Super edge-connectivity of dense digraphs and graphs

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Abstract


Super-λ is a more refined network reliability index than edge-connectivity. A graph is super-λ if every minimum edge-cut set is trivial (the set of edges incident at a node with the minimum degree δ). This paper establishes the relation between diameter and super-λ: enlarging the order n under the given maximum degree Δ and diameter D not only maximizes edge-connectivity, but also minimizes the number of minimum edge-cut sets, thus attaining super-λ. The following sufficient conditions for a digraph and graph G to be super-λ are derived.

- Digraph G is super-λ if \( n > \delta ((\Delta - 1)^{D-1} - 1) / (\Delta - 1) + \Delta ^{D-1} \).
- Graph G is super-λ if \( n > \delta ((\Delta - 1)^{D-1} + 1) / (\Delta - 1) + (\Delta - 1)^{D-1} \).

These conditions are the best possible. From these, the de Bruijn digraph B(\(d,D\)), the Kautz digraph K(\(d,D\)), and most of the densest known graphs (listed in [3, 10]) are shown to be super-λ. Also, the digraph \(G_d(n,d)\) proposed in [25] as a maximally connected \(d\)-regular digraph with quasiminimal diameter (at most one larger than the lower bound) is proved to be super-λ for any \(d > 2\) and any order \(n > d^3\).

1. Introduction

A processor interconnection network or a communications network is conveniently modeled by a graph or a digraph (directed graph) \(G = (V,E)\), in which the node set \(V\) corresponds to processors or switching elements, and the edge set \(E\) corresponds to communication links. Two fundamental considerations in the design of such networks are overall reliability and maximum transmission delay [27, 4]. This paper identifies a new relation between an overall reliability index and a maximum transmission delay index.
1.1. Reliability indices

Overall reliability can usually be measured by the connectivity $\kappa(G)$ or the edge-connectivity $\lambda(G)$ of the graph, which respectively correspond to the minimum number of nodes or edges whose break-down disrupts communication between a pair of nodes. For a node $v$ in a graph, the degree is the number of nodes which are adjacent to node $v$, and for a node $v$ in a digraph, the outdegree (indegree) is the number of nodes which are adjacent from (to) node $v$. Note that $E$ may contain self loops (a self loop is an edge from a node to itself), but they are not counted in the degree. The maximum degree $\Delta(G)$ (minimum degree $\delta(G)$) of a graph $G$ is the maximum (minimum) degree of the nodes in $G$. For a digraph $G$, the maximum degree $\Delta(G)$ (minimum degree $\delta(G)$) is similarly defined as the maximum (minimum) of outdegree and indegree of its nodes. These parameters satisfy the inequality

$$\kappa(G) \leq \lambda(G) \leq \delta(G) \leq \Delta(G).$$

A graph (digraph) $G$ is called to be maximally connected if $\kappa(G) = \delta(G)$ and maximally edge-connected if $\lambda(G) = \delta(G)$. Terminology not defined here can be found in [2] or [8].

One might be interested in more refined indices of reliability. Even two graphs with the same maximum edge-connectivity $\lambda$, may be considered to have different reliabilities, since the number of minimum edge-cut sets is different. Let us consider the probability $P(G, q)$ that the graph $G$ is disconnected when the edge failures are statistically independent with equal probability $q$ and the nodes are reliable. Hereafter, only edge failures will be considered, except for the case especially denoted. Denote by $m_k$ the number of edge-cut sets of order $k$. The probability that the network fails, $P(G, q)$, of a $q$-edge graph $G$ with edge failure probability $q$ is

$$P(G, q) = \sum_{k=1}^{a} m_k q^k (1-q)^{q-k}.$$

The problem of constructing an $n$-node, $q$-edge graph $G$ with minimum $P(G, q)$ has been considered in [1, 5, 16]. The key idea behind that work is the intuitive notion that when $q$ is small the term $m_\lambda q^\lambda (1-q)^{q-\lambda}$ dominates; thus one wishes to find graphs that “maximize $\lambda$ and minimize $m_\lambda$”.

