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Local embeddability of real analytic path geometries

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ABSTRACT

An almost complex structure $\mathfrak{J}$ on a 4-manifold X may be described in terms of a rank 2 vector bundle $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$. We call a pair of line subbundles L_1, L_2 of $\Lambda^2 TX^*$ a splitting of $\mathfrak{J}$ if $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$. A hypersurface $M \subset X$ satisfying a nondegeneracy condition inherits a CR-structure from $\mathfrak{J}$ and a path geometry from the splitting (L_1, L_2). Using the Cartan–Kähler theorem we show that locally every real analytic path geometry is induced by an embedding into $\mathbb{C}^2$ equipped with the splitting generated by the real and imaginary part of $dz^1 \wedge dz^2$. As a corollary we obtain the well-known fact that every 3-dimensional nondegenerate real analytic CR-structure is locally induced by an embedding into $\mathbb{C}^2$.

1. Introduction

Motivated by the well-known fact (see for instance [6]) that an almost complex structure $\mathfrak{J}$ on a 4-manifold X admits a description in terms of a rank 2 vector bundle $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$, we introduce the notion of a splitting of an almost complex structure: A pair of line subbundles L_1, L_2 of $\Lambda^2 TX^*$ is called a *splitting* of $\mathfrak{J}$ if $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$. A hypersurface $M \subset X$ satisfying a nondegeneracy condition inherits a CR-structure from $\mathfrak{J}$ and a path geometry from the splitting (L_1, L_2). The purpose of this Note is to show that locally every real analytic path geometry is induced by an embedding into $\mathbb{R}^4 \simeq \mathbb{C}^2$ equipped with the splitting generated by the real and imaginary part of $dz^1 \wedge dz^2$. This will be done using the Cartan–Kähler theorem. As a corollary we obtain the well-known fact that every 3-dimensional nondegenerate real analytic CR-structure is locally induced by an embedding into $\mathbb{C}^2$. It follows with Nirenberg's example of a smooth non-embeddable 3-dimensional CR-manifold that the real analyticity in our main statement is necessary.

The notation and terminology for the Cartan–Kähler theorem and exterior differential systems are chosen to be consistent with [4,7]. Moreover we adhere to the convention of summing over repeated indices.

2. Preliminaries

2.1. Pairs of 2-forms

Throughout this section, let V denote an oriented 4-dimensional real vector space. Fix a volume form $\varepsilon \in \Lambda^4 V^*$ which induces the given orientation. Given two 2-forms $\omega, \phi \in \Lambda^2 V^*$, we may write $\omega \wedge \phi = \langle \omega, \phi \rangle \varepsilon$ for some unique real

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number $\langle\omega,\phi\rangle$. Clearly the map $(\omega,\phi)\mapsto\langle\omega,\phi\rangle$ defines a symmetric bilinear form on the 6-dimensional real vector space Λ^2V^* which is easily seen to be nondegenerate and of signature $(3,3)$. Replacing ε with another orientation compatible volume form gives a bilinear form which is a positive multiple of $\langle\cdot,\cdot\rangle$. Consequently, the wedge product may be thought of as a conformal structure of split signature on Λ^2V^*.

Definition. A pair of 2-forms $\omega,\phi\in\Lambda^2V^*$ is called *elliptic* if

$$\langle\omega,\omega\rangle\langle\phi,\phi\rangle>\langle\omega,\phi\rangle^2.$$

It is a natural problem to classify the pairs of elliptic 2-forms on V. This is a special case of a more general problem: Let $\omega\in\Lambda^2V^*$ be a symplectic 2-form whose stabilizer subgroup will be denoted by $\mathrm{Sp}(\omega)\subset\mathrm{GL}(V)$. The natural representation of $\mathrm{Sp}(\omega)$ on Λ^2V^* decomposes as $\Lambda^2V^*=\{\omega\}\oplus\omega^\perp$ where both summands are irreducible $\mathrm{Sp}(\omega)$-modules.[2] Here $\omega^\perp$ is the 5-dimensional linear subspace of Λ^2V^* consisting of 2-forms orthogonal to ω. One can ask to classify the orbits of $\mathrm{Sp}(\omega)$ on $\omega^\perp$. This has been carried out in [8]. In the elliptic case one obtains:

