}) \cong Q \otimes_{\hat{R}} \mathfrak{a}$, where $\mathfrak{a}$ is a left ideal of $\hat{R}$ considered, if necessary, with its induced filtration. Then $Q/J \cong Q \otimes_{\hat{R}} (\hat{R}/\mathfrak{a})$. Inasmuch as Q is a flat right $\hat{R}$-module (see, e.g., [16, Corollary 6.26]), $p\text{-dim}_Q(Q/J) \leqslant p\text{-dim}_{\hat{R}}(\hat{R}/\mathfrak{a})$ and, thereby, gl-dim $Q \leqslant$ gl-dim $\hat{R}$.

2. AUSLANDER REGULARITY AND DIMENSIONS

If R is a left and right Noetherian ring having finite global dimension μ, it is well known that there exists, for any finitely generated R-module M, a smallest natural number $j = j(M)$ such that $\operatorname{Ext}_R^j(M, R) \neq 0$. This is called the *grade of M* and the ring is said to be (*Auslander*) *regular* in case the following so-called *Auslander condition* holds:

(A) If M is a finitely generated R-module then, for any $0 \leqslant k \leqslant \mu$ and any nonzero submodule N of $\operatorname{Ext}_R^k(M, R)$, $j(N) \geqslant k$.

Our aim is to show that being regular is a microlocal property for a left and right Zariski ring whose associated graded ring is commutative. We start with a lemma in which $1_{R\text{-filt}}$ denotes the identity functor in R-filt.

LEMMA 2.1. *Let R be a filtered ring, $S \subset R$ a subset in the standard situation, $Q = Q_S^\mu(R)$, and U: Q-filt $\to R$-filt the forgetful functor. The canonical morphisms $\varphi_M: M \to Q_S^\mu(M)$, for $M \in R$-filt, define a natural transformation between $1_{R\text{-filt}}$ and $U \circ Q_S^\mu(-)$.*

Proof. Take a morphism in R-filt, $f: M \to N$. Just by definition of

$Q_S^\mu(f)$ (see [16, p. 993]), we have that $U(Q_S^\mu(f)) \circ \varphi_M = \varphi_N \circ f$. This proves the assertion.

In the proposition below the ring of integers $\mathbb{Z}$ is considered to be filtered by means of its trivial filtration (i.e., $F_n\mathbb{Z} = 0$, for $n < 0$, and $F_n\mathbb{Z} = \mathbb{Z}$, for $n \geqslant 0$).

PROPOSITION 2.2. *Let R, S, and Q be as in the above lemma and $N \in R$-filt. The two contravariant functors $\mathrm{HOM}_R(-, Q_S^\mu(N))$ and $\mathrm{HOM}_Q(Q_S^\mu(-), Q_S^\mu(N))$, from R-filt to $\mathbb{Z}$-filt, are naturally isomorphic via the assignment $f \to Q_S^\mu(f)$.*

Proof. Let $M \in R$-filt and $f \in \mathrm{Hom}_R(M, Q_S^\mu(N))$. Then $Q_S^\mu(f) \in \mathrm{Hom}_Q(Q_S^\mu(M), Q_S^\mu(Q_S^\mu(N)))$. Since $Q_S^\mu(Q_S^\mu(N)) = Q_S^\mu(N)$, we see that $Q_S^\mu(f)$ is actually an element of $\mathrm{Hom}_Q(Q_S^\mu(M), Q_S^\mu(N))$. We claim that the assignment $f \to Q_S^\mu(f)$ is a morphism in Z-filt. The sum is clearly preserved so that we only have to check that if $f \in F_p \mathrm{Hom}_R(M, Q_S^\mu(N))$ then $Q_S^\mu(f) \in F_p \mathrm{HOM}_Q(Q_S^\mu(M), Q_S^\mu(N))$. But, from the universal property of the microlocalization functor (see [16, Theorem II]), one gets that $Q_S^\mu(M(-p)) = Q_S^\mu(M)(-p)$ and, by the properties of the shifted filtered modules, there is no loss of generality in assuming that f has degree zero. Then the claim follows from the functoriality of $Q_S^\mu(-)$. Let us now define a map $\mathrm{HOM}_Q(Q_S^\mu(M), Q_S^\mu(N)) \to \mathrm{HOM}_R(M, Q_S^\mu(N))$ by $g \to U(g) \circ \varphi_M$, where U: Q-filt $\to R$-filt is the forgetful functor. It can be easily proved that it is a morphism in $\mathbb{Z}$-filt. We see that it is the inverse of the above one. Let $f \in \mathrm{HOM}_R(M, Q_S^\mu(N))$, which can be assumed to be of degree zero. By the previous lemma, $U(Q_S^\mu(f)) \circ \varphi_M = \varphi_{Q(N)} \circ f$, where $Q(N)$ denotes $Q_S^\mu(N)$. Since $\varphi_{Q(N)}$ is the identity map of $Q_S^\mu(N)$ as a consequence of the stabilization of the microlocalization functor (i.e., $Q_S^\mu(Q_S^\mu(-)) \cong Q_S^\mu(-)$), we get that $U(Q_S^\mu(f)) \circ \varphi_M = f$. Conversely, let $g \in \mathrm{HOM}_Q(Q_S^\mu(M), Q_S^\mu(N))$, which can be chosen again of degree zero. Then $Q_S^\mu(U(g) \circ \varphi_M) = Q_S^\mu(U(g)) \circ Q_S^\mu(\varphi_M)$. But $Q_S^\mu(\varphi_M)$ is the identity map of Q and $Q_S^\mu(U(g)) = g$. So $Q_S^\mu(U(g) \circ \varphi_M) = g$, as desired. The naturality is the only thing left to be proved. This is proposed as an exercise to the reader.