As a more refined index than edge-connectivity, super edge-connectivity is proposed in [1, 5]. Let $G = (V, E)$ be a maximally edge-connected graph (digraph), i.e., $\lambda = \delta$. Then, any set of edges incident at (from or to) a node of degree (outdegree or indegree) $\delta$ is certainly a minimum edge-cut set of size $\lambda$. In this context, such edge sets are called trivial. Note that the deletion of any trivial edge set in a graph isolates a node of degree $\delta$. Therefore, it is defined that a graph (digraph) $G$ is super-$\lambda$ if every minimum edge-cut set is trivial. If $G$ is super-$\lambda$, then $\lambda = \delta$, but, as is easily seen, the converse is not true. Let $m_\lambda$ denote the number of edge-cut sets of size $\lambda$ and $n_\delta$ denote the number of nodes with minimum degree; then $m_\lambda \geq n_\delta$. A super-$\lambda$ graph attains $m_\lambda = n_\delta$. A precise justification for the design problem of super-$\lambda$ graphs is given in [5].
1.2. Maximum transmission delay index

The maximum transmission delay can be measured by the diameter $D$ of the graph. The diameter $D$ is the maximum of the distance over all pairs of nodes, where the distance between two nodes $u$ and $v$ in a graph $G$ (from node $u$ to $v$ in a digraph $G$), denoted by $\text{dis}(u,v)$, is the length of a shortest path between $u$ and $v$ (from $u$ to $v$). The minimum diameter graph (digraph) problem is to find a graph (digraph) $G$ whose diameter $D$ is minimum for the given order (the number of nodes) $n$ and maximum degree $\Delta$. Conversely, this problem can be regarded as finding a graph (or digraph) with a maximum order $n$ for the given $\Delta$ and $D$, which is known as the $(\Delta, D)$ graph (digraph) problem. A theoretical upper bound on the order $n$ in the $(\Delta, D)$ graph problem is given by Moore (see [8]) as

\[
n \leq \frac{\Delta(\Delta-1)^{\frac{D}{2}} - 2}{\Delta - 2} \quad \text{for} \quad \Delta > 2.\]

Similarly, an upper bound on $n$ for the digraph case is

\[
n \leq \frac{\Delta^{D+1} - 1}{\Delta - 1} \quad \text{for} \quad \Delta > 1.\]

These bounds are called the Moore bounds. The graphs (digraphs) satisfying the equality are called the Moore graphs (digraphs). The Moore graphs can exist only if $D = 1$ or $D = 2$ and $\Delta = 3, 7,$ or possibly for 57, while the Moore digraphs can exist only if $D = 1$ [9]. A nearly optimum solution of the $(\Delta, D)$ graph (digraph) problem is often called a dense graph (digraph). Many dense graphs have been reported recently (see for example [3, 10]), while the de Bruijn digraph $B(d, D)$ and the Kautz digraph $K(d, D)$ have been proposed as dense digraphs in [11, 14, 21].

From the Moore bound, the lower bound on the diameter $D$ for the minimum diameter digraph problem is derived as

\[
D \geq \lceil \log_\Delta (n(\Delta - 1) + 1) \rceil - 1, \tag{1}
\]

where $\Delta > 1$ and $\lceil x \rceil$ denotes the smallest integer not less than $x$ [18]. The minimum diameter digraph problem has also been discussed in [18, 19, 22], where the generalized de Bruijn digraph $G_{\delta}(n,d)$ and the generalized Kautz digraph $G_{\ell}(n,d)$ are proposed as digraphs with quasiminimal diameters (at most one larger than the lower bound) for any order $n$ and maximum degree $d$.

1.3. Relations between reliability indices and diameter

Until recently, diameter and reliability indices such as connectivity have been treated independently, and little work has been carried out on the reliability of these dense or quasiminimal diameter graphs (digraphs). Imase, Soneoka and Okada [20] clarified the relation between the connectivity and the diameter of a digraph and
presented the following sufficient conditions for a digraph to be maximally edge-connected or maximally connected.

\[ \lambda = \delta, \text{ if } n > (\delta - 1) \left( \frac{A^{D-1} - 1}{A - 1} + A + 1 \right), \]

\[ \kappa = \delta, \text{ if } n > (\delta - 1) \left( \frac{A^{D-1}}{A - 1} + A \right). \]

Related work on undirected graphs has been done in [12, 26]. In [26], the following conditions have been derived.

\[ \lambda = \delta, \text{ if } n > (\delta - 1) \left( \frac{(A - 1)^{D-1} + A - 3}{A - 2} + A - 1 \right), \]

\[ \kappa = \delta, \text{ if } n > (\delta - 1)(A - 1)^{D-1} + 2 \text{ and } \delta > 2. \]

These bounds are useful for proving most of the densest known graphs (listed in [3, 10]) to be maximally connected.