Lemma 1. *(See [8].) Let $\omega,\phi\in\Lambda^2V^*$ be a pair of elliptic orthogonal 2-forms, then there exists a positive real number κ and a basis e^i of V^* such that*

$$\omega=e^1\wedge e^3-e^2\wedge e^4,\qquad\phi=\kappa\big(e^1\wedge e^4+e^2\wedge e^3\big).$$

The constant κ is an $\mathrm{Sp}(\omega)$-invariant and thus parametrizes the set of elliptic $\mathrm{Sp}(\omega)$-orbits. Ellipticity will be useful because of the following:

Lemma 2. *Let W be 3-dimensional real vector space. Then the pullback of an elliptic pair of 2-forms $\omega,\phi\in\Lambda^2V^*$ with any injective linear map $A:W\to V$ gives two linearly independent 2-forms on W.*

Proof. The ellipticity condition is equivalent to every nonzero linear combination of (ω,ϕ) being symplectic. Suppose (ω,ϕ) is an elliptic pair of 2-forms. Then for every choice of real numbers $(\lambda_1,\lambda_2)\neq0$, the 2-form $\tau=\lambda_1\omega+\lambda_2\phi$ is symplectic. Since there are no isotropic subspaces of dimension greater than 2 in the symplectic vector space (V,τ), it follows that $A^*\tau=\lambda_1A^*\omega+\lambda_2A^*\phi\neq0$ for every linear injective map $A:W\to V$. $\square$

2.2. Splittings of complex structures

Let $\mathcal{C}^+(V)$ denote space of complex structures on V which are compatible with the orientation, i.e. its points $J\in\mathrm{End}(V)$ satisfy $\varepsilon(v_1,Jv_1,v_2,Jv_2)\geqslant0$ for all vectors $v_1,v_2\in V$. Moreover let $G_2^+(\Lambda^2V^*,\wedge_+)$ denote the submanifold of the Grassmannian of oriented 2-planes in Λ^2V^* to whose elements the wedge product restricts to be positive definite. Given a $(2,0)$-form $\alpha\in\Lambda^{2,0}V^*$ with respect to some $J\in\mathcal{C}^+(V)$, let $\Lambda_J\in G_2^+(\Lambda^2V^*,\wedge_+)$ denote the 2-dimensional linear subspace spanned by $\mathrm{Re}(\alpha)$, $\mathrm{Im}(\alpha)$ and orient Λ_J by declaring $\mathrm{Re}(\alpha),\mathrm{Im}(\alpha)$ to be positively oriented. Clearly Λ_J and its orientation are independent of the chosen $(2,0)$-form α and one thus obtains a map $\psi:\mathcal{C}^+(V)\to G_2^+(\Lambda^2V^*,\wedge_+)$ given by $J\mapsto\Lambda_J$. Note that $G=\mathrm{GL}^+(V)$ acts smoothly and transitively from the left on $\mathcal{C}^+(V)$ via $(A,J)\mapsto A^{-1}JA$. Every element of $G_2^+(\Lambda^2V^*,\wedge_+)$ admits a positively oriented elliptic conformal basis. It follows with Lemma 1 that via pushforward, $\mathrm{GL}^+(V)$ acts smoothly and transitively from the left on $G_2^+(\Lambda^2V^*,\wedge_+)$ as well.