Remark 2.3. We say that N is a *filtered R–R-bimodule* if it is a bimodule with a filtration FN such that $F_nN \cdot F_pR \subseteq F_{n+p}N$ and $F_pR \cdot F_nN \subseteq F_{n+p}N$, for all $n, p \in \mathbb{Z}$. If, in the above proposition, N is a filtered R–R-bimodule whose left and right microlocalizations with respect to S coincide, then the two mentioned functors are actually functors from R-filt to filt-Q and the result is still valid in this situation. In particular, that is the case when $N = R$.

Let now $\underline{M}$: $\cdots \to M_{n-1} \xrightarrow{d_n} M_n \xrightarrow{d_n+1} M_{n+1} \to \cdots$ be a cochain complex in R-filt (resp. filt-R). Then the cohomology modules $H^n(\underline{M}) =$

$\operatorname{Ker} d_{n+1}/\operatorname{Im} d_n$ ($n \in \mathbb{Z}$) can be filtered in a natural manner by taking the filtration induced in $\operatorname{Ker} d_{n+1}$ by M_n and providing $H^n(\underline{M})$ with its quotient filtration. If R is Noetherian and satisfies the property that any good filtration induces a good filtration on each submodule, then, in case each M_n is filt-finitely generated, the filtration in $H^n(\underline{M})$ is good, for every $n \in \mathbb{Z}$.

EXAMPLE 2.4. Let R be a left and right Zariski ring, N a filt-finitely generated R–R-bimodule, and M a filt-finitely generated left R-module. If $\underline{L}$: $\cdots \to L_n \to L_{n-1} \to \cdots \to L_1 \to L_0 \to 0 \to \cdots$ is a *strict finitely generated filt-free resolution* of M (i.e., it is exact in R-Mod, all the L_n are finitely generated filt-free and all the homomorphisms are strict filtered), then $\underline{L}^*$: $\cdots \to 0 \to \operatorname{Hom}_R(L_0, N) \to \operatorname{Hom}_R(L_1, N) \to \cdots \to \operatorname{Hom}_R(L_n, N) \to \cdots$ is a cochain complex in filt-R (recall that $\operatorname{Hom}_R(L_n, N) = \operatorname{HOM}_R(L_n, M)$, for each $n \in \mathbb{Z}$, in the present situation!). Its cohomology modules are $H^n(\underline{L}^*) = \operatorname{Ext}^n_R(M, N)$, for each $n \geqslant 0$. It is an easy exercise to see that the filtration in $\operatorname{Hom}_R(L_n, N) = \operatorname{HOM}_R(L_n, N)$ is good so that $\operatorname{Ext}^n_R(M, N)$ is provided with a good filtration, for every $n \in \mathbb{Z}$.

