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CONFIGURATION SPACES OF POINTS ON THE CIRCLE AND HYPERBOLIC DEHN FILLINGS

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A purely combinatorial compactification of the configuration space of $n\,(\geqslant 5)$ distinct points with equal weights in the real projective line was introduced by M. Yoshida. We geometrize it so that it will be a real hyperbolic cone-manifold of finite volume with dimension $n-3$. Then, we vary weights for points. The geometrization still makes sense and yields a deformation. The effectivity of deformations arisen in this manner will be locally described in the existing deformation theory of hyperbolic structures when $n-3=2, 3$.

1. INTRODUCTION

Let $X(n)$ be a space of configurations of n distinct points in the real projective line $\mathbf{RP}^1$ up to projective automorphisms. $X(n)$ can be expressed by the point set,

$$X(n) = ((\mathbf{RP}^1)^n - \mathbf{D})/\mathrm{PGL}(2, \mathbf{R})$$

where $\mathbf{D}$ is the big diagonal set,

$$\mathbf{D} = \{(\alpha_1, \ldots, \alpha_n) \in (\mathbf{RP})^n \mid \alpha_i = \alpha_j \text{ for some } i \neq j\}$$

and $\mathrm{PGL}(2, \mathbf{R})$ acts on $(\mathbf{RP}^1)^n$ diagonally. We assume that the number of points is at least five throughout this paper (for the case $n = 4$, see [7]).

There are two obvious observations. $X(n)$ is not connected since we are not allowed to have collisions of points. Each component can be labeled by a circular permutation of n letters up to reflection. In particular, the number of connected components is $(n-1)!/2$. Also a configuration can be normalized by sending three consecutive points to $\{0, 1, \infty\}$ so that the other points lie in the open unit interval (0, 1). Hence, each component of $X(n)$ can be identified with the set of ordered $n-3$ points in (0, 1), and in particular is homeomorphic to a cell of dimension $n-3$.

M. Yoshida introduced a purely combinatorial compactification $X_Y(n)$ of $X(n)$ in [12] by deleting the following smaller set $\mathbf{D}_*$ instead of the big diagonal $\mathbf{D}$,

$$\mathbf{D}_* = \{(\alpha_1, \ldots, \alpha_n) \in (\mathbf{RP}^1)^n \mid \alpha_{i_1} = \alpha_{i_2} = \cdots = \alpha_{i_{[(n+1)/2]}} \text{ for some distict } i_1, \ldots, i_{[(n+1)/2]}\}$$

where $[x]$ denotes the maximal integer which does not exceed x. When n is even, $X_Y(n)$ is not compact in fact, but there is a natural combinatorial interpretation of the ends.

On the other hand, Thurston gave a family of incomplete complex hyperbolic metrics on the space of configurations of n distinct points on the complex projective line $\mathbf{CP}^1$ in [10]. Their completions become complex hyperbolic cone-manifolds of complex dimension $n-3$.

By considering the real slice of Thurston's complex hyperbolization, we can give a natural real hyperbolic polyhedral structure on each component of $X(n)$. Moreover, the boundary of each polyhedron will be coded by degenerate configurations so that the hyperbolization gives rise to a geometric identification of pairs of faces on the components.

The first purpose of this paper is to show that such identifications make $X(n)$ lie as an open dense subset in a connected real hyperbolic cone-manifold homeomorphic to Yoshida's compactification in Theorem 1.

We will see more precisely that the resultant of face identification is compact if n is odd, and noncompact of finite volume otherwise. If $n \leqslant 6$, it is nonsingular. If $n \geqslant 7$, it is singular and its cone angle θ_n along a codimension two singular stratum satisfies the identity $\cos(\theta_n/6) = 1/(2\cos(2\pi/n))$. Hence, θ_n lies in $(\pi, 2\pi)$ and approaches 2π as $n \to \infty$.

Some related works on geometric interpretation of $X(n)$ from more analytic viewpoints can be found in [2, 11, 13].

The second purpose of this paper is to relate our geometrization with the existing deformation theory of hyperbolic structures by relaxing the construction when $n = 5, 6$. The discussion for the first purpose is based on the hypothesis that the points on a configuration all have equal weights. We will see what happens if we perturb weights of points slightly.

When $n = 5$, the geometrized configuration space is a nonorientable hyperbolic surface homeomorphic to a connected sum of five copies of the projective plane, $\#^5\mathbf{RP}^2$. We will assign a deformed hyperbolic structure on $\#^5\mathbf{RP}^2$ to each perturbation of weights. The freedom for varying weights on five points is of four-dimensional, and the space of marked hyperbolic structures on $\#^5\mathbf{RP}^2$ is homeomorphic to $\mathbf{R}^9$. We will show that this assignment is differentiable and that the derivative at equal weights has full rank in Theorem 2.

When $n = 6$, the geometrized configuration space is a complete hyperbolic 3-manifold of finite volume with ten cusps. A small perturbation gives rise to not a deformation in the usual sense, but a resultant of hyperbolic Dehn filling. The space of hyperbolic Dehn fillings of a complete hyperbolic 3-manifold can be locally identified with the space of representations of a fundamental group up to conjugacy. It has a structure of an algebraic variety of complex dimension = the number of cusps, and is smooth at the complete structure. Hence, in our case, the space of Dehn fillings is locally biholomorphic to $\mathbf{C}^{10}$. Then we will show again that the assignment is differentiable and the derivative at equal weights has full rank in Theorem 3.

We carry out the geometrization in the next section, and relate it with the deformation theory in the section after.

2. GEOMETRIZATION

2.1. Hyperbolic polyhedral structure on $X(n)$

To geometrize $X(n)$, we consider Euclidean n-gons with vertices marked by integers from 1 to n, where the marking may not be cyclically monotone. Let $X_{n,c}$ be the set of all marked equiangular n-gons up to mark preserving (possibly orientation reversing) congruence, and X_n a further quotient of $X_{n,c}$ by similarities.

For any $\alpha \in (\mathbf{RP}^1)^n - \mathbf{D}$, we assign the unit disc in $\mathbf{C}$ with n points specified on the boundary. By the Schwarz–Christoffel mapping or its complex conjugate, we can map α to an *equiangular* n-gon up to mark preserving similarity—i.e. an element of X_n. This induces a map from $X(n)$ to X_n since a projective transformation on the unit disc does not change the image of the map. It is also injective because if two configurations α and β map to the same element of $X_{n,c}$ by f_α and f_β then $f_\beta \circ f_\alpha^{-1}$ is a mark preserving projective automorphism of the unit disk. By the Carathéodory theorem, this map is surjective. Therefore we have proved

LEMMA 1. *There is a canonical homeomorphism between* $X(n)$ *and* X_n.

The aim of this subsection is to construct a hyperbolic polyhedral structure on a component of $X(n)$ through the above identification by taking the real part of Thurston's geometrization in [10]. Similar discussions with the rest of this subsection can be found also in [3, 6].

As we have noted in the introduction, the connected component of $X(n)$ consists of the configurations with a fixed circular permutation of markings up to reflection. For the simplicity of index, we shall concentrate on the component U of $X_{n,c}$ which is labeled by $12\cdots n$. Let $U_s \subset X_n$ be the set of its mark-preserving similarity classes. Note that U_s corresponds to the component of $X(n)$ labeled by $12\cdots n$ also. We identify U_s with the set U_1 which, by definition, consists of the set of equiangular polygons having its area one (see Fig. 1). We describe polygons as follows.

The elements of U can be described by the vector of side lengths $(x_1, x_2, \dots, x_n)$ where x_j is the length of the edge between the vertices marked by j and $j+1$. Since they represent an equiangular n-gon, they satisfy

$$x_1 + x_2\zeta_n + \cdots + x_n\zeta_n^{n-1} = 0$$

where $\zeta_n = \exp(2\pi i/n)$. We set

$$\mathscr{E}_n := \{(x_1, x_2, \dots, x_n) | x_1 + x_2\zeta_n + \cdots + x_n\zeta_n^{n-1} = 0\}$$

$$\mathscr{E}_n^+ := \mathscr{E}_n \cap \bigcap_{j=1}^{n} \{x_j > 0\}.$$

Note that $U = \mathscr{E}_n^+$.

For each element P of $\mathscr{E}_n^+$, we denote by Area(P) the area of P. It is a function from $\mathscr{E}_n^+$ to $\mathbf{R}$.

LEMMA 2. *The function* Area *is extended to a quadratic form of signature* $(1, n-3)$ *on* $\mathscr{E}_n$.

Proof. Suppose that P is an element of U. Place the edge x_n of P on the real axis of complex plane and extend other edges until they touch the real axis. We set edge length y_j, z_k, a_l and triangles B_s, C_t as in the figure (see Fig. 2). (If n is odd, then no $A_{n/2}$ appears.) Let A be the triangle $P \cup_s B_s \cup_t C_t$.

Suppose that n is odd and $n = 2m+1$. By the sine rule relatively to B_j,

$$\frac{a_j}{\sin j\theta_n} = \frac{y_j}{\sin\theta_n} = \frac{a_{j-1} + x_j}{\sin(j+1)\theta_n}$$

for $1 \leqslant j \leqslant m-1$ where $\theta_n = 2\pi/n$. Thus, we have

$$y_j \sin(j+1)\theta_n - y_{j-1}\sin(j-1)\theta_n = x_j \sin\theta_n.$$

The same argument for C_j shows that

$$z_j \sin(j+1)\theta_n - z_{j-1}\sin(j-1)\theta_n = x_{n-j}\sin\theta_n.$$

If we set

$$w_0 := x_n + \sum_j y_j + \sum_j z_j,$$

the above equalities show that the coordinate $(z_{m-1}, \dots, z_1, w_0, y_1, \dots, y_{m-1})$ is obtained from $(x_1, \dots, x_n)$ by a linear isomorphism.