Proposition 1. *The map $\psi:\mathcal{C}^+(V)\to G_2^+(\Lambda^2V^*,\wedge_+)$, $J\mapsto\Lambda_J$ is a G-equivariant diffeomorphism.*

Proof. Clearly the map ψ is G-equivariant. To prove that ψ is a diffeomorphism it is sufficient to show that $G_J=G_{\psi(J)}$ for all $J\in\mathcal{C}^+(V)$ where G_J and $G_{\psi(J)}$ denote the stabilizer subgroups of G with respect to J and $\psi(J)$ respectively. Choose $J\in\mathcal{C}^+(V)$, then we have $G_J\subset G_{\psi(J)}$. Write

$$J(v)=-e^2(v)e_1+e^1(v)e_2-e^4(v)e_3+e^3(v)e_4$$

for some basis (e_i) of V and dual basis (e^i) of V^*. Then

$$\omega=e^1\wedge e^3-e^2\wedge e^4=\frac{1}{2}w_{kl}e^k\wedge e^l,\qquad\phi=e^1\wedge e^4+e^2\wedge e^3=\frac{1}{2}f_{kl}e^k\wedge e^l$$

is a positively oriented conformal basis of Λ_J. Consequently every $A\in G_{\psi(J)}$ satisfies $A^*\omega=x\omega+y\phi$ and $A^*\phi=-y\omega+x\phi$, for some real numbers $(x,y)\neq0$. The matrix representation a of A with respect to the basis (e_i) thus satisfies

[2] We denote by $\{\cdot\}$ the linear span of the elements within. In the case of smooth differential forms, the coefficients are smooth real-valued functions.

$$a^t wa = xw + yf, \qquad a^t fa = -yw + xf.$$

From this one easily concludes $awf = wfa$ which is equivalent to A commuting with J. □

Proposition 1 motivates the following:

Definition. A *splitting* of a complex structure J on V is a pair of lines $L_1, L_2 \in \mathbb{P}(\Lambda^2 V^*)$ such that $\Lambda_J = L_1 \oplus L_2$.

Call two 4-dimensional real vector spaces V, V' equipped with complex structures J, J' and splittings (L_1, L_2), (L_1', L_2') equivalent, if there exists a complex linear map $A : V \to V'$ such that $A^*(L_i') = L_i$ for $i = 1, 2$.

On $V = \mathbb{R}^4$ let $\omega_0 = e^1 \wedge e^3 - e^2 \wedge e^4$ and $\phi_0 = e^1 \wedge e^4 + e^2 \wedge e^3$ where $e^1, \dots, e^4$ denotes the standard basis of $(\mathbb{R}^4)^*$. Define $L_1 = \{\omega_0\}$ and $L_2 = \{\alpha\omega_0 + \phi_0\}$ for some nonnegative real number α. Orient $L_1 \oplus L_2$ by declaring ω_0, ϕ_0 to be a positively oriented basis and let J_0 be the associated complex structure. Then $S_\alpha = (L_1, L_2)$ is a splitting of J_0.

Proposition 2. *Every pair (V, J) equipped with a splitting (L_1, L_2) is equivalent to $(\mathbb{R}^4, J_0)$ equipped with the splitting S_α for some unique $\alpha \in \mathbb{R}_0^+$.*

Proof. Let $L_1 = \{\omega\}$ and $L_2 = \{\omega'\}$ for some 2-forms $\omega, \omega' \in \Lambda^2 V^*$. Since the wedge product restricts to be positive definite on $L_1 \oplus L_2$ we have $\omega \wedge \omega > 0$ and there exists a real number α, such that $\omega' = \alpha\omega + \phi$ for some 2-form ϕ satisfying $\omega \wedge \phi = 0$ and $\phi \wedge \phi > 0$. After possibly rescaling ω' we can assume that $\phi \wedge \phi = \omega \wedge \omega$ and that α is nonnegative. It follows with Lemma 1 that there exists a linear map $A : V \to \mathbb{R}^4$ which identifies ω with ω_0 and ϕ with ϕ_0, in particular A is complex linear. To prove uniqueness of α suppose $A : \mathbb{R}^4 \to \mathbb{R}^4$ satisfies $A^*\omega_0 = x\omega_0$ and $A^*(\alpha\omega_0 + \phi_0) = y(\beta\omega_0 + \phi_0)$ for some real numbers $x, y \neq 0$ and some nonnegative real numbers α, β. Then $A^*(\omega_0 \wedge \omega_0) = x^2\omega_0 \wedge \omega_0$ and consequently