PROPOSITION 2.5. *Let R be a left and right Zariski ring, $S \subset R$ a subset in the standard situation, and N a filt-finitely generated R–R-bimodule whose left and right microlocalizations with respect to S coincide. For any finitely generated filt-free left R-module L, the map Φ_L: $\operatorname{Hom}_R(L, N) \otimes_R Q \to \operatorname{Hom}_R(L, Q^\mu_S(N))$, defined by $\Phi_L(f \otimes q)(x) = \varphi_N(f(x))\, q$, is an isomorphism in* filt-Q, *which is natural in the finitely generated filt-free component L.*

Proof. Let L be filtered by $F_n L = \bigoplus_{1 \leqslant i \leqslant k} F_{n-p_i} R \cdot e_i$, for each $n \in \mathbb{Z}$, where $\{e_i / 1 \leqslant i \leqslant k\}$ is an R-basis of L. It is not hard to prove that, when we identify $\operatorname{Hom}_R(L, N)$ (resp. $\operatorname{Hom}_R(L, Q^\mu_S(N))$) with N^k (resp. $Q^\mu_S(N)^k$), the filtration of the first module is identified with the filtration $F_p N^k = \bigoplus_{1 \leqslant i \leqslant k} F_{p+p_i} N$ (resp. $F_p Q^\mu_S(N)^k = \bigoplus_{1 \leqslant i \leqslant k} F_{p+p_i} Q^\mu_S(N)$), for each $p \in \mathbb{Z}$. Then $\operatorname{Hom}_R(L, N)$ (resp. $\operatorname{Hom}_R(L, Q^\mu_S(N))$) is identified with the filtered right R-module $\bigoplus_{1 \leqslant i \leqslant k} N(p_i)$ (resp. $\bigoplus_{1 \leqslant i \leqslant k} Q^\mu_S(N)(p_i)$). Thus we only have to prove that $N(p) \otimes_R Q \cong Q^\mu_S(N)(p)$ in filt-Q, for any $p \in \mathbb{Z}$. The first member of the expression is $(N \otimes_R Q)(p)$ (see [9, Lemma D.VIII.1]) so that the mentioned isomorphism follows in a straightforward way from the isomorphism $N \otimes_R Q \cong Q^\mu_S(N)$ (see, e.g., [1, Corollary 3.20]), which is an isomorphism in filt-Q. We leave as an exercise to check that the isomorphism $\operatorname{Hom}_R(L, N) \otimes_R Q \cong \operatorname{Hom}_R(L, Q^\mu_S(N))$ provided by the above identifications is the desired one.

As a consequence of all the above lemmas and propositions, we get the key result of this part.

THEOREM 2.6. *Let R be a left and right Zariski ring, $S \subset R$ a subset in the standard situation, and N a filt-finitely generated R–R-bimodule whose left and right microlocalizations with respect to S coincide. Then, for each filt-finitely generated left R-module M and each $n \geqslant 0$, $Q_S^\mu(\mathrm{Ext}_R^n(M, N)) \cong \mathrm{Ext}_Q^n(Q_S^\mu(M), Q_S^\mu(N))$ in a natural way.*

Proof. Let us note that the statement makes sense due to the fact that the microlocalization of $\mathrm{Ext}_R^n(M, N)$ does not depend on the good filtration taken in it. Let $\underline{L}$: $\cdots \to L_n \xrightarrow{d_n} L_{n-1} \to \cdots \xrightarrow{d_2} L_1 \xrightarrow{d_1} L_0 \to 0 \to \cdots$ be a strict finitely generated filt-free resolution of M (i.e., $M = \mathrm{Coker}\, d_1$ with its quotient filtration). We have now the following cochain complexes in filt-Q: $\mathrm{Hom}_Q(Q_S^\mu(\underline{L}), Q_S^\mu(N))$, $\mathrm{Hom}_R(\underline{L}, Q_S^\mu(N))$, $\mathrm{Hom}_R(\underline{L}, N) \otimes_R Q$, and $Q_S^\mu(\mathrm{Hom}_R(\underline{L}, N))$. By the right Zariskian condition of R and the fact that $\mathrm{Hom}_R(\underline{L}, N)$ is a complex in filt-R, $\mathrm{Hom}_R(\underline{L}, N) \otimes_R Q$ and $Q_S^\mu(\mathrm{Hom}_R(\underline{L}, N))$ are naturally isomorphic complexes of filt-Q (see [1, Corollary 3.20]). By using now Proposition 2.2 and Proposition 2.5, we get that the above four complexes of filt-Q are isomorphic. If we take their cohomology modules with filtrations defined as in Example 2.4, we get the desired result.

Remark 2.7. In the above proof, we have actually shown that, when $\underline{L}$ is a strict finitely generated filt-free resolution of $M \in R$-filt and the right R-module $\mathrm{Ext}_R^n(M, N)$ is given the good filtration coming from it, then $Q_S^\mu(\mathrm{Ext}_R^n(M, N))$ is isomorphic, as filtered right Q-module, to the right Q-module $\mathrm{Ext}_Q^n(Q_S^\mu(M), Q_S^\mu(N))$, which is considered with the filtration deduced from the strict finitely generated filt-free resolution $Q_S^\mu(\underline{L})$ of $Q_S^\mu(M) \in Q$-filt.