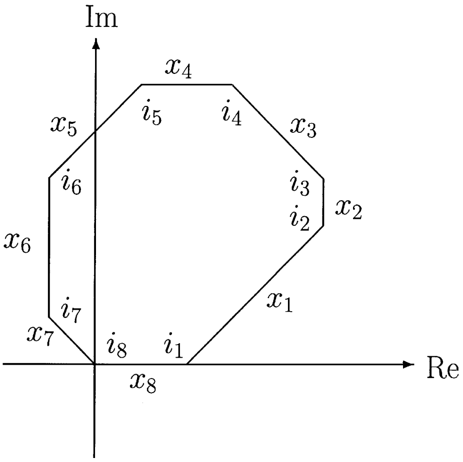


Fig. 1. An equiangular octagon.

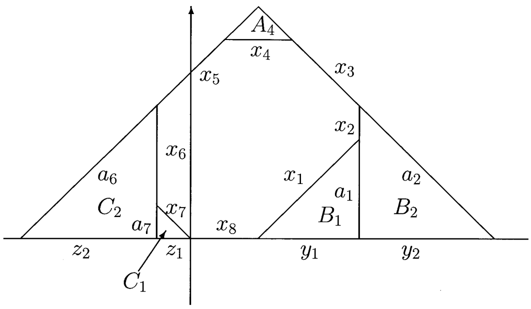


Fig. 2.

Since the areas of the triangles A, B_j, C_j are as follows,

$$\text{Area } A = w_0^2 \frac{1}{4} \tan \frac{\theta_n}{2}$$

$$\text{Area } B_j = \frac{1}{2} y_j (a_{j-1} + x_j) \sin j\theta_n = y_j^2 \frac{\sin j\theta_n \sin (j+1)\theta_n}{2 \sin \theta_n}$$

$$\text{Area } C_j = \frac{1}{2} z_j (a_{n-j+1} + x_{n-j}) \sin j\theta_n = z_j^2 \frac{\sin (-j)\theta_n \sin (-j+1)\theta_n}{2 \sin \theta_n}$$

we define the new coordinate by setting

$$X_0 = w_0 \sqrt{\frac{1}{4} \tan \frac{\theta_n}{2}}$$

$$Y_j = y_j \sqrt{\frac{\sin j\theta_n \sin (j+1)\theta_n}{2 \sin \theta_n}}$$

$$Z_j = z_j \sqrt{\frac{\sin (-j)\theta_n \sin (-j+1)\theta_n}{2 \sin \theta_n}}.$$

$(Z_{m-1}, \dots, Z_1, X_0, Y_1, \dots, Y_{m-1})$ is obtained from $(z_{m-1}, \dots, z_1, w_0, y_1, \dots, y_{m-1})$ by a linear isomorphism. The area of the polygon is

$$\text{Area} = X_0^2 - \sum_j Y_j^2 - \sum_j Z_j^2.$$

If n is even and $n = 2m + 2$, add $X_{m+1} := x_{m+1}\sqrt{(1/4)\tan\theta_n/2}$ and $(Z_{m-1}, \dots, Z_1, X_0, Y_1, \dots, Y_{m-1}, X_{m+1})$ is the coordinate. In this case, the area is

$$\text{Area} = X_0^2 - X_{m+1}^2 - \sum_j Y_j^2 - \sum_j Z_j^2.$$

The function Area: $\mathscr{E}_n^+ \to \mathbf{R}$ defines a quadratic form of $\mathscr{E}_n$ which we also denote by Area and the above calculation shows that its signature is $(1, n-3)$. □

$\mathscr{E}_n$ together with Area becomes a Minkowski space. Let $\mathscr{P}_n$ be the intersection $\text{Area}^{-1}(1) \cap \{X_0 > 0\}$. Then $\mathscr{P}_n$ is the hyperbolic space and U_1 is canonically homeomorphic to

$$\text{Area}^{-1}(1) \cap \mathscr{E}_n^+ = \text{Area}^{-1}(1) \cap \bigcap_{i=1}^{n} \{x_i > 0\} \subset \mathscr{P}_n.$$

The region is bounded by $\mathscr{P}_n \cap \{x_i = 0\}$ for $i = 1, 2, \dots, n$. Since $\{x_i = 0\}$ is a hyperplane containing the origin in the Minkowski space $\mathscr{E}_n$, the intersection with the hyperboloid $\mathscr{P}_n$ is the hyperbolic hyperplane. It implies that:

LEMMA 3. *There is a canonical homeomorphism between U_1 and an interior of the $(n-3)$-dimensional hyperbolic polyhedron obtained by taking closure of* $\text{Area}^{-1}(1) \cap \mathscr{E}_n^+$.

We denote this closed hyperbolic polyhedron by Δ_n and its faces corresponding to $\{x_i = 0\} \cap \Delta_n$ by F_i. To see some geometric properties of Δ_n, we recall a lemma in hyperbolic geometry. The Lorentz bilinear form $q(\ ,\ )$ on the Minkowski space $\mathscr{E}_n$ is defined by a quadratic form Area by setting

$$q(x, y) = \tfrac{1}{2}(\text{Area}(x + y) - \text{Area}(x) - \text{Area}(y)).$$

LEMMA 4 (Thurston [10, Proposition 2.4.5]). (1) *Let p_1 and p_2 be in hyperbola $\mathscr{P}_n$ and $d(p_1, p_2)$ the hyperbolic distance between p_1 and p_2. Then*

$$\cosh d(p_1, p_2) = q(p_1, p_2).$$

(2) *Let n_1 and n_2 be normal vectors of the hyperplanes H_1 and H_2 in $\mathscr{P}_n$.*

(a) *If H_1 and H_2 intersect in $\mathscr{P}_n$, then their dihedral angle $\angle(H_1, H_2)$ satisfies*

$$\cos \angle(H_1, H_2) = \frac{q(n_1, n_2)}{\sqrt{q(n_1, n_1)}\sqrt{q(n_2, n_2)}}.$$

(b) *If H_1 and H_2 do not intersect in $\mathscr{P}_n$, then their shortest distance $d(H_1, H_2)$ satisfies*

$$\cosh d(H_1, H_2) = \frac{q(n_1, n_2)}{\sqrt{q(n_1, n_1)}\sqrt{q(n_2, n_2)}}.$$

Then we have

Lemma 5. *The faces of Δ_n intersect as follows.*

(1) $|i-j| \geqslant 2 \Rightarrow F_i \perp F_j$,
(2) *If $n = 5$ or 6, then $F_j \cap F_{j+1} = \emptyset$,*
(3) *If $n \geqslant 7$,*

$$\cos(\omega_n) = \frac{1}{2\cos 2\pi/n},$$

where ω_n is the dihedral angle between F_j and F_{j+1}.

Proof. (1) Suppose that $i - j \geqslant 2$. We change the way we extend the edges of P to form triangles B_i and C_i as we have done to get the new coordinate of $\mathscr{E}_n$. Extend two edges adjacent to the edge x_i and x_j, respectively, so that we obtain an $(n-2)$-gon P' (see Fig. 3).

Call the triangles B_1, B_2. For any linear isomorphism on $\mathscr{E}_n$ whose first and second new coordinates are $Y_1 = \sqrt{\text{Area}(B_1)}$ and $Y_2 = \sqrt{\text{Area}(B_2)}$, the quadratic form Area is written as:

$$\text{Area} = (\text{quadratic form without } Y_1 \text{ and } Y_2) - Y_1^2 - Y_2^2.$$

$\{x_i = 0\}$ and $\{x_j = 0\}$ in this coordinate is $\{Y_1 = 0\}$ and $\{Y_2 = 0\}$, respectively, and we can choose normal vectors by $n_1 = (1, 0, \ldots, 0)$ and $n_2 = (0, 1, 0, \ldots, 0)$, respectively. Therefore $q(n_i, n_j) = 0$ and F_i is orthogonal to F_j by Lemma 4(2).

(2) Suppose that $n = 5$. The intersection of F_5 and F_1 must consist of ("generalized" and "degenerate") equiangular pentagons as in the figure (see Fig. 4). There are two possibilities depending on the sign of x_2. In both cases the areas are negative so that they cannot intersect in $\mathscr{P}_5$.

The case $n = 6$ is similar. If we set $x_i = x_{i+1} = 0$, then there are four patterns of equiangular hexagons according to the sign of x_{i-2} and x_{i-1}. The areas all are negative.

(3) By the symmetry of Δ_n we may assume that $j = 1$. We denote $\{x_i = 0\}$ by E_i for $i = 1, 2$. E_1 is equal to $\{Y_1 = 0\}$ in the new coordinate. The n-gon with $x_2 = 0$ is depicted in Fig. 5.