$$A^*\big(\omega_0 \wedge (\alpha\omega_0 + \phi_0)\big) = \alpha x^2 \omega_0 \wedge \omega_0 = xy\beta\omega_0 \wedge \omega_0,$$

which is equivalent to $\alpha x = \beta y$. We also have

$$A^*\big((\alpha\omega_0 + \phi_0) \wedge (\alpha\omega_0 + \phi_0)\big) = x^2\big(\alpha^2 + 1\big)\omega_0 \wedge \omega_0 = y^2\big(\beta^2 + 1\big)\omega_0 \wedge \omega_0,$$

which implies $x^2 = y^2$ and thus $\alpha^2 = \beta^2$. Since $\alpha, \beta \geqslant 0$, the claim follows. □

For a splitting (L_1, L_2), the unique nonnegative real number α provided by Proposition 2 will be called the *degree* of the splitting. A splitting of degree 0 will be called *orthogonal*.

3. Local embeddability of real analytic path geometries

3.1. Splittings of almost complex structures

Let X be a smooth 4-manifold and $\mathfrak{J}$ be an almost complex structure with associated rank 2 vector bundle $\Lambda_{\mathfrak{J}} \subset \Lambda^2 TX^*$ whose fibre at $p \in X$ is the linear subspace $\Lambda_{\mathfrak{J}_p} \subset \Lambda^2 T_pX^*$ associated to $\mathfrak{J}_p : T_pX \to T_pX$. A *splitting* of $\mathfrak{J}$ consists of a pair of smooth line bundles $L_1, L_2 \subset \Lambda^2 TX^*$ so that $\Lambda_{\mathfrak{J}} = L_1 \oplus L_2$.

3.2. Induced structure on hypersurfaces

A *CR-structure* on a 3-manifold M consists of a rank 2 subbundle $D \subset TM$ and a vector bundle endomorphism $I : D \to D$ which satisfies $I^2 = -\mathrm{Id}_D$. A CR-structure (D, I) is called *nondegenerate* if D is nowhere integrable, i.e. a contact plane field. A closely related notion is that of a path geometry (see for instance [7] for a motivation of the following definition). A path geometry on a 3-manifold M consists of a pair of line subbundles (P_1, P_2) of TM which span a contact plane field. A CR-structure (D, I) and a path geometry (P_1, P_2) on M will be called *compatible* if $D = P_1 \oplus P_2$ and $I(P_1) = P_2$.

Let (L_1, L_2) be a splitting of the almost complex structure $\mathfrak{J}$ on X and (ω, ϕ) a pair of 2-forms defined on some open subset $\tilde{U} \subset X$ which span (L_1, L_2). Then the pair (ω, ϕ) is elliptic, i.e. (ω_p, ϕ_p) is elliptic for every point $p \in \tilde{U}$. Suppose $M \subset X$ is a hypersurface. Then Lemma 2 implies that the 2-forms (ω, ϕ) remain linearly independent when pulled back to $M \cap \tilde{U}$. This is useful because of the following:

Lemma 3. *Let β_1, β_2 be smooth linearly independent 2-forms on a 3-manifold M. Then there exists a local coframing $\eta = (\eta^1, \eta^2, \eta^3)^t$ on M such that $\beta_1 = \eta_2 \wedge \eta_1$ and $\beta_2 = \eta_2 \wedge \eta_3$.*

Recall that a (local) *coframing on* M consists of three smooth linearly independent 1-forms defined on (some proper open subset of) M.

Proof of Lemma 3. Let $x : U \to \mathbb{E}^3$ be local coordinates on M with respect to which $\beta_1|_U = b_1 \cdot \star \mathrm{d}x$ and $\beta_2|_U = b_2 \cdot \star \mathrm{d}x$ for some smooth $b_i : U \to \mathbb{R}^3$ where $\star$ denotes the Hodge-star of Euclidean space $\mathbb{E}^3$. Define $e = (b_1 \times b_2)/|b_1 \times b_2| : U \to \mathbb{R}^3$ and