To prove some of the remaining graphs to be maximally connected, another type of sufficient condition has been derived for undirected graphs [26], by introducing another basic graph parameter, girth \( g \) (the length of the shortest cycle). The similar sufficient condition for digraphs to be maximally connected has been derived by Fabrega and Fiol [13], in which they have also derived the girth type sufficient conditions for digraphs and graphs to be super-\( \lambda \).

In this direction, this paper establishes the relation between the diameter and super-\( \lambda \): enlarging the order \( n \) under the given maximum degree \( A \) and diameter \( D \) not only maximizes edge-connectivity, but also minimizes the number of minimum edge-cut sets, that is, attaining super-\( \lambda \). The following sufficient conditions for digraph and graph \( G \) to be super-\( \lambda \) are derived.

Digraph \( G \) is super-\( \lambda \), if \( n > \delta \left( \frac{A^{D-1} - 1}{A - 1} + 1 \right) + A^{D-1}, \)

graph \( G \) is super-\( \lambda \), if \( n > \delta \left( \frac{(A - 1)^{D-1} - 1}{A - 2} + 1 \right) + (A - 1)^{D-1}. \)

We show that these bounds are the best possible, at least for diameter \( D = 2, 3 \) digraphs, and for \( D = 2, 3, 4 \) and 6 undirected graphs. From these sufficient conditions, the de Bruijn digraph \( B(d, D) \) and the Kautz digraph \( K(d, D) \), and most of the densest known graphs (listed in [3, 10]) are proved to be super-\( \lambda \). Also, the maximally connected \( d \)-regular digraphs with quasiminimal diameter proposed in [25] are proved to be super-\( \lambda \) for any \( d > 2 \) and any order \( n > d^3 \).
2. Sufficient conditions for super-\(\lambda\) digraphs and graphs

2.1. Digraph case

**Theorem 2.1.** Let \(G\) be a digraph of minimum degree \(\delta\), maximum degree \(\Delta\), order \(n\) and diameter \(D\), then

\[ G \text{ is super-}\(\lambda\), if } n > \delta \left( \frac{\Delta^{D-1} - 1}{\Delta - 1} + 1 \right) + \Delta^{D-1}. \]

**Proof.** For a digraph \(G = (V, E)\), let \(T \subset E\) be an arbitrary minimum edge-cut set of \(G\) (\(|T| = \lambda\)). The node set \(V\) can be partitioned into two disjoint nonempty sets \(Y\) and \(Y'\) such that \(G - T\) contains no edge from \(Y\) to \(Y'\) and every edge of \(T\) has initial node in \(Y\) and terminal node in \(Y'\). Let \(Y_0 (Y_0')\) be the set of the initial (terminal) nodes of the edges of \(T\). Let \(K = \max_{y \in Y} \text{dis}(y, Y_0), K' = \max_{y' \in Y'} \text{dis}(Y_0', y')\), \(Y_i = \{ y \in Y \mid \text{dis}(y, Y_0) = i \} (1 \leq i \leq K)\), and \(Y'_i = \{ y' \in Y' \mid \text{dis}(Y_0', y') = i \} (1 \leq i \leq K')\), where \(\text{dis}(u, W) = \min \{ \text{dis}(u, v) \mid v \in W \}\) and \(\text{dis}(W, u) = \min \{ \text{dis}(u, v) \mid u \in W \}\). Thus, \(|Y_0| \leq |T|, |Y_i| \leq |T|, |Y_{i+1}| \leq |Y_i| (0 \leq i \leq K - 1)\), and \(|Y'_i| \leq |Y'_i| (0 \leq i \leq K' - 1)\).