By the sine rule for the two triangles in the figure, we have

$$\frac{y_1}{\sin 3\theta_n} = \frac{y_2}{\sin \theta_n}.$$

The ratio of the square of the areas of two triangles are

$$\frac{Y_2^2}{Y_1^2} = \frac{y_2}{y_1} = \frac{1}{3 - 4\sin^2\theta_n}$$

and constant. Thus,

$$E_2 = \{Y_1 = Y_2\sqrt{3 - 4\sin^2\theta_n}\}.$$

Therefore, normal vectors of these two hyperplanes can be chosen by

$$n_1 = (0, \ldots, 0, 1, 0, \ldots, 0)$$

$$n_2 = (0, \ldots, 0, 1, -\sqrt{3 - 4\sin^2\theta_n}, \ldots, 0).$$

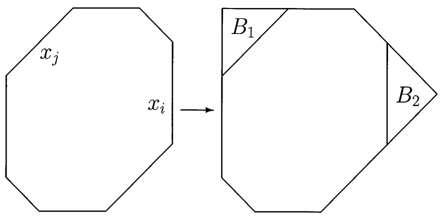


Fig. 3.

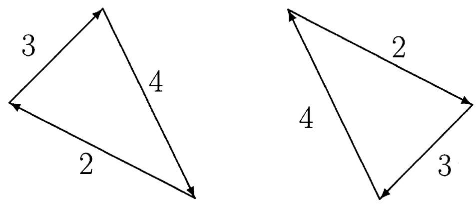


Fig. 4. Degenerated pentagons.

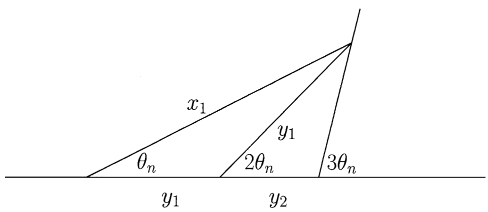


Fig. 5.

Then by Lemma 4(1), we have

$$\cos \omega_n = \frac{q(n_1, n_2)}{\sqrt{q(n_1, n_1)\, q(n_2, n_2)}} = \frac{1}{2\cos\theta_n}. \qquad \square$$

Remark 1. The formula in the Lemma tells us the angle:

n	7	8	9	10	$\cdots$	$n \to \infty$
ω_n (deg)	36.6845	45	49.2542	51.8273	$\cdots$	$\omega_n \nearrow 60$

Therefore $\omega_n < 60$ for $n = 7, 8, \dots$.

Remark 2. Δ_5 is bounded by 5 geodesics and F_i and $F_{i \pm 2}$ are orthogonal, i.e. a hyperbolic right angle pentagon. The same argument shows that Δ_6 is a hyperbolic trigonal bipyramid with angles between two intersecting faces $= \pi/2$ and three ideal vertices.

2.2. *Identification of the boundary of* $X_n \approx X(n)$

The number of hyperplanes $\{x_i = 0\}$ with consecutive indices which can meet on the boundary of Δ_n is at most $[(n-1)/2] - 1$, otherwise $[(n-1)/2]$ consecutive edges of the corresponding equiangular n-gon degenerate so that the area cannot be positive.

Such a degeneration corresponds to a collision of $[(n+1)/2]$ consecutive points on the configuration which is ruled out in Yoshida's $X_Y(n)$. Thus, the boundary of Δ_n corresponds to the set of degenerate configurations added in $X_Y(n)$. On the other hand, our geometrization assigns not only a projective class of a configuration to each point in the interior on Δ_n, but a corresponding degenerate configuration to each point on the boundary of Δ_n. Gluing $(n-1)!/2$ copies of Δ_n together along the faces which represent the same degenerate configurations, and we obtain $\overline{X_n}$ which is homeomorphic to Yoshida's compactification. $X_n \approx X(n)$ now lives in $\overline{X_n}$ as an open dense subset.

We finish this section by discussing how singularities appear in $\overline{X_n}$ by looking at gluing rule of polyhedral blocks.

The point lying on a face of codimension one in Δ_n corresponds to a configuration with a collision of two points. Hence, we may label this face by a circular permutation of $n-2$ numbers and a group of two numbers up to reflection. For example, the label $(12)3\cdots n$ means that each point on that face represents a collision of the points marked by 1 and 2. Then the number of polyhedral blocks which share such a face is two according to how we approach to that degenerate configuration from nondegenerate ones. Hence, the gluing does not yield any singularity along such a face. This proves that our geometrization is a hyperbolic cone-manifold by the definition of cone-manifolds.

The point lying on a face of codimension two in Δ_n corresponds to a configuration with either a pair of collisions of two points or a collision of three points. We may label such a face by grouping marks involved in the collision together, such as $(12)3(45)6\ldots n$, $(123)4\ldots n$, etc.

In the first case, the number of polyhedral blocks which share such a face is four according to how we approach to that degenerate configuration. On the other hand, the dihedral angle of two faces which share this codimension two face is $\pi/2$ by Lemma 5(1). Hence, again the gluing does not yield any singularity along such a face.

These two observations show that $\overline{X_n}$ is nonsingular when $n=5$. Actually, it is a hyperbolic surface which consists of 12 hyperbolic right angle pentagons. Since each vertex belongs to four pentagons, the number of faces (pentagon), edges, vertices are 12, 30, 15, respectively, and Euler characteristic is -3. It follows that it is a nonorientable surface homeomorphic to a connected sum of five copies of $\mathbf{RP}^2$.

Let us discuss the case when $n=6$ before going into the other case. X_6 consists of the interior of 60 hyperbolic hexahedra. We are not allowed to have a collision of three successive points in this case. Hence, the gluing does not yield any singularity along face of codimension at most two. Then consider a point on the face of codimension three. Since $n-3=3$, such a face is a vertex and corresponds to a triple of collisions of two points. The number of components of X_6 which share such a vertex is eight. On the other hand, the neighborhood of the vertex of Δ_6 is isometric to a neighborhood of the vertex of the intersection between the non-negative orthant in the Poincaré model of $\mathbf{H}^3$ in $\mathbf{R}^3$. Hence, again the gluing does not yield any singularity. Moreover, since horospherical cut of an ideal vertex in Δ_6 is always square, the gluing yields a complete end. Therefore $\overline{X_6}$ is a complete hyperbolic 3-manifold.

We can derive a few more information about geometry of $\overline{X_6}$. Since Δ_6 is scissors congruent to a quarter of the regular ideal octahedron, whose volume is $3.66386\cdots$, the volume of $\overline{X_6}$ is $54.957\cdots$. $\overline{X_6}$ admits a natural action of the symmetry group of degree 6 by permuting labels of points. It turns out to be a full isometry group since the quotient is congruent to the smallest orbifold with appropriate date found by Adams in [1]. We will see in the next section that $\overline{X_6}$ has ten cusps.

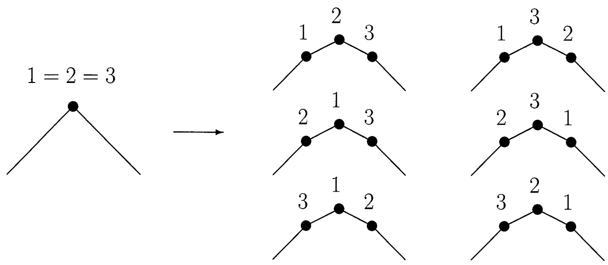


Fig. 6.

Suppose now that $n \geqslant 7$ and let us consider the neighborhood of a degenerate configuration with a collision of three successive points. This always happens when $n \geqslant 7$. There are six polyhedral blocks which share that configuration on the boundary according to the permutations of three numbers involved in the collision as in the Fig. 6. But by Lemma 5, the dihedral angle of each piece is less than $2\pi/6$, thus it gives rise to the singular points. Hence we have

THEOREM 1. *$\overline{X_n}$ is a hyperbolic cone-manifold and homeomorphic to $X_Y(n)$.*

- *When $n = 5$ or 6, it is nonsingular.*
- *When $n \geqslant 7$, the singular set is nonempty.*

Remark 3. Every configuration appeared in $\overline{X_n}$ has at least three labeled points which are disjointly placed on the circle. Normalize neighbor configurations of any particular degenerate one by sending such labeled points to $\{0, 1, \infty\}$ by means of a projective automorphism. Then its neighborhood in $\overline{X_n}$ is parameterized by the position of other $n-3$ points. This observation shows that $\overline{X_n}$ is topologically a manifold.

3. DEFORMATION

3.1. Perturbation

In the previous section, the configuration space $X(n)$ is identified with the space of marked equiangular n-gons up to similarity. The identification is given by the Schwarz–Christoffel mapping with all external angles fixed to $2\pi/n$. If we perturb these external angles, the images of Schwarz–Christoffel mapping change and one can expect that the hyperbolic structure of the configuration space will deform accordingly.

Let Θ_n be the set of n-tuples of real numbers $\theta = (\theta_1, \ldots, \theta_n)$ satisfying the relations

$$\sum_{j=1}^{n} \theta_j = 2\pi \quad \text{and} \quad 0 < \theta_i + \theta_j < \pi \quad (i, j \in \{1, \ldots, n\}).$$

The indices should be understood modulo n throughout the sequel.

Fix an element $\theta = (\theta_1, \ldots, \theta_n)$ of Θ_n. Choose $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbf{RP}^1)^n - \mathbf{D}$, and we assign to α the unit disc in $\mathbf{C}$ with n points specified on the boundary. Then we map it conformally to an n-gon P whose vertices are the images of the specified points and the external angle of the image of the jth point α_j is θ_j. By the Schwarz–Christoffel formula, P is defined up to mark preserving similarity. Let $X_{n,\theta}$ be the space of mark preserving similarity classes of Euclidean n-gons with external angles $\{\theta_1, \ldots, \theta_n\}$ compatible with markings.