The foregoing theorem proves very useful in the study of the behaviour of the global dimension of $\hat{R}$ with respect to microlocalizations. Before tackling that problem, we give the following lemma, which can be easily deduced from [11, Corollary 3.34].

LEMMA 2.8. *Let A be a left and right Noetherian ring. If A has finite global dimension, then* $\text{inj-dim}_A A = \text{gl-dim}\, A$ (*resp.* $\text{inj-dim}\, A_A = \text{gl-dim}\, A$), *where* inj-dim *denotes the injective dimension.*

If R is a filtered ring whose associated graded ring $G(R)$ is commutative, then we can take $\mathrm{Spec}^{\mathrm{gr}}(G(R))$ (resp. $\mathrm{Max}^{\mathrm{gr}}(G(R))$) to be the set of prime graded (resp. maximal graded) ideals of $G(R)$. For each $\mathfrak{p} \in \mathrm{Spec}^{\mathrm{gr}}(G(R))$ (and hence for every $\mathfrak{p} \in \mathrm{Max}^{\mathrm{gr}}(G(R))$), $h(G(R)) \setminus \mathfrak{p} = \{u \in h(G(R)) / u \notin \mathfrak{p}\}$ is an Ore set of $G(R)$. We write $Q_{\mathfrak{p}}^\mu(R)$ for the microlocalization of R with respect to any multiplicative subset S of R such that $\sigma(S) = h(G(R)) \setminus \mathfrak{p}$ (see Corollary 1.2).

THEOREM 2.9. *Let R be a filtered ring such that $G(R)$ is Noetherian commutative. The following statements hold:*

(a) *If $\hat{R}$ has finite global dimension, then* gl-dim $\hat{R} = \mathrm{Sup}\{$gl-dim $Q^{\mu}_{\mathfrak{p}}(R)/\mathfrak{p} \in \mathrm{Spec}^{gr}(G(R))\} = \mathrm{Sup}\{$gl-dim $Q^{\mu}_{\mathfrak{m}}(R)/\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))\}$.

(b) *If $G(R)$ is* gr-*semilocal, i.e., it has only a finite number of maximal graded ideals, then* K-dim $\hat{R} = \mathrm{Sup}\{K$-dim $Q^{\mu}_{\mathfrak{p}}(R)/\mathfrak{p} \in \mathrm{Spec}^{gr}(G(R))\} = \mathrm{Sup}\{K$-dim $Q^{\mu}_{\mathfrak{m}}(R)/\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))\}$.

Proof. Without loss of generality, we can assume that R is complete so that $R = \hat{R}$ and it is a left and right Zariski ring. Let $\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))$ and $\mathfrak{p} \in \mathrm{Spec}^{gr}(G(R))$ such that $\mathfrak{p} \subseteq m$. Then $Q^{\mu}_{\mathfrak{p}}(Q^{\mu}_{\mathfrak{m}}(R)) = Q^{\mu}_{\mathfrak{p}}(R)$ (see [1, Corollary 3.18]). By Corollary 1.6, dim $Q^{\mu}_{\mathfrak{p}}(R) \leqslant$ dim $Q^{\mu}_{\mathfrak{m}}(R)$ for any of the two dimensions considered. Hence we only need to check the equalities concerning $\mathrm{Max}^{gr}(G(R))$. On the other hand, the inequality dim $R \geqslant \mathrm{Sup}\{\dim Q^{\mu}_{\mathfrak{m}}(R)/\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))\}$ is also a direct consequence of Corollary 1.6. so we only have to prove the converse inequality for both dimensions.

(a) For the global dimension case, the previous lemma ensures that we can replace gl-dim R by inj-dim ${}_RR$. Let us put $n =$ inj-dim ${}_RR$. Then $\mathrm{Ext}^n_R(M, R) \neq 0$ for some finitely generated left R-module M. We give M a good R-filtration so that $\mathrm{Ext}^n_R(M, R)$ gets a good filtration in the manner explained in Example 2.4. If $Q^{\mu}_{\mathfrak{m}}(\mathrm{Ext}^n_R(M, R))$ were zero for every $\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))$, then its associated graded module $G(\mathrm{Ext}^n_R(M, R))_{\mathfrak{m}}$ would be zero for every $\mathfrak{m}$. As in the ungraded case, this implies that $G(\mathrm{Ext}^n_R(M, R))$ is zero. The fact that R is Zariski would entail that $\mathrm{Ext}^n_R(M, R) = 0$, thus contradicting the hypothesis. Hence we can pick up an $\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))$ such that $Q^{\mu}_{\mathfrak{m}}(\mathrm{Ext}^n_R(M, R)) \neq 0$ and, by Theorem 2.6, $\mathrm{Ext}^n_Q(Q^{\mu}_m(M), Q^{\mu}_{\mathfrak{m}}(R)) \neq 0$, which shows that gl-dim $Q^{\mu}_m \geqslant n$.