**Case 1:** \(1 \leq K \leq D - 2\) (and therefore \(D \geq 3\)).

\[
|T| = \sum_{i=0}^{K} |Y_i| + \sum_{i=0}^{K'} |Y'_i| \\
\leq |Y_0| \sum_{i=0}^{K} A^i + |Y_0'| \sum_{i=0}^{K'} A^i \\
\leq |T| \frac{A^{K+1} + A^{K'+1} - 2}{\Delta - 1} \\
\leq |T| \frac{A^{D-1} + A^2 - 2}{\Delta - 1} \quad (2)
\]

Since \(n > \delta (A^{D-1} - 1)/(\Delta - 1) + A^{D-1}\), we get \(|T| > \delta\); contradicting \(|T| = \lambda \leq \delta\).

**Case 2:** \(K = 0\). This indicates \(Y = Y_0\). Thus, \(1 \leq |Y| \leq |T| < \delta\). For \(y \in Y\), let \(E(y) = \{(y, y') \mid y' \in Y'\}\), and \(\deg_+(y)\) be the outdegree of \(y \in Y\).

\[
|T| = \sum_{y \in Y} |E(y)| \geq \sum_{y \in Y} (\deg_+(y) - (|Y| - 1)).
\]

Since \(\deg_+(y) \geq \delta\) and \(|T| \leq \delta\), \(\delta \geq |T| \geq |Y| (|Y| - (|Y| - 1))\). Thus, \(|Y| = 1\) or \(|Y| = \delta\). When \(|Y| = \delta\), there is a node \(v\) in \(Y_0\) which has exactly one adjacent node in \(Y'_0\). Since \(\text{dis}(v, y') \leq D\) for any \(y' \in Y'\), \(|Y'_0| \leq \Delta^{D-1}\). Thus,

\[
n \leq |Y| + \sum_{i=0}^{K-1} |Y_i| + |Y'_0| \leq |Y| + |Y_0| \sum_{i=0}^{K-1} A^i + |Y'_0| \\
\leq \delta + \delta \frac{A^{D-1} - 1}{\Delta - 1} + A^{D-1};
\]
contradicting the precondition. Therefore, there is a minimum edge-cut set $T$ only if $|Y'| = 1$.

**Case 3:** $K = D - 1$. If $K = D - 1$, then $K' = 0$ and by applying Case 2 to the reverse digraph of $G$, we may conclude that $|Y'| = 1$.

Therefore, $G$ is proved to be super-$\lambda$ under the above condition. □

**Remark 2.2.** This bound is the best possible at least for diameter 2 and 3.

Indeed, there exists a $\Delta$-regular digraph $G$ of diameter 2, edge-connectivity $\lambda = \Delta$, and order $n = 3\Delta$, which has a nontrivial minimum edge-cut set. The nodes of $G$ are partitioned into three sets, $Y_0 = \{u_i \mid 1 \leq i \leq \Delta\}$, $Y_0' = \{u_i \mid 1 < i < \Delta\}$, and $Y_1' = \{w_i \mid 1 \leq i \leq \Delta\}$. The nodes in $Y_0$ constitute a complete digraph $K_\Delta$. There are two matchings from $Y_0$ to $Y_0'$, $(u_i, u_i)$, and from $Y_1'$ to $Y_0$, $(w_i, u_i)$. A node $u_i$ in $Y_0'$ is adjacent to all the nodes $w_j (j = 1, \ldots, \Delta)$ in $Y_1'$, and $u_i$ in $Y_0'$ is adjacent from all the nodes $w_j (j \neq i)$ in $Y_1'$. It is clear that the matching from $Y_0$ to $Y_0'$ is a nontrivial minimum edge-cut set.