Then the same argument in Lemma 1 shows that

LEMMA 6. *There is a canonical homeomorphism between* $X(n)$ *and* $X_{n,\theta}$.

As in the previous section, we get a hyperbolic polyhedral structure on each component of $X_{n,\theta}$—we start from the parameterization $(x_1, \ldots, x_n)$ of polygons by edge length and the function Area turns out to be a quadratic form of signature $(1, n-3)$ again. Each component is identified with the intersection of hyperbola $\mathrm{Area}^{-1}(1)$ and n halfspaces. The components of $X_{n,\theta}$ are no longer congruent each other. However, the gluing rule still makes sense, and we obtain a hyperbolic cone-manifold $\overline{X_{n,\theta}}$ as well by identifying the boundary of $X_{n,\theta} \approx X(n)$ with corresponding weights. $\overline{X_{n,\theta}}$ contains $X_{n,\theta}$ as an open dense subset. We will see what $\overline{X_{n,\theta}}$ looks like when $n = 5, 6$.

3.2. Deformations of $\overline{X_5}$

Fix an element $\theta = (\theta_1, \ldots, \theta_5)$ of Θ_5. Let $p = \langle i_1\, i_2\, i_3\, i_4\, i_5 \rangle$ be a circular permutation of $\{1, \ldots, 5\}$ up to reflection and $U_{p,\theta}$ the component of $X_{5,\theta}$ which consists of all pentagons whose marking correspond to p. The external angle of the vertex marked by i is θ_i by definition. Each element of $U_{p,\theta}$ is parameterized by the side lengths $(x_{i_1 i_2}, x_{i_2 i_3}, \ldots, x_{i_5 i_1})$ where $x_{i_a i_b}$ is the length of the edge between the vertices marked by i_a and i_b.

We modify the notation in the previous section and set

$$\mathscr{E}_{p,\theta} := \left\{ (x_{i_1 i_2}, \ldots, x_{i_5 i_1}) \,|\, x_{i_1 i_2} + x_{i_2 i_3} \exp(\sqrt{-1}\theta_{i_2}) + \cdots + x_{i_5 i_1} \exp\Big(\sqrt{-1} \sum_{j=2}^{5} \theta_{i_j}\Big) = 0 \right\}$$

$$X := \sqrt{\mathrm{Area}\, A}, \qquad Y := \sqrt{\mathrm{Area}\, B}, \qquad Z := \sqrt{\mathrm{Area}\, C}$$

where A, B and C are the triangles A, B_1 and C_1 in Fig. 2, respectively. By the same argument in the proof of Lemma 2, (X, Y, Z) is a coordinate of $\mathscr{E}_{p,\theta}$.

Set $\mathscr{P}_5 = \mathrm{Area}^{-1}(1) \cap \{X > 0\}$, then $U_{p,\theta}$ is homeomorphic to $\mathscr{P}_5 \cap \bigcap_{a=1}^{5} \{x_{i_a i_{a+1}} > 0\}$ i.e. a hyperbolic pentagon. We denote by $\Delta_{p,\theta}$ this hyperbolic pentagon. We also simply denote by $(i_1 i_2) i_3 i_4 i_5$ its edge which corresponds to the degenerate pentagons by the collision of the vertices i_1 and i_2. Similarly, we use $(i_1 i_2)\,(i_3 i_4)\; i_5$ to represent the verticies of $\Delta_{p,\theta}$.

We next calculate the length of the edge $i_1 i_2 i_3 (i_4 i_5)$.

LEMMA 7. *Let* $\theta = (\theta_1, \ldots, \theta_5)$ *be an element of* Θ_5 *and* $L(i_1 i_2 i_3 (i_4 i_5); \theta)$ *the length of the edge* $i_1 i_2 i_3 (i_4 i_5)$ *of* $\Delta_{p,\theta}$. *Then we have*

$$\cosh L(i_1 i_2 i_3 (i_4 i_5); \theta) = \sqrt{\frac{\sin \theta_{i_1} \sin \theta_{i_3}}{\sin(\theta_{i_1} + \theta_{i_2}) \sin(\theta_{i_2} + \theta_{i_3})}}\,.$$

Proof. Suppose that the pentagon $\Delta_{p,\theta}$ is placed in the hyperboloid of XYZ space as explained above such that $i_1 i_2 i_3 (i_4 i_5)$ and $i_1 (i_2 i_3)\, i_4 i_5$ are in $\{Y = 0\}$ and $\{Z = 0\}$, respectively. By Lemma 5, the end points of $i_1 i_2 i_3 (i_4 i_5)$ are $i_1 (i_2 i_3)(i_4 i_5)$ and $(i_1 i_2) i_3 (i_4 i_5)$. We denote their coordinates by (x', y', z') and (x'', y'', z'').

Observe that $i_1 (i_2 i_3)(i_4 i_5)$ is in $\{Y = 0\}$ and $\{Z = 0\}$, so $(x', y', z') = (1, 0, 0)$.

Next we calculate x''. The pentagon P with $x_{i_2 i_3} = 0$ and $x_{i_4 i_5} = 0$ is depicted in the Fig. 7. Calculate the areas of A and P by using the edge e_1 as the common base edge:

$$\mathrm{Area}(P) : \mathrm{Area}(A) = e_2 \sin(\theta_{i_1} + \theta_{i_2}) : e_3 \sin \theta_{i_1}.$$

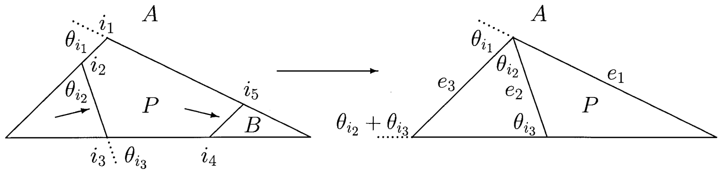


Fig. 7. Degeneration of $i_1 i_2$ and $i_4 i_5$.

Since $\mathrm{Area}(P) = 1$ at (x'', y'', z'') and using the sine rule:

$$\frac{e_2}{\sin(\theta_{i_2} + \theta_{i_3})} = \frac{e_3}{\sin\theta_{i_3}}$$

we have

$$x'' = \sqrt{\mathrm{Area}(A)} = \sqrt{\frac{\sin\theta_{i_1}\sin\theta_{i_3}}{\sin(\theta_{i_1} + \theta_{i_2})\sin(\theta_{i_2} + \theta_{i_3})}}\,. \tag{1}$$

By Lemma 4,

$$\cosh L(i_1 i_2 i_3(i_4 i_5); \theta) = q((x', y', z'), (x'', y'', z'')) = q((1, 0, 0), (x'', y'', z'')) = x''. \tag{2}$$

The identities (1) and (2) prove the lemma. □

Now we investigate how the hyperbolic structure changes when θ is perturbed. First, we comment on the topological type of the new $\overline{X_{5,\theta}}$. For each θ near $\theta_0 = (2\pi/5, 2\pi/5, \ldots, 2\pi/5)$, the same argument in Lemma 5(1) shows that $\Delta_{p,\theta}$ is a right-angled pentagon, and hence $\overline{X_{5,\theta}}$ is still a hyperbolic surface with the same topology as $\overline{X_5} \approx \#^5\mathbf{RP}^2$.

The surface is nonorientable and does not support any complex structure at all. However, we can still establish the analogue of Teichmüller theory. In fact, if we choose a maximal family of mutually disjoint nonparallel simple closed curves, then the set of hyperbolic structures is parameterized by their lengths and twisting amount for 2-sided ones. Hence the Teichmüller space $\mathscr{T}(\#^5\mathbf{RP}^2)$ is homeomorphic to $\mathbf{R}^9$. Following Teichmüller theory, we call this coordinate a Fenchel–Nielsen coordinate.

To find a system of mutually disjoint, nonparallel simple closed curves, let us enjoy some patch work. Here is part of our surface.

The end points of each solid lines in the Fig. 8 are identified in $\overline{X_{5,\theta}}$ to form closed geodesics and they are mutually disjoint. Reading the labels in the figure, we have simple closed curves $(i_4 i_5)$ in $\overline{X_{5,\theta}}$ which consists of three edges $i_1 i_2 i_3(i_4 i_5)$, $i_2 i_1 i_3(i_4 i_5)$ and $i_2 i_3 i_1(i_4 i_5)$.

We choose $(j5)$, $j = 1, \ldots, 4$ as a system of four mutually disjoint, nonparallel simple closed curves, and let $L(ij;\theta)$ denote the length of (ij) where $i, j \in \{1, \ldots, 5\}$ and $\theta \in \Theta_5$. In the next lemma, we calculate $L(ij;\theta)$. For $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in \Theta$, define a function $N(i_1 i_2 i_3;\theta)$ by

$$N(i_1 i_2 i_3;\theta) = \frac{\sin\theta_{i_1}\sin\theta_{i_2}\sin\theta_{i_3} - \sin(\theta_{i_1} + \theta_{i_2} + \theta_{i_3})\,(\sin\theta_{i_1}\sin\theta_{i_2} + \sin\theta_{i_2}\sin\theta_{i_3} + \sin\theta_{i_3}\sin\theta_{i_1})}{\sin(\theta_{i_1} + \theta_{i_2})\sin(\theta_{i_2} + \theta_{i_3})\sin(\theta_{i_3} + \theta_{i_1})}\,.$$

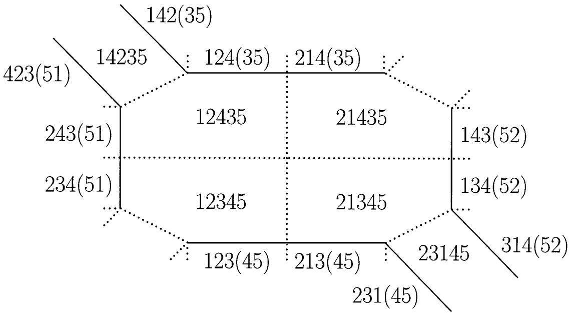


Fig. 8.