(b) For the Krull dimension case, let us consider, for a filt-finitely generated left R-module M, the map $\mathfrak{L}({}_RM) \to \Pi\{\mathfrak{L}({}_QQ^{\mu}_{\mathfrak{m}}(M))/\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))\}$, given by $N \to (Q^{\mu}_{\mathfrak{m}}(N))_{\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))}$, where the induced filtration is considered on each N. It is an order-preserving lattice homomorphism when the componentwise order is taken in the product. Furthermore, it is injective. Then the Krull dimension of the lattice $\mathfrak{L}({}_RM)$ is smaller than that of $\Pi\{\mathfrak{L}({}_QQ^{\mu}_{\mathfrak{m}}(M)/\mathfrak{m} \in \mathrm{Max}^{gr}(G(R))\}$. If $\mathrm{Max}^{gr}(G(R))$ is finite, the Krull dimension of the product is just the supremum of the Krull dimensions of its components, and we are done.

From the above result we get the microlocal condition of the Auslander regularity of $\hat{R}$.

THEOREM 2.10. *Let R be a filtered ring such that $G(R)$ is Noetherian commutative. If $\hat{R}$ has finite global dimension, then the following assertions are equivalent:*

(a) *$\hat{R}$ is Auslander regular.*

(b) *$Q_S^\mu(R)$ is Auslander regular, for every subset $S \subset R$ in the standard situation.*

(c) *$Q_{\mathfrak{m}}^\mu(R)$ is Auslander regular, for every $\mathfrak{m} \in \mathrm{Max}^{\mathrm{gr}}(G(R))$.*

Proof. As in the proof of the above theorem, we can assume that R is complete so that $R = \hat{R}$.

(a)$\Rightarrow$(b) Since gl-dim $Q_S^\mu(R)$ is finite, we only have to check the Auslander condition. Let U be a finitely generated left $Q_S^\mu(R)$-module. Then $U \cong W/V$, for some finitely generated free left $Q_S^\mu(R)$-module W and some submodule V. When W is made into a finitely generated filt-free $Q_S^\mu(R)$-module, it is isomorphic to $Q_S^\mu(L)$ for a certain finitely generated filt-free left R-module L. Now Theorem 1.5 applies and we can think of V as being of the type $Q_S^\mu(N)$, for some R-submodule N of L on which the induced filtration is taken. Thus $U \cong Q_S^\mu(L)/Q_S^\mu(N) \cong Q_S^\mu(L/N)$ and it is not restrictive to assume that $U = Q_S^\mu(M)$, for some filt-finitely generated left R-module M. If $0 \leqslant k \leqslant \mu_S = \text{gl-dim}\, Q_S^\mu(R)$ and $0 \neq Y \subseteq \mathrm{Ext}^k_{Q_S^\mu(R)}(Q_S^\mu(M), Q_S^\mu(R)) \cong Q_S^\mu(\mathrm{Ext}^k_R(M, R))$ is a submodule, then Theorem 1.5 may be used again to see that $Y \cong Q_S^\mu(N)$, for some nonzero R-submodule N of $\mathrm{Ext}^k_R(M, R)$. Then $0 \leqslant k \leqslant \mu_S \leqslant \mu = \text{gl-dim}\, R$ and the Auslander condition for R ensures that $j(N_R) \geqslant k$, i.e., $\mathrm{Ext}^i_R(N, R) = 0$ for $i \leqslant k-1$. Therefore $\mathrm{Ext}^i_Q(Q_S^\mu(N), Q_S^\mu(R)) \cong Q_S^\mu(\mathrm{Ext}^i_R(N, R)) = 0$, showing that $j(Y_Q) \geqslant k$, where $Q = Q_S^\mu(R)$.

(b)$\Rightarrow$(c) Clear.