Similarly, there exists a $\Delta$-regular digraph $G'$ of diameter 3, edge-connectivity $\lambda = \Delta$, and order $n = 2\Delta(\Delta + 1)$, which has a nontrivial minimum edge-cut set. The nodes of $G'$ are partitioned into four sets, $Y_0 = \{u_i \mid 1 \leq i \leq \Delta\}$, $Y_0' = \{u_i \mid 1 \leq i \leq \Delta\}$, $Y_1' = \{w_{i,j} \mid 1 \leq i \leq \Delta, 1 \leq j \leq \Delta\}$, and $Y_2' = \{x_{i,j} \mid 1 \leq i \leq \Delta, 1 \leq j \leq \Delta\}$. The nodes in $Y_0$ constitute a complete digraph $K_\Delta$. There is a matching from $Y_0$ to $Y_0'$, $(u_i, u_i)$. A node $u_i$ in $Y_0'$ is adjacent to all the nodes $w_{i,j}$ $(j = 1, \ldots, \Delta)$ in $Y_1'$, a node $w_{i,j}$ in $Y_1'$ is adjacent to all the nodes $x_{j,k}$ $(k = 1, \ldots, \Delta)$ in $Y_2'$, and a node $x_{i,j}$ in $Y_2'$ is adjacent to $u_i$ in $Y_0$ and all the $u_k$ $(k \neq i)$ in $Y_0'$ if $j = 1$, otherwise $x_{i,j}$ is adjacent to all the $w_{k,j}$ $(k = 1, \ldots, \Delta)$. It is clear that the matching from $Y_0$ to $Y_0'$ is a nontrivial minimum edge-cut set (see Fig. 1).

![Fig. 1. $\Delta$-regular digraph of $D = 3$, $n = 2\Delta(\Delta + 1)$ with a nontrivial minimum edge-cut ($\Delta = 3$).](image-url)

From this theorem, it can be directly proved that the de Bruijn digraph $B(d, D)$ and the Kautz digraph $K(d, D)$ are super-$\lambda$ when $d \geq 3$ and $D \geq 2$. (The result for the Kautz digraph $K(d, D)$ has been known in [13].) The de Bruijn digraph $B(d, D)$ with maximum degree $\Delta = d$ and diameter $D$ is the digraph whose nodes are labeled with words of length $D$ on an alphabet of $d$ letters [11, 14]. There is an edge from $(x_1, x_2, \ldots, x_D)$ to all the vertices $(x_2, \ldots, x_D, \alpha)$, where $\alpha$ is any letter of the alphabet. This digraph has the minimum degree $\delta = d - 1$ and the order $n = d^D$. Since $n = d^D > d^{D-1} + d - 2 + d^{D-1} = 2d^{D-1} + d - 2$ for $d \geq 3$ and $D \geq 2$, $B(d, D)$ is super-$\lambda$. Similarly, the Kautz digraph $K(d, D)$ with maximum degree $\Delta = d$ and diameter $D$ is the digraph whose nodes are labeled with words of length $D$ on an alphabet of $d + 1$ letters, such that two consecutive letters are different [14, 21]. There is an edge from $(x_1, x_2, \ldots, x_D)$ to all the nodes $(x_2, \ldots, x_D, \alpha)$ where $\alpha$ is any letter of the alphabet different from $x_D$. This digraph has the minimum degree $\delta = d$ and the order $n = d^D + d^{D-1}$. Since $n = d^D + d^{D-1} > d((d^{D-1} - 1)/(d - 1) + 1) + d^{D-1}$ for $d \geq 3$ and $D \geq 2$, $K(d, D)$ is super-$\lambda$.

When $D = 1$, $B(d, D)$ and $K(d, D)$ are obviously super-$\lambda$ from their constructions. $B(d, 1)$ and $K(d, 1)$ respectively correspond to the complete digraph $K_d$ with each node having a self loop and the complete digraph $K_{d+1}$. In the case of $d = 2$, $B(2, D)$ can be shown to be super-$\lambda$ in a similar way to the proof of Theorem 2.1, because $|Y| = \delta = d - 1 = 1$. However, $K(2, D)$ is not super-$\lambda$, because it has $K_2$ as a subgraph.

2.2. Undirected graph case

In the same manner as the proof of the digraph case, the following theorem can be proved.