Then we have

LEMMA 8. *For* $\{i_1, i_2, i_3, i_4, i_5\} = \{1, \ldots, 5\}$ *and* $\theta = (\theta_i, \theta_2, \theta_3, \theta_4, \theta_5) \in \Theta_5$:

$$\cosh(L(i_4 i_5; \theta)) = N(i_1 i_2 i_3; \theta).$$

To prove this lemma, we prepare a lemma for trigonometric computation.

LEMMA 9. *For any* $\alpha, \beta, \gamma \in \mathbf{R}$, *we have*

$$\sin\alpha \sin\gamma - \sin(\alpha+\beta)\sin(\beta+\gamma) = -\sin\beta\sin(\alpha+\beta+\gamma). \tag{3}$$

Proof. The addition rule says the identity, $(\cos(a-b) - \cos(a+b))/2 = \sin a \sin b$. Apply this relation for each term in equation (3). □

Proof of Lemma 8. Put $\Delta_{p,\theta}$ in the hyperbola of XYZ space again (see Fig. 9(a)). Suppose that $e_1 := i_1 i_2 i_3 (i_4 i_5)$ corresponds to $\{Y = 0\}$ and $e_2 := (i_1 i_2) i_3 i_4 i_5$ corresponds to $\{Z = 0\}$. Developing across e_2, we meet $i_2 i_1 i_3 i_4 i_5$ and the edge next to e_1 is $e_3 := i_2 i_1 i_3 (i_4 i_5)$. Let $p_a = (x_a, y_a, z_a)$ be the end points of e_a $(a = 1, 3)$ which is not $(1, 0, 0)$. For simplicity, we denote $\sin\theta_{i_j}$, $\sin(\theta_{i_j} + \theta_{i_k})$, $\sin(\theta_{i_j} + \theta_{i_k} + \theta_{i_l})$ by s_j, s_{jk}, s_{jkl} respectively. Then by Lemma 7

$$x_1 = \sqrt{\frac{s_1 s_3}{s_{12} s_{23}}}, \quad y_1 = \sqrt{x_1^2 - 1} = \sqrt{\frac{s_1 s_3 - s_{12} s_{23}}{s_{12} s_{23}}} = \sqrt{\frac{-s_2 s_{123}}{s_{12} s_{23}}}, \quad z_1 = 0$$

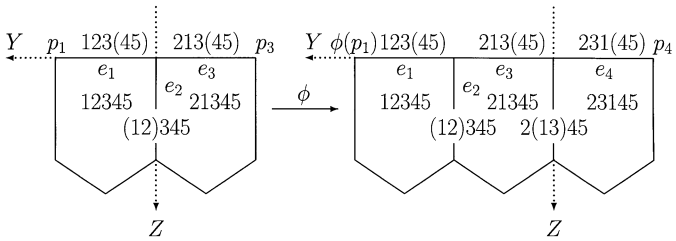
$$x_3 = \sqrt{\frac{s_2 s_3}{s_{21} s_{13}}}, \quad y_3 = -\sqrt{x_3^2 - 1} = -\sqrt{\frac{s_2 s_3 - s_{21} s_{13}}{s_{21} s_{13}}} = -\sqrt{\frac{-s_1 s_{123}}{s_{21} s_{13}}}, \quad z_3 = 0.$$

We used equation (3) for y_1 and y_3.

Now apply the hyperbolic isometry ϕ on the hyperboloid which fixes the geodesic through e_1 (and e_3) setwise and sends p_3 to $(1, 0, 0)$ (Fig. 9(b)). Denote by (x_1', y_1', z_1') the coordinate of $\phi(p_1)$. Since ϕ preserves q

$$\begin{aligned} x_1' &= x_1' \cdot 1 - y_1' \cdot 0 - z_1' \cdot 0 = q(\phi(p_1), \phi(p_3)) = q(p_1, p_3) = x_1 x_3 - y_1 y_3 \\ &= \sqrt{\frac{s_1 s_3 s_2 s_3}{s_{12} s_{23} s_{21} s_{13}}} + \sqrt{\frac{s_2 s_{123} s_1 s_{123}}{s_{12} s_{23} s_{21} s_{13}}} \end{aligned}$$

$$y_1' = \sqrt{{x_1'}^2 - 1} = \sqrt{\frac{s_1 s_3 s_2 s_3 + s_2 s_{123} s_1 s_{123} - 2 s_1 s_2 s_3 s_{123} - s_{12} s_{23} s_{21} s_{13}}{s_{12} s_{23} s_{21} s_{13}}}.$$



(a) (b)

Fig. 9.

In the above equation, we used $\sqrt{s_j^2} = s_j$, $\sqrt{s_{jj+1}^2} = s_{jj+1}$ and $\sqrt{s_{jj+1j+2}^2} = -s_{jj+1j+2}$ because $0 < \theta_{i_j} + \theta_{i_{j+1}} < \pi$ and $\pi < \theta_{i_j} + \theta_{i_{j+1}} + \theta_{i_{j+2}} < 2\pi$. By (3), $s_{12}s_{23}s_{21}s_{13} = (s_1 s_3 + s_2 s_{123})(s_2 s_3 + s_1 s_{123})$, hence

$$= \sqrt{\frac{s_1 s_3 s_2 s_3 + s_2 s_{123} s_1 s_{123} - 2 s_1 s_2 s_3 s_{123} - (s_1 s_3 + s_2 s_{123})\ (s_2 s_3 + s_1 s_{123})}{s_{12} s_{23} s_{21} s_{13}}}$$

$$= \sqrt{\frac{-(s_1 + s_2)^2 s_3 s_{123}}{s_{12} s_{23} s_{21} s_{13}}},$$

$$z_1' = 0.$$

Develop again across $i_2(i_1 i_3) i_4 i_5$ to meet the component $i_2 i_3 i_1 i_4 i_5$ and the edge next to e_3 is $e_4 = i_2 i_3 i_1 (i_4 i_5)$. Let $p_4 = (x_4, y_4, z_4)$ be the end points of e_4 which is not $(1, 0, 0)$. Then again by Lemma 7

$$x_4 = \sqrt{\frac{s_2 s_1}{s_{23} s_{31}}}, \quad y_4 = -\sqrt{x_4^2 - 1} = -\sqrt{\frac{s_2 s_1 - s_{23} s_{31}}{s_{23} s_{31}}} = -\sqrt{\frac{-s_3 s_{123}}{s_{23} s_{31}}}, \quad z_4 = 0.$$

Hence, the hyperbolic cosine of the length of $e_1 e_3 e_4$ is

$$\begin{aligned}
&\cosh d(\phi(p_1), p_4)\\
&= x_1' x_4 - y_1' y_4 - z_1' z_4\\
&= \left(\sqrt{\frac{s_1 s_3 s_2 s_3}{s_{12} s_{23} s_{21} s_{13}}} + \sqrt{\frac{s_2 s_{123} s_1 s_{123}}{s_{12} s_{23} s_{21} s_{13}}}\right)\sqrt{\frac{s_2 s_1}{s_{23} s_{31}}} + \sqrt{\frac{-(s_1 + s_2)^2 s_3 s_{123}}{s_{12} s_{23} s_{21} s_{13}}}\sqrt{\frac{-s_3 s_{123}}{s_{23} s_{31}}}\\
&= \frac{s_1 s_2 s_3 - s_{123}(s_1 s_2 + s_2 s_3 + s_3 s_1)}{s_{12} s_{23} s_{31}}.
\end{aligned}$$

□

THEOREM 2. *The map from* Θ_5 *to* $\mathscr{T}(\#^5\mathbf{RP}^2)$ *defined by the assignment*: $\theta \mapsto \overline{X_{n,\theta}}$ *is a local embedding at* $\theta_0 = (2\pi/5, \ldots, 2\pi/5)$.