(c)$\Rightarrow$(a) Let M be a finitely generated left R-module and give it a good filtration. If $0 \leqslant k \leqslant \mu = \text{gl-dim}\, R$ and $N \neq 0$ is a submodule of $\mathrm{Ext}^k_R(M, R)$, then the induced filtration on N is good and $Q_{\mathfrak{m}}^\mu(N)$ is a $Q_{\mathfrak{m}}^\mu(R)$-submodule of $Q_{\mathfrak{m}}^\mu(\mathrm{Ext}^k_R(M, R)) \cong \mathrm{Ext}^k_{Q_{\mathfrak{m}}^\mu(R)}(Q_{\mathfrak{m}}^\mu(M), Q_{\mathfrak{m}}^\mu(R))$, for every maximal graded ideal $\mathfrak{m}$ of $G(R)$. If $k > \mu_{\mathfrak{m}} = \text{gl-dim}\, Q_{\mathfrak{m}}^\mu(R)$, then the latter $Q_{\mathfrak{m}}^\mu(R)$-module is zero and thereby $Q_{\mathfrak{m}}^\mu(N) = 0$. This implies that $Q_{\mathfrak{m}}^\mu(\mathrm{Ext}^j_R(N, R)) = 0$ for every $j \geqslant 0$, by Theorem 2.6. If $k \leqslant \mu_{\mathfrak{m}}$, then the Auslander condition for $Q_{\mathfrak{m}}^\mu(R)$ yields $\mathrm{Ext}^j_{Q_{\mathfrak{m}}^\mu(R)}(Q_{\mathfrak{m}}^\mu(N), Q_{\mathfrak{m}}^\mu(R)) = 0$ for $j \leqslant k-1$. By combining the two possibilities ($k \leqslant \mu_{\mathfrak{m}}$ and $k \geqslant \mu_{\mathfrak{m}}$), we get that $Q_{\mathfrak{m}}^\mu(\mathrm{Ext}^j_R(N, R)) = 0$ for each $\mathfrak{m} \in \mathrm{Max}^{\mathrm{gr}}(G(R))$ and each $j \leqslant k-1$. The Zariskian condition of R entails that $\mathrm{Ext}^j_R(N, R) = 0$ for $j \leqslant k-1$ and thus $j(N_R) \geqslant k$.

3. Applications and Examples

In this part our goal is to apply the previous results to the study of the global and Krull dimensions of certain filtered rings.

A. *Discrete Filtered Rings*

Recall that a graded ring $A = \bigoplus_{n \in \mathbb{Z}} A_n$ is called *left limited* when there exists a $p \in \mathbb{Z}$ (in fact $p \leqslant 0$) such that $A_n = 0$ for any $n < p$. When $p = 0$ in this definition, A is called a *positively graded ring*. If A is an arbitrary graded ring, then $A^+ = \bigoplus \{A_n / n \geqslant 0\}$ is a positively graded subring of A. Analogous terminology is used for graded modules and graded ideals.

Lemma 3.1. *Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a left limited commutative graded ring. Then any maximal graded ideal $\mathfrak{m}$ of A is of the type $(\bigoplus_{n<0} A_n) \oplus \mathfrak{m}_0 \oplus (\bigoplus_{n>0} A_n)$, where $\mathfrak{m}_0$ is a maximal ideal of A_0.*

Proof. Let $\mathfrak{m} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{m}_n$ and put $S = h(A) \backslash \mathfrak{m}$. Since S is a multiplicatively closed subset and any element of $\bigoplus_{n<0} A_n$ is nilpotent, S cannot have elements of negative degree. Hence $\mathfrak{m}_n = A_n$ for any $n < 0$. Let now $\{y_i / i \in I\}$ be a set of homogeneous elements that generate $\bigoplus_{n<0} A_n$ as A_0-module. Then $\mathfrak{m} = \mathfrak{m}^+ + \sum_{i \in I} A y_i$, where $\mathfrak{m}^+ = \bigoplus \{\mathfrak{m}_n / n \geqslant 0\}$. We prove that $\mathfrak{m}^+$ is a maximal graded ideal of $A^+ = \bigoplus \{A_n / n \geqslant 0\}$ and the problem is reduced to the case of a positively graded ring. Suppose that m^+ is not so and let $\mathfrak{m}^+ \subset \mathfrak{n}^+ \subset A^+$ be strict inclusions, where $\mathfrak{n}^+$ is a graded ideal of A^+. This entails that $\mathfrak{n}^+$ is not contained in $\mathfrak{m}$, whence $\mathfrak{m}$ is strictly contained in $\mathfrak{n}^+ + \sum_{i \in I} A y_i$. We can use the fact that $\{y_i / i \in I\}$ generates $\bigoplus \{A_n / n < 0\}$ as A_0-module to see that $\mathfrak{n}^+ + \sum_{i \in I} A y_i$ is a graded ideal of A (it is generated by homogeneous elements!). From the maximality of $\mathfrak{m}$ we get that $\mathfrak{n}^+ + \sum_{i \in I} A y_i = A$ so that $1 = n^+ + \sum_{i \in I} a_i y_i$, for certain $n^+ \in \mathfrak{n}^+$ and some finite family $\{a_i / i \in I\}$ (i.e., $a_i = 0$ for all $i \in I$ except for a finite number of them). By taking the zeroth homogeneous component of that equality, we can assume that n^+ and each $a_i y_i$ have degree zero. But then $\sum_{i \in I} a_i y_i$ is a homogeneous element of degree zero of $\mathfrak{m}$ and hence of $\mathfrak{m}^+ \subset \mathfrak{n}^+$. This implies that $1 \in \mathfrak{n}^+$, which contradicts the choice of $\mathfrak{n}^+$. If now everything is assumed positively graded, then $\mathfrak{m} \subseteq \mathfrak{m}_0 \oplus (\bigoplus_{n \geqslant 1} A_n)$ and the latter is clearly a graded ideal of A. Therefore $\mathfrak{m} = \mathfrak{m}_0 \oplus (\bigoplus_{n \geqslant 1} A_n)$, as desired.