**Theorem 2.3.** Let $G$ be an undirected graph of minimum degree $\delta$, maximum degree $\Delta$, order $n$, and diameter $D$, then

$$G \text{ is super-$\lambda$, if } n > (A - 1)^{D-1} - 1 + (\Delta - 1)^{D-1}. $$

**Remark 2.4.** This bound is the best possible at least for diameter 2, 3, 4, and 6.

Indeed, there exists a regular graph $G$ of diameter $D = 2$ (3, 4, or 6), degree $\Delta$, edge-connectivity $\lambda = \Delta$, and order

$$n = \Delta \left( \frac{(\Delta - 1)^{D-1} - 1}{\Delta - 2} + 1 \right) + (\Delta - 1)^{D-1}$$

$$= 2 \left( \frac{(\Delta - 1)^{D-1}}{\Delta - 2} + \Delta - 1 \right)$$

$$- 2 \sum_{i=0}^{D-1} q^i + q \quad (q = \Delta - 1),$$
which has a nontrivial minimum edge-cut set. This graph \( G \) is constructed from the \((q + l)\)-regular graph \( B_q(P_q, Q_q, H_q) \) by replacing a node with a complete graph \( K_q+1 \). Here, the graph \( B_q \) is a complete bipartite graph \( K_q+1, q+1 \), and the graph \( P_q(Q_q, H_q) \) is called a Moore graph of minimum degree \( q+1 \) and girth 6 (8, 12), which has diameter 3 (4, 6) and order \( n = 2 \sum_{i=0}^{D-1} q^i \) [8]. It is clear that the graph \( G \) has a nontrivial minimum edge-cut set disconnecting the \( K_q+1 \) from the others.

For each value of the maximum degree and diameter, the entries in Table 1 are the order in the densest known graphs [3, 10]. Theorem 2.3 shows that the densest known graphs corresponding to the double-starred entries in Table 1 are super-\( \lambda \), while the starred entries represent the densest known graphs shown to be maximally edge-connected by the sufficient condition derived in [26].

3. Super-\( \lambda \) digraphs with quasiminimal diameter

This section shows that the digraph \( G^{m}_{D}(n, d) \), proposed as a maximally connected \( d \)-regular digraph with a quasiminimal diameter in [25], is also super-\( \lambda \). The

Table 1. Densest known graphs (* stands for the maximally edge-connected dense graphs; ** stands for the super-\( \lambda \) dense graphs; the number in ( ) represents the Moore bound)

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digraph $G^*_B(n,d)$ is constructed from the generalized de Bruijn digraph $G_B(n,d) = (V,E)$ ($n \geq d$) [18, 22], which is defined as:

$$V = \{0, 1, \ldots, n-1\},$$
$$E = \{(u,v) \mid v = u \cdot d + a \mod n, \ a = 0, 1, \ldots, d-1\}. \tag{3}$$

It has been shown in [22] that "$d + \gcd(n,d - 1) - 1$ nodes of $G_B(n,d)$ have a self loop, where $\gcd(p,q)$ is the greatest common divisor of $p$ and $q$". Thus, the connectivity and edge-connectivity of $G_B(n,d)$ are not larger than $d - 1$. The maximally connected $d$-regular digraph $G^*_B(n,d)$ is constructed from $G_B(n,d)$ by removing all the self loops and adding a cycle that will connect the nodes originally with a self loop. Figure 2 shows a digraph $G^*_B(12,3)$ constructed from $G_B(12,3)$.

The diameter $D$, connectivity $\kappa$, and edge-connectivity $\lambda$ of $G^*_B(n,d)$ have been shown in [24, 25].

**Property 3.1.** $D(G^*_B(n,d)) \leq \lceil \log_d n \rceil$. Namely, $d^{D(G^*_B)} - 1 < n$.

From (1), this means that $D(G^*_B(n,d))$ is quasiminimal.

**Property 3.2.** $G^*_B(n,d)$ is a $d$-regular digraph satisfying that

$$\lambda(G^*_B(n,d)) = d \text{ for any } n \text{ and } d \geq 3,$$
$$\kappa(G^*_B(n,d)) = d \text{ for any } n > d^3 \text{ and } d \geq 3.$$

The following theorem will be proved.

**Theorem 3.3.** $G^*_B(n,d)$ is super-$\lambda$ if $n > d^3$ and $d \geq 3$.

The following notation and properties of $G_B(n,d)$ will be needed in the proof of Theorem 3.3.