Proof. Choose four closed geodesics (15), (25), (35) and (45). By Fig. 8, they are mutually disjoint and nonparallel, so that their lengths will be a part of a Fenchel–Nielsen coordinate. Taking these components of the map from Θ to $\mathscr{T}(\#^5\mathbf{RP}^2)$, we get

$$\begin{aligned}
\Phi\colon \Theta_5 &\to \mathbf{R}^4\\
\theta = (\theta_1, \ldots, \theta_5) &\mapsto (L(15;\theta), L(25;\theta), L(35;\theta), L(45;\theta)).
\end{aligned}$$

It is enough to show that the Jacobian of Φ does not vanish at θ_0. As a basis of the tangent space of Θ at θ_0, we take four paths $p_j(t) = (\theta_{j1}(t), \ldots, \theta_{j5}(t))\ j = 1, \ldots, 4$ passing through the barycenter θ_0 defined by $\theta_{jj}(t) = \frac{2\pi}{5} + t, \theta_{jj+1}(t) = \frac{2\pi}{5} - t, \theta_{jk}(t) = \frac{2\pi}{5}$ for $k \neq j, j+1$. Then

$$\Phi(p_1(t)) = (a(t), b(t), c(t), c(t))$$

$$\Phi(p_2(t)) = (c(t), a(t), b(t), c(t))$$

$$\Phi(p_3(t)) = (c(t), c(t), a(t), b(t))$$

$$\Phi(p_4(t)) = (b(t), b(t), b(t), d(t))$$

where

$$\cosh a(t) = N(234; p_1(t)), \qquad \cosh b(t) = N(134; p_1(t))$$

$$\cosh c(t) = N(124; p_1(t)), \qquad \cosh d(t) = N(123; p_4(t)).$$

Then since $a'(0) = -b'(0)$ is non-zero and $c'(0) = d'(0) = 0$, it is easy to see that the Jacobian of Φ at θ_0 is $-3(a'(0))^4 \neq 0$. □

3.3. *Deformations of* $\overline{X_6}$

As in the previous case, we use the notation Θ_6, $p = \langle i_1 i_2 i_3 i_4 i_5 i_6 \rangle$, $U_{p,\theta}$ and $\mathscr{P}_6$, etc. Also set

$$\mathscr{E}_{p,\theta} := \left\{ (x_{i_1 i_2}, \ldots, x_{i_6 i_1}) \,|\, x_{i_1 i_2} + x_{i_2 i_3} \exp(\sqrt{-1}\theta_{i_2}) + \cdots + x_{i_6 i_1} \exp(\sqrt{-1} \sum_{j=2}^{6} \theta_{i_j}) = 0 \right\},$$

$$X := \sqrt{\mathrm{Area}\, A}, \quad Y := \sqrt{\mathrm{Area}\, B}, \quad Z := \sqrt{\mathrm{Area}\, C}, \quad W := \sqrt{\mathrm{Area}\, D}$$

where A,B,C,D are A,B_1,C_1,A_3 in Fig. 2. Then (X,Y,Z,W) is a coordinate of $\mathscr{E}_{p,\theta}$. $U_{p,\theta}$ is homeomorphic to the region $\Delta_{p,\theta} = \mathscr{P}_6 \cap \bigcap_{a=1}^{6} \{x_{i_a i_{a+1}} \geqslant 0\}$ which is a hyperbolic hexahedron of finite volume.

Let us describe how $\Delta_{p,\theta}$ deforms when we perturb θ (see Fig. 10). We denote the face of $\Delta_{p,\theta}$ which corresponds to the set of degenerate hexagons by collisions of the points i_j and i_{j+1} by $(i_j i_{j+1}) i_{j+2} i_{j+3} i_{j+4} i_{j+5}$ or $(i_j i_{j+1})$ if there is no confusion.

When the weights are equal, namely $\theta_0 = (2\pi/6, \ldots, 2\pi/6)$, Δ_{p,θ_0} has three ideal vertices. Observe that the four faces containing an ideal vertex has labels of type $(i_k i_{k+1})$, $(i_{k+1} i_{k+2})$, $(i_{k+3} i_{k+4})$, $(i_{k+4} i_{k+5})$ for some $k \in \{0, \ldots, 5\}$. We denote this vertex by $(i_k i_{k+1} i_{k+2})(i_{k+3} i_{k+4} i_{k+5})$.

If we perturb the angles so that $\theta_{i_k} + \theta_{i_{k+1}} + \theta_{i_{k+2}} < \pi$, then three vertices i_k, i_{k+1}, i_{k+2} of the hexagon can collide, and $(i_k i_{k+1})$ and $(i_{k+1} i_{k+2})$ intersects in $\mathscr{P}_6$. If $\theta_{i_{k+3}} + \theta_{i_{k+4}} + \theta_{i_{k+5}} < \pi$, then $(i_{k+3} i_{k+4})$ and $(i_{k+4} i_{k+5})$ intersects (see Fig. 10). Let us use the notation $(i_1 i_2 i_3) i_4 i_5 i_6$ and $i_1 i_2 i_3 (i_4 i_5 i_6)$ to indicate edges appeared by these perturbations, respectively.

We shall calculate the dihedral angles between this newly intersecting faces. Note that dihedral angles around other (old) edges are $\pi/2$ by the same argument as in Lemma 5.

LEMMA 10. *Suppose that* $\theta = (\theta_1, \ldots, \theta_6)$ *be an element of* Θ_6 *and* $\theta_{i_1} + \theta_{i_2} + \theta_{i_3} < \pi$. *Let* ω *be the dihedral angle between the faces* $(i_1 i_2) i_3 i_4 i_5 i_6$ *and* $i_1 (i_2 i_3) i_4 i_5 i_6$. *Then we have*

$$\cos \omega = \sqrt{\frac{\sin \theta_{i_1} \sin \theta_{i_3}}{\sin(\theta_{i_1} + \theta_{i_2}) \sin(\theta_{i_2} + \theta_{i_3})}}.$$

To prove this lemma, we shall use the next identity.

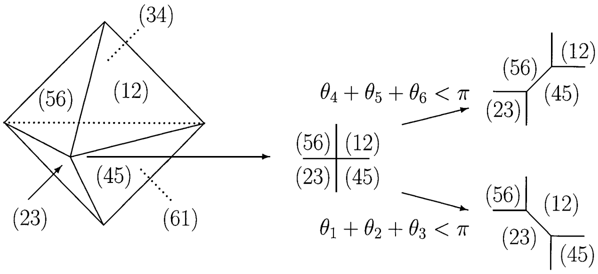


Fig. 10. Cusp of $\overline{X_6}$.

Lemma 11. *Suppose that* $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6=2\pi$. *Then*

$$\sin(\alpha_1+\alpha_2)\sin\alpha_4-\sin(\alpha_5+\alpha_6)\sin\alpha_3=\sin(\alpha_1+\alpha_2+\alpha_3)\sin(\alpha_3+\alpha_4). \tag{4}$$

Proof. By the addition rule, we have

$$\begin{aligned}
&\sin(\alpha_1+\alpha_2)\sin\alpha_4-\sin(\alpha_5+\alpha_6)\sin\alpha_3\\
&\quad=\sin(\alpha_1+\alpha_2+\alpha_3-\alpha_3)\sin\alpha_4-\sin(\alpha_4+\alpha_5+\alpha_6-\alpha_4)\sin\alpha_3\\
&\quad=\sin(\alpha_1+\alpha_2+\alpha_3)\cos\alpha_3\sin\alpha_4-\cos(\alpha_1+\alpha_2+\alpha_3)\sin\alpha_3\sin\alpha_4\\
&\qquad-\sin(\alpha_4+\alpha_5+\alpha_6)\cos\alpha_4\sin\alpha_3+\cos(\alpha_4+\alpha_5+\alpha_6)\sin\alpha_4\sin\alpha_3,
\end{aligned}$$

and by $\sum\alpha_i=2\pi$,

$$\begin{aligned}
&\quad=\sin(\alpha_1+\alpha_2+\alpha_3)\cos\alpha_3\sin\alpha_4-\cos(\alpha_1+\alpha_2+\alpha_3)\sin\alpha_3\sin\alpha_4\\
&\qquad+\sin(\alpha_1+\alpha_2+\alpha_3)\cos\alpha_4\sin\alpha_3+\cos(\alpha_1+\alpha_2+\alpha_3)\sin\alpha_4\sin\alpha_3\\
&\quad=\sin(\alpha_1+\alpha_2+\alpha_3)\sin(\alpha_3+\alpha_4),
\end{aligned}$$

which proves the identity. □

Proof of Lemma 10. We find the defining equations of two faces as hyperplanes in $\mathscr{E}_{p,\theta}$ in terms of the coordinate $XYZW$. We again adopt the notation s_j, s_{jk}, s_{jkl} for $\sin\theta_{i_j}$, $\sin\theta_{i_ji_k}$, $\sin\theta_{i_ji_ki_l}$. We first recall the transformation from $\mathscr{E}_{p,\theta}$ to $XYZW$. We have

$$Y=\sqrt{\operatorname{Area}B}=\sqrt{\frac{s_1s_2}{2s_{12}}}\,x_{i_1i_2},\quad W=\sqrt{\operatorname{Area}D}=\sqrt{\frac{s_3s_4}{2s_{34}}}\,x_{i_3i_4},$$

$$\begin{aligned}
X=\sqrt{\operatorname{Area}A}&=\sqrt{\frac{s_{12}s_{34}}{2s_{56}}}\left(x_{i_2i_3}+\frac{s_1}{s_{12}}\,x_{i_1i_2}+\frac{s_4}{s_{34}}\,x_{i_3i_4}\right)\\
&=\sqrt{\frac{s_{12}s_{34}}{2s_{56}}}\left(x_{i_2i_3}+\sqrt{\frac{2s_1}{s_2s_{12}}}\sqrt{\operatorname{Area}B}+\sqrt{\frac{2s_4}{s_3s_{34}}}\sqrt{\operatorname{Area}D}\right).
\end{aligned}$$

Thus, the defining equation of $\{x_{i_1i_2}=0\}$ and $\{x_{i_2i_3}=0\}$ is

$$Y=0\quad\text{and}\quad X=\sqrt{\frac{s_{34}s_1}{s_{56}s_2}}\,Y+\sqrt{\frac{s_{12}s_4}{s_{56}s_3}}\,Z.$$