Corollary 3.2. *Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a left limited commutative graded ring. The following assertions hold:*

(a) *Every maximal graded ideal of A is a maximal ideal in the ungraded sense.*

(b) $A_k \cdot A_{-k} \subseteq J(A_0)$ *for any* $k \neq 0$, *where* $J(A_0)$ *denotes the Jacobson radical of* A_0.

If R is a discrete filtered ring whose associated graded ring $G(R)$ is Noetherian commutative, then Lemma 3.1 tells us that, for any maximal graded ideal $\mathfrak{n}$ of $G(R)$, $h(G(R))\backslash \mathfrak{n} = G(R)_0 \backslash \mathfrak{n}_0$, where $\mathfrak{n}_0$ is a maximal ideal of $G(R)_0 = F_0 R/F_{-1} R$. Since $\mathfrak{n}_0$ must be of the type $\mathfrak{m}/F_{-1}(R)$, for some (left and right) maximal ideal $\mathfrak{m}$ of $F_0 R$, we get that $S_{\mathfrak{m}} = F_0 R \backslash \mathfrak{m}$ is a multiplicatively closed subset such that $\sigma(S_{\mathfrak{m}}) = h(G(R))\backslash \mathfrak{n}$. We can use the Zariskian condition of R and Proposition 17 of [17] to see that $S_{\mathfrak{m}}$ is actually a (left and right) Ore subset of R and thus of $F_0 R$. We write $R_{\mathfrak{m}}$ for $S_{\mathfrak{m}}^{-1} R$, hoping that no confusion will arise with the localization at maximal ideals ($\mathfrak{m}$ is not an ideal of R!). The following result is now a direct consequence of Lemma 3.1 and Theorem 2.7, bearing in mind that $Q^{\mu}_{\mathfrak{m}}(R) = R_{\mathfrak{m}}$ in this case.

THEOREM 3.3. *Let* R *be a discrete filtered ring such that* $G(R)$ *is Noetherian commutative. The following sentences are true*:

(a) *If* R *has finite global dimension, then* gl-dim R *is the supremum of the set* $\{\text{gl-dim } R_{\mathfrak{m}} / \mathfrak{m} \in \operatorname{Max}(F_0 R)\}$.

(b) *If* $F_0 R$ *is semilocal, i.e., has only a finite number of maximal ideals, then the above equality holds for Krull dimension.*

EXAMPLE 3.4 (Rings of differential operators). Let A be a Noetherian commutative ring containing a field K of characteristic zero. A K-derivation of A is a K-linear map $\delta: A \to A$ such that $\delta(ab) = \delta(a)b + a\delta(b)$. If $\delta_1, ..., \delta_n$ are n A-algebraically independent K-derivations of A, the ring $D_{\delta_1, ..., \delta_n}(A)$, or $D(A)$ if the δ_i's have been previously fixed, is the subring of the ring $\operatorname{End}_K(A)$ of K-endomorphisms of A generated by the multiplications by elements of A and the δ_i's. The ring $D(A)$ admits a canonical filtration: $\sum_p D(A) = \{\sum_{|\alpha| \leqslant p} a_\alpha \delta^\alpha / a_\alpha \in A$ for each $\alpha \in \mathbb{N}^n\}$ for $p \geqslant 0$ and $\sum_p D(A) = 0$ for $p < 0$ (here $\delta^\alpha = \delta_1^{\alpha_1} \cdot \cdots \cdot \delta_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ when $\alpha = (\alpha_1, ..., \alpha_n)$). Under certain conditions of regularity in A (see, e.g., [3, Theorem 3.1.2]), the associated graded ring $G_\Sigma(R)$ is (Noetherian) commutative with A as zeroth homogeneous component. The above theorem tells us that gl-dim $D(A) = \operatorname{Sup}\{\text{gl-dim } D(A_{\mathfrak{m}}) / \mathfrak{m} \in \operatorname{Max}(A)\}$, provided that the first member of the equality is finite. This is just Lemma 3.1.7 of [3]. The corresponding equality for Krull dimension also holds when A is assumed to be semilocal.