![Fig. 2. $G_B(12,3)$ and $G^*_B(12,3)$](image_url)
Definition 3.4. For a digraph $G = (V, E)$ and a node subset $V' \subseteq V$,

- $S(V') \overset{\text{def}}{=} \{ u \mid v \in V' \text{ and } (u, v) \in E \}$,
- $P(V') \overset{\text{def}}{=} \{ u \mid v \in V' \text{ and } (v, u) \in E \}$,
- $S'(V') \overset{\text{def}}{=} S(S' - 1(V'))$, and
- $P'(V') \overset{\text{def}}{=} P(P' - 1(V'))$.

where $S^0(V') = V'$ and $P^0(V') = V'$.

In other words, $S'(V')$ is the set of nodes to which there is a $t$-length walk from some node $v$ in $V'$, and $P'(V')$ is the set of nodes from which there is a $t$-length walk to some node $v$ in $V'$.

In the digraph $G_B(12, 3)$ shown in Fig. 2,

- $S(0) = \{0, 1, 2\}$, $P(0) = \{0, 4, 8\}$,
- $S'(0) = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, and $P'(0) = \{0, 1, 2, 4, 5, 6, 8, 9, 10\}$.

It is not so hard to see the following properties of $G_B(n, d)$ and the proofs can be found in [20].

**Property 3.5.** For any node $v$ in $G_B(n, d) = (V, E)$, if $1 < D(G_B)$,

$|S'(v)| = |P'(v)| = d'$.

**Property 3.6.** Let $v$ be a node in $G_B(n, d)$. If $V' \subseteq S'^{-1}(v)$ and $1 \leq t < D(G_B)$, then $|S(V')| = d \cdot |V'|$, and if $V' \subseteq P'^{-1}(v)$ and $1 \leq t < D(G_B)$, then $|P(V')| = d \cdot |V'|$.

**Proof of Theorem 3.3.** Let $G_B^{**}$ be a digraph obtained from $G_B^*(n, d)$ by returning all the removed self loops (retaining the added cycle as well). To prove that $G_B^{**}(n, d)$ is super-$\lambda$, it is enough to prove that $G_B^{**}$ has only trivial minimum edge-cut sets, because self loops do not contribute to the minimum edge-cut set. Remark that $G_B^{**}$ also remains $d$-regular (self loops are not counted in the degree).

For $G_B^{**} = (V, E)$, let $T \subseteq E$ be an arbitrary minimum edge-cut set of $G_B^{**}$. From Property 3.2, $|T| = d$. $V$ can be partitioned into two disjoint nonempty sets $Y$ and $Y'$ such that $G_B^{**} - T$ contains no edges from $Y$ to $Y'$ and every edge of $T$ has initial node in $Y$ and terminal node in $Y'$. Let $Y_0(Y_0)$ be the set of the initial (terminal) nodes of the edges of $T$. Let $K = \max_{y \in Y} \text{dis}(y, Y_0)$, $K' = \max_{y' \in Y'} \text{dis}(Y_0, y')$, $Y_i = \{ y \in Y \mid \text{dis}(y, Y_0) = i \} (1 \leq i \leq K)$, and $Y'_i = \{ y' \in Y' \mid \text{dis}(Y_0, y') = i \} (1 \leq i \leq K')$. Thus, $|Y_0| \leq |T|$, $|Y_0| \leq |T|$, $|Y_i| \leq d$, $|Y'_i| \leq d$, $|Y_i| (1 \leq i \leq K - 1)$, and $|Y'_i| (1 \leq i \leq K' - 1)$.

Denote $D(G_B^{**})$ as $D$, and remark that $K + K' + 1 \leq D$. Since $G_B^{**}$ is $d$-regular, the number of edges from $Y$ to $Y'$ is equal to that from $Y'$ to $Y$. Thus, $K \leq K'$ can be supposed without loss of generality.