Hence, we can choose normal vectors n_1, n_2 for each hyperplane by $n_1 = (0, 1, 0, 0)$ and $n_2 = (1, \sqrt{s_{34}s_1/s_{56}s_2}, 0, \sqrt{s_{12}s_4/s_{56}s_3})$, respectively. Then by Lemma 4

$$\cos\omega = \frac{q(n_1, n_2)}{\sqrt{q(n_1, n_1)q(n_2, n_2)}} = \frac{\sqrt{\dfrac{s_{34}s_1}{s_{56}s_2}}}{\sqrt{-1 + \dfrac{s_{34}s_1}{s_{56}s_2} + \dfrac{s_{12}s_4}{s_{56}s_3}}}$$

$$= \left(\sqrt{-\frac{s_{56}s_2}{s_{34}s_1} + 1 + \frac{s_{12}s_4s_2}{s_3s_{34}s_1}}\right)^{-1} = \left(\sqrt{1 + \frac{s_2}{s_1s_3} \times \frac{s_{12}s_4 - s_{56}s_3}{s_{34}}}\right)^{-1},$$

by the identities (4) and (3),

$$= \left(\sqrt{1 + \frac{s_2s_{123}}{s_1s_3}}\right)^{-1} = \left(\sqrt{1 + \frac{s_{12}s_{23} - s_1s_3}{s_1s_3}}\right)^{-1} = \sqrt{\frac{s_1s_3}{s_{12}s_{23}}}$$

which concludes the proof. □

As mentioned in the sentences before the lemma, each dihedral angle around the old edges is $\pi/2$ so they fit together without producing any singularity. But the hyperbolic structure at the new edges can be singular in $\overline{X_{6,\theta}}$. A cross-section perpendicular to the new edge will be a cone, obtained by taking a two-dimensional hyperbolic sector of some angle and identifying the two bounding rays emanating from the center. Such a singular structure appears in the hyperbolic Dehn filling theory in [9] and is called a cone singularity.

Before investigating what the singularity looks like, we describe a polygonal decomposition of cusps. The faces of hexahedra Δ_{p,θ_0}'s lie in the same component of cusps in $\overline{X_6}$ if and only if the labels are identical as a partition of six numbers. Hence, the number of cusps is equal to the number of partitions of $\{1, 2, \ldots, 6\}$ into a pair of three numbers, $= \binom{6}{3}/2 = 10$. We may use the notation $(i_1i_2i_3)\,(i_4i_5i_6)$ to indicate a component of cusps also if there is no confusion. Now, since the total amount of ideal vertices of hexahedra in X_6 is $3 \times (6-1)!/2 = 180$, $180/10 = 18$ components of ideal vertices of Δ_{p,θ_0}'s come to a component of cusps in $\overline{X_6}$. Figure 11 shows how they come to. Replacing the mark j by i_j in Fig. 11, we get the picture of the cusp labeled by $(i_1i_2i_3)(i_4i_5i_6)$.

To calculate the cone angle for each cone singularity, we look at the boundary of a small equidistant neighborhood of the singular locus (or the ideal vertex). Note that the boundary is a torus obtained by gluing rectangles, each of which is a new face of some $\Delta_{p,\theta}$ appeared by the truncation. Since the old faces of $\Delta_{p,\theta}$ intersect in right angle, not only a combinatorial but conformal pattern of its polygonal decomposition is described by Fig. 11. We then see that six edges with the same label (i_5i_6) form a nontrivial loop on the boundary. If $\theta_{i_1} + \theta_{i_2} + \theta_{i_3} < \pi$, it winds once around the singular locus labeled by $(i_1i_2i_3)i_4i_5i_6$. If $\theta_{i_1} + \theta_{i_2} + \theta_{i_3} > \pi$, it is homotopic to a loop winding the singular locus labeled by $i_1i_2i_3(i_4i_5i_6)$ twice.

LEMMA 12. *Suppose that* $\{i_1, \ldots, i_6\} = \{1, \ldots, 6\}$, $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \in \Theta_6$ *and* $\theta_{i_1} + \theta_{i_2} + \theta_{i_3} < \pi$. *Let* $\angle((i_1i_2i_3); \theta)$ *be the cone angle about the singular locus labeled by* $(i_1i_2i_3)i_4i_5i_6$ *in* $\overline{X_{6,\theta}}$. *Then we have*

$$\cos(\angle((i_1i_2i_3); \theta)/2) = N(i_1i_2i_3; \theta).$$

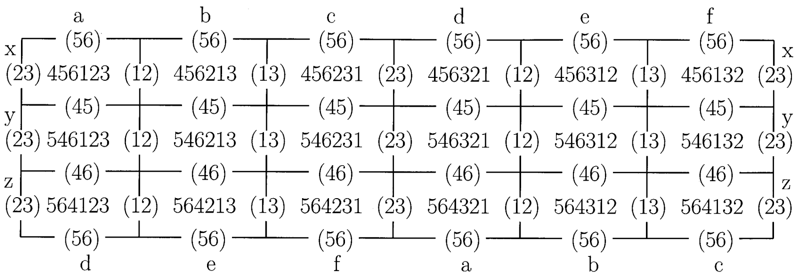


Fig. 11. Cusp of $\overline{X_6}$: edges with the same letters are identified.

Proof. Regarding the mark j as i_j in Fig. 11, we can identify the angle with the sum of dihedral angles about the new edges of six successive hexahedra in the horizontal direction. However because of the gluing rule shown in Fig. 11, the half can be computed by summing three successive ones. To compute it, look at the faces of hexahedra which appear as sections to the singular locus. The corresponding Fig. 12 in this case to Fig. 9 shows how hexahedra are developed about the singular locus in $\mathscr{P}_6$. Choose unit normal vectors to each face sharing the singular locus. By replacing point vectors on the hyperbola in the proof of Lemma 8 by normal vectors to the faces in the Minkowski space, we can proceed the computation of sum of three dihedral angles in the light of similarity between Lemmas 7 and 10. The conclusion follows from Lemma 4 (2a) here instead of (1). □

We very briefly recall some foundations of the hyperbolic Dehn filling theory based on Thurston [9], Neumann-Zagier [8] and Culler-Shalen [4]. Let N be an orientable complete hyperbolic 3-manifold of finite volume with s cusps, and $\rho_0 : \pi_1(N) \to \mathrm{SL}(2, \mathbf{C})$ a lift of the holonomy representation of N. The algebro geometric quotient of all $\mathrm{SL}(2, \mathbf{C})$-representations of $\Pi = \pi_1(N)$ is called a character variety and denoted by $X(\Pi)$. The set of Dehn filled deformations of N is locally parameterized by a neighborhood of ρ_0 in $X(\Pi)$. The following structure theorem is fundamental, which appeared in this form for example in [5], though the claim could be derived from the arguments in [9, 4, 8].

LEMMA 13. *$X(\pi_1(N))$ is a smooth manifold near ρ_0 of complex dimension s. If $m_1, \ldots, m_s$ are meridional curves for cusps, then the map $f: X(\pi_1(N)) \to \mathbf{C}^s$ defined by*

$$f(\chi) = (\chi_\rho(m_1), \ldots, \chi_\rho(m_s))$$

is a local diffeomorphism near ρ where $\chi_\rho(m_i) = \operatorname{trace} \rho(m_i)$.

Going back to our setting and let $\mathscr{L}$ be the union of singular loci (or ideal vertices). $\overline{X_{6,\theta}} - \mathscr{L}$ is homeomorphic to $\overline{X_6}$ and $\overline{X_{6,\theta}}$ is its Dehn filled resultant. $\overline{X_{6,\theta}} - \mathscr{L}$ carries a nonsingular but incomplete hyperbolic metric. Let $\rho_\theta : \Pi = \pi_1(\overline{X_{6,\theta}} - \mathscr{L}) \to \mathrm{SL}(2, \mathbf{C})$ be a lift of the holonomy representation of $\overline{X_{6,\theta}} - \mathscr{L}$.

To see how $\overline{X_{6,\theta}}$ is deformed, we only need by Lemma 13 to compute the trace of a holonomy image of ρ_θ at some meridional elements. To define appropriate meridional elements, assume for the moment that $i_1 + i_2 + i_3 < \pi$. Then the cusp labeled by $(i_1 i_2 i_3)(i_4 i_5 i_6)$ becomes a singular locus in $\overline{X_{6,\theta}}$ labeled by $(i_1 i_2 i_3) i_4 i_5 i_6$, and there is a natural meridional element winding once around the singular locus. We denote it by $m_{i_1 i_2 i_3}$. It is homotopic to a loop on the boundary of a tubular neighborhood of the singular locus labeled by either $(i_4 i_5)$, $(i_5 i_6)$ or $(i_4 i_6)$ (see Fig. 11). Note that $m_{i_1 i_2 i_3}$ is a meridional element if $i_1 + i_2 + i_3 < \pi$ but no longer meridional if $i_1 + i_2 + i_3 > \pi$ in any sense.

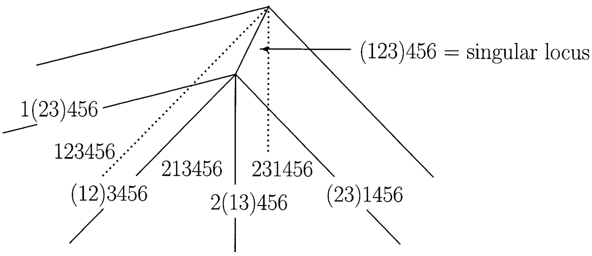


Fig. 12.