B. *Strongly Filtered Rings*

If R is a *strongly filtered ring* (i.e., $F_mR \cdot F_nR = F_{m+n}R$ for all $m, n \in \mathbb{Z}$) whose associated graded ring is commutative, then $G(R)$ is strongly graded and, consequently, there is an equivalence of categories $G(R)$-gr $\cong G(R)_0$-Mod under which $G(R)$ and $G(R)_0$ are corresponding objects (see [9, Theorem A.I.3.4]). This implies that any maximal graded ideal $\mathfrak{n}$ of $G(R)$ is of the type $\mathfrak{n} = G(R)\bar{\mathfrak{m}}$, where $\bar{\mathfrak{m}}$ is a maximal ideal of $G(R)_0$. Since the latter ring is just $F_0R/F_{-1}R$, $\bar{\mathfrak{m}} = \mathfrak{m}/F_{-1}R$ for some (left and right) maximal ideal $\mathfrak{m}$ of F_0R containing $F_{-1}R$. If we consider the multiplicatively closed subsets $U = h(G(R))\backslash\mathfrak{n}$ and $V = G(R)_0\backslash\bar{\mathfrak{m}}$ of $G(R)$, one easily gets that $U^{-1}G(R) = V^{-1}G(R)$. Now $S_{\mathfrak{m}} = F_0R\backslash\mathfrak{m}$ is a multiplicative subset of R such that $\sigma(S_{\mathfrak{m}}) = V$. By Theorem 1.3, $Q^{\mu}_{\mathfrak{n}}(R) = Q^{\mu}_{S_{\mathfrak{m}}}(R)$. Furthermore, $S_{\mathfrak{m}}$ is strongly multiplicatively closed in the terminology of [17] and, when considered as a subset of $\hat{R}$, the conditions of Proposition 17 of [17] are satisfied, because $\hat{R}$ is a Zariski ring. So $S_{\mathfrak{m}}$ is a (left and right) Ore subset of $\hat{R}$ and $Q^{\mu}_{S_{\mathfrak{m}}}(R)$ is the completion $(S_{\mathfrak{m}}{}^{-1}\hat{R})^{\wedge}$ of $S_{\mathfrak{m}}{}^{-1}\hat{R}$. This proves the following result.

THEOREM 3.5. *Let R be a strongly filtered ring whose associated graded ring is Noetherian commutative. Then* gl-dim $\hat{R} = \operatorname{Sup}\{\text{gl-dim}(S_{\mathfrak{m}}{}^{-1}\hat{R})^{\wedge} /$ $\mathfrak{m} \in \operatorname{Max}(F_0R)$ *and* $F_{-1}R \subseteq \mathfrak{m}\}$, *provided that the first member of the equality is finite.*

If the set of left (or right) maximal ideal of F_0R containing $F_{-1}R$ is finite, then the corresponding equality holds for the Krull dimension.

Remark 3.6. If in the above theorem R is also assumed to be Zariski, then we can substitute R for $\hat{R}$ so that $R_{\mathfrak{m}} = S_{\mathfrak{m}}{}^{-1}R$ appears instead of $S_{\mathfrak{m}}{}^{-1}\hat{R}$. This is due to the fact that the conditions of Proposition 17 in [17] are satisfied by $S_{\mathfrak{m}}$ and R. In particular, this is the case when R is complete. We have not obtained an answer for the general problem and the following question remains open:

Let R be a strongly filtered ring whose associated graded ring is Noetherian commutative and let $\mathfrak{m}$ be a left (or right) ideal of F_0R containing $F_{-1}R$. Is $S_{\mathfrak{m}} = F_0R\backslash\mathfrak{m}$ a left (resp. right) Ore subset of, say, F_0R?

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