**Case 1:** $K = 0$. This case indicates $Y = Y_0$. In a similar way to the proof of Case 2 of Theorem 2.1, we can get $|Y| = 1$ or $|Y| = d = d$. Assume that $|Y| = d$. Since $G_B^*(n, d)$ is $d$-regular and $|T| = d$, the subgraph induced by the node set $Y$ is a
complete digraph of order $d$, $K_d$. Since $G^*_B(n, d)$ is constructed from $G_B(n, d)$ by removing all self loops and adding a cycle that will connect the nodes originally with a self loop, it is easy to see that $G_B(n, d)$ has either a complete digraph $K_3$ or an edge $(i, j)$ whose initial node $i$ and terminal node $j$ have self loops. In the first case, for the three nodes in $K_3$, $u$, $v$, and $w$, there exist two walks of length 2 from $u$ to itself, $u, v, u$ and $u, w, u$. In the second case, there exist two walks of length 2 from $i$ to $j$, that is, $i, i, j$ and $i, j, j$. Either of these cases contradicts the property that “the terminal node of every $t$-length walk ($t < D(G_B)$) from a node is distinct”, which can be easily derived from Property 3.5, for $D(G_B) \geq 4$, that is, for $n > d^3$.

Case 2: $K = 1$. Let $y$ be a node of $Y$ such that $\text{dis}(y, Y_0) = 1$. Since $S(y) \subseteq Y$ and $|S(y)| = d$ in the subgraph $G_B$ of $G^*_B$, $|S(y) \cap Y| \geq d$ in $G^*_B$. For any $V' \subseteq Y$, since $S(V') \subseteq Y \cup Y'$ and $|Y' \cap Y'| \leq |T| = d$,

$$|S(V') \cap Y'| = |S(V')| - |S(V') \cap Y'| \geq |S(V')| - |Y'| \geq |S(V')| - d.$$

From these, $D \geq 4$, and Property 3.6,

$$|S^2(y) \cap Y| \geq |S(S(y) \cap Y) \cap Y| \geq |S(S(y) \cap Y)| - d \geq d^2 - d.$$

$$|Y| \geq |S^3(y) \cap Y| \geq |S(S^2(y) \cap Y) \cap Y| \geq |S(S^2(y) \cap Y)| - d \geq d(d^2 - d) - d = d^3 - d^2 - d.$$

On the other hand,

$$|Y| = |Y_0| + |Y_1| \leq |T| + |T| d = d^2 + d.$$

Since $d \geq 3$, these are a contradiction.

Case 3: $K \geq 2$ (and therefore $D \geq 5$). From (2) in the proof of Theorem 2.1, $K + K' + 1 \leq D$, and $|T| = \delta = \delta = d$, we can get

$$n \leq |T| \frac{\Delta^k + 1 + \Delta^{k'} - 2}{\Delta - 1} \leq d^D - 2 \frac{d^{D-2} + d^3 - 2}{d - 1}.$$

Since $d \geq 3$ and $D \geq 5$, this contradicts $n > d^{D-1}$. \qed

4. Conclusions

This paper considers the relation between super-$\lambda$ and diameter, and shows that enlarging the order $n$ under the given maximum degree $\Delta$ and diameter $D$ not only maximizes edge-connectivity, but also minimizes the number of minimum edge-cut sets, that is, attaining super-$\lambda$. Sufficient conditions for digraphs and graphs to be super-$\lambda$ are derived, which are shown to be the best possible. Also, the digraph $G^*_B(n, d)$, proposed in [25] as a maximally connected $d$-regular digraph with a quasiminimal diameter, is proved to be super-$\lambda$ for any $d > 2$ and any order $n > d^3$. 
Establishing the relation between diameter and analogous vulnerability indices for node failure remains for further study. Analogously, a graph $G$ is defined to be super-$\kappa$ if every minimum node-cut set is trivial (the set of adjacent nodes of a node of degree $\delta$), that is, isolating a node of degree $\delta$ [5]. For digraphs, a similar definition can be considered. This definition allows the minimum node-cut set to create many isolated nodes, for example $K_{n,n}$. A more restrictive definition can be considered; $G$ is hyper-$\kappa$ if every minimum node-cut set creates exactly two components, one of which is an isolated node of degree $\delta$. However, we remark that one can generate examples which show that neither super-$\kappa$ nor hyper-$\kappa$ implies minimizing the total number of distinct minimum node-cut sets, and that neither converse is valid [5, 6, 15, 17, 23].

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References