For example, $\rho_\theta(m_{123})$ in Fig. 11 acts as a translation of six blocks in horizontal direction, however, it is actually a rotation, a parabolic translation or a hyperbolic translation according to whether $\theta_1+\theta_2+\theta_3$ is less than, equal to or greater than π. Having this picture in mind, we prove

LEMMA 14. *Suppose that* $\{i_1, \dots, i_6\} = \{1, \dots, 6\}$, $\theta = (\theta_i, \dots, \theta_6) \in \Theta_6$, *and denote by* ρ_θ *the holonomy representation with respect to* θ. *Then we have*

$$\chi_{\rho_\theta}(m_{i_1 i_2 i_3}) = 2N(i_1 i_2 i_3; \theta).$$

Proof. Suppose that $\theta_{i_1}+\theta_{i_2}+\theta_{i_3}<\pi$ and let ω be $\exp(\sqrt{-1}\angle((i_1 i_2 i_3)/2;\theta))$. Then $\rho_\theta(m_{i_1 i_2 i_3})$ is an elliptic element of rotation ω^2, and its action on $\mathbf{C}\cup\{\infty\}$ is conjugate to

$$z \mapsto \omega^2 z = \frac{\omega z + 0}{0\cdot z + \omega^{-1}}.$$

Hence $\rho_\theta(m_{i_1 i_2 i_3})$ is conjugate to

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$

By Lemma 12,

$$\chi_{\rho_\theta}(m_{i_1 i_2 i_3}) = \omega + \omega^{-1} = 2\cos(\angle((i_1 i_2 i_3);\theta)/2) = 2N(i_1 i_2 i_3;\theta).$$

Suppose that $\theta_{i_1}+\theta_{i_2}+\theta_{i_3}=\pi$. Then $(i_1 i_2 i_3)(i_4 i_5 i_6)$ is an ideal vertex and $\rho_\theta(m_{i_1 i_2 i_3})$ is a parabolic element. By easy computation, we have

$$\chi_{\rho_\theta}(m_{i_1 i_2 i_3}) = 2 = 2N(i_1 i_2 i_3;\theta).$$

Suppose that $\theta_{i_1}+\theta_{i_2}+\theta_{i_3}>\pi$. Then $\rho_\theta(m_{i_1 i_2 i_3})$ is a hyperbolic element translating the face labeled by, say, $(i_2 i_3)$ to one to be identified in $\overline{X_{6,\theta}}$. Regarding i_j as j in Fig. 11, we can identify the faces in question with ones on the right-hand and left-hand sides. Let δ be a distance between them.

δ is equal to the sum of lengths of edges labeled by $i_1 i_2 i_3(i_4 i_5 i_6)$ in six successive hexahedra in the horizontal direction in Fig. 11. Again because of the gluing rule in Fig. 11, the half of δ can be computed by summing three successive ones. Then developing the faces of three hexahedra involving the edges labeled by $i_1 i_2 i_3(i_4 i_5 i_6)$ as in Fig. 9, we now realize that the computation we carried out in Lemma 12 measures the distance between faces labeled by (i_2, i_3) on a side and $(i_2 i_3)$ on the middle in Fig. 11 because of Lemma 4 (2b). More precisely, we have

$$\cosh(\delta/2) = N(i_1 i_2 i_3;\theta).$$

The difference is that the value of Lorentz bilinear forms for normal vectors is greater than 1 in this case, but less than 1 in the previous case.

The action of such a hyperbolic motion is conjugate to

$$z \mapsto \lambda^2 z = \frac{\lambda z + 0}{0 \cdot z + \lambda^{-1}}$$

where λ is a real number > 1 and $\delta = \log \lambda^2$. Hence, $\rho_\theta(m_{i_1 i_2 i_3})$ is conjugate to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

and

$$\chi_{\rho_\theta}(m_{i_1 i_2 i_3}) = \lambda + \lambda^{-1} = e^{\delta/2} + e^{-\delta/2} = 2\cosh(\delta/2) = 2N(i_1 i_2 i_3; \theta).$$

This completes the proof. □

THEOREM 3. *The map from Θ_6 to $X(\Pi)$ defined by the assignment: $\theta \mapsto \overline{X_{6,\theta}}$ is a local embedding at $\theta_0 = (\pi/3, \ldots, \pi/3)$.*

Proof. To apply Lemma 13, we choose meridional loops for five cusps, say m_{145}, m_{234}, m_{235}, m_{245}, m_{345}. Denote by ρ_θ a lift of the holonomy of $\overline{X_{6,\theta}} - \mathscr{L}$ in $\mathrm{SL}(2, \mathbf{C})$. Let $\Phi : \Theta_6 \to \mathbf{C}^5$ be a map defined by

$$\theta \mapsto (\chi_{\rho_\theta}(m_{145}), \chi_{\rho_\theta}(m_{234}), \chi_{\rho_\theta}(m_{235}), \chi_{\rho_\theta}(m_{245}), \chi_{\rho_\theta}(m_{345})).$$

It suffices to show that Φ is locally injective around θ_0. Since the image of Φ is contained in the real part $\mathbf{R}^5 \subset \mathbf{C}^5$, we regard Φ as a map from Θ to $\mathbf{R}^5$. We shall show that the Jacobian of Φ does not vanish.

As a basis of the tangent space of Θ at θ_0, we take the following five paths. $p_j(t) = (\theta_{j1}(t), \ldots, \theta_{j6}(t))$ $(j = 1, \ldots, 5)$ passing through θ_0 defined by

$$\theta_{jj}(t) = \frac{\pi}{3} - t, \qquad \theta_{j6}(t) = \frac{\pi}{3} + t, \qquad \theta_{jk}(t) = \frac{\pi}{3} \quad \text{for } k \neq j, 6.$$

By Lemma 14 we have

$$\begin{array}{lcl} \Phi(p_1(t)) & = & (\; f(t), \; 2, \; 2, \; 2, \; 2 \;) \\ \Phi(p_2(t)) & = & (\; 2, \; f(t), \; f(t), \; f(t), \; 2 \;) \\ \Phi(p_3(t)) & = & (\; 2, \; f(t), \; f(t), \; 2, \; f(t) \;) \\ \Phi(p_4(t)) & = & (\; f(t), \; f(t), \; 2, \; f(t), \; f(t) \;) \\ \Phi(p_5(t)) & = & (\; f(t), \; 2, \; f(t), \; f(t), \; f(t) \;) \end{array}$$

where

$$f(t) = 2N\left(123; \left(\frac{\pi}{3} - t, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right)\right)$$

$$= 2\,\frac{\sin\dfrac{\pi}{3}\sin\left(\dfrac{\pi}{3} - t\right) - \sin\dfrac{\pi}{3}\sin t - 2\sin t \sin\left(\dfrac{\pi}{3} - t\right)}{\sin^2\left(\dfrac{2\pi}{3} - t\right)}.$$

Then $f'(0) = -6\sqrt{3}$. Therefore, the Jacobian of Φ at θ_0 is $-3\times(-6\sqrt{3})^5$ which concludes the proof of the theorem. □

Remark 4. Our computation is valid only when θ is close to θ_0. Ref. [7] presents an expanded background for the present deformation which might help to prove global injectivity.

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REFERENCES

1. Adams, C., Noncompact hyperbolic 3-orbifolds of small volume. In *Topology '90*, eds. Apanasov *et al.* de Gruyter, Berlin, 1992, pp. 1–15.
2. Apéry, F. and Yoshida, M., Pentagonal structure of the configuration space of five points in the real projective line. *Kyushu Journal of Mathematics*, 1998, **52**, 1–14.
3. Bavard, C. and Ghys, E., Polygones du plan et polyedres hyperboliques. *Geometriae Dedicata*, 1992, **43**, 207–224.
4. Culler, M. and Shalen, P., Varieties of group representations and splittings of 3-manifolds. *Annals of Mathematics*, 1983, **117**, 109–146.
5. Hodgson, C. and Kerckhoff, S., Rigidity of hyperbolic cone-manifolds and hyperbolic Dehn surgery. *Journal of Differential Geometry*, 1998, **48**, 1–59.
6. Kojima, S. and Yamashita, Y., Shapes of stars. *Proceedings of the American Mathematical Society*, 1993, **117**, 845–851.
7. Morin, B. and Nishi, H., Hyperbolic structures on the configuration space of six points in the projective line. Preprint, KYUSHU-MPS-1997-24.
8. Neumann, W. and Zagier, D., Volumes of hyperbolic 3-manifolds. *Topology*, 1985, **24**, 307–332.
9. Thurston, W., *Geometry and Topology of 3-Manifolds*, Lecture Notes. Princeton University Press, Princeton, 1977/78
10. Thurston, W., Shapes of polyhedra. Research Report GCG 7, 1986.
11. Yamazaki, T. and Yoshida, M., On Hirzebruch's examples of surfaces with $c_1^2 = 3c_2$. *Mathematische Annalen*, 1984, **266**, 421–431.
12. Yoshida, M., The democratic compactification of configuration spaces of point sets on the real projective line. *Kyushu Journal of Mathematics*, 1996, **50**, 493–512.
13. Yoshida, M., The real loci of the configuration space of six points on the projective line and a Picard modular 3-fold. Preprint, KYUSHU-MPS-1997-14.

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