$$\eta_1 = (b_1 \times e) \cdot \mathrm{d}x, \qquad \eta_2 = e \cdot \mathrm{d}x, \qquad \eta_3 = (b_2 \times e) \cdot \mathrm{d}x,$$

then (η^1, η^2, η^3) have the desired properties. □

A local coframing on M obtained via Lemma 3 and some (local) choice of 2-forms (ω, ϕ) spanning (L_1, L_2) will be called *adapted* to the structure induced by the splitting (L_1, L_2). Independent of the particular adapted local coframings are the line subbundles P_1 and P_2 of TM, locally defined by

$$P_1 = \{\eta_1, \eta_2\}^{\perp}, \qquad P_2 = \{\eta_2, \eta_3\}^{\perp}.$$

Call a hypersurface $M \subset X$ *nondegenerate* if $D = P_1 \oplus P_2$ is a contact plane field. Summarizing, we have shown:

Proposition 3. *A nondegenerate hypersurface $M \subset X$ inherits a path geometry from the splitting (L_1, L_2).*

Remark. Fixing a $(2, 0)$-form on X allows to define a coframing on a hypersurface $M \subset X$. For the construction of the coframing and its properties see [3].

3.3. Local embeddability

We conclude by using the Cartan–Kähler theorem to show that locally every real analytic path geometry is induced by an embedding into $\mathbb{C}^2$ equipped with the splitting $(\{\omega_0\}, \{\phi_0\})$. Here $\omega_0 = \mathrm{Re}(\mathrm{d}z^1 \wedge \mathrm{d}z^2)$ and $\phi_0 = \mathrm{Im}(\mathrm{d}z^1 \wedge \mathrm{d}z^2)$ where $z = (z^1, z^2)$ are standard coordinates on $\mathbb{C}^2$. Writing $z^1 = x^1 + \mathrm{i}x^2$ and $z^2 = x^3 + \mathrm{i}x^4$ for standard coordinates $x = (x^i)$ on $\mathbb{R}^4$, we have

$$\omega_0 = \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \mathrm{d}x^2 \wedge \mathrm{d}x^4, \qquad \phi_0 = \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3.$$

In [5], as an application of his method of equivalence, Cartan has shown how to associate a Cartan geometry to every path geometry.

Definition. Let G be a Lie group and $H \subset G$ a Lie subgroup with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. A *Cartan geometry of type* (G, H) on a manifold M consists of a right principal H-bundle $\pi : B \to M$ together with a 1-form $\theta \in \mathcal{A}^1(B, \mathfrak{g})$ which satisfies the following conditions:

(i) $\theta_b : T_b B \to \mathfrak{g}$ is an isomorphism for every $b \in B$,
(ii) $\theta(X_v) = v$ for every fundamental vector field X_v, $v \in \mathfrak{h}$,
(iii) $(R_h)^* \theta = \mathrm{Ad}_{\mathfrak{g}}(h^{-1}) \circ \theta$.

Here $\mathrm{Ad}_{\mathfrak{g}}$ denotes the adjoint representation of G. The 1-form θ is called the *Cartan connection* of the Cartan geometry $(\pi : B \to M, \theta)$.

Denote by $H \subset \mathrm{SL}(3, \mathbb{R})$ the Lie subgroup of upper triangular matrices. In modern language Cartan's result is as follows (for a proof see [2,7]):

Theorem 1 *(Cartan). Given a path geometry (M, P_1, P_2), then there exists a Cartan geometry $(\pi : B \to M, \theta)$ of type $(\mathrm{SL}(3, \mathbb{R}), H)$ which has the following properties: Writing*

$$\theta = \begin{pmatrix} \theta^0_0 & \theta^0_1 & \theta^0_2 \\ \theta^1_0 & \theta^1_1 & \theta^1_2 \\ \theta^2_0 & \theta^2_1 & \theta^2_2 \end{pmatrix},$$

(i) *for any section $\sigma : M \to B$, the 1-form $\phi = \sigma^* \theta$ satisfies $P_1 = \{\phi^2_1, \phi^2_0\}^{\perp}$ and $P_2 = \{\phi^1_0, \phi^2_0\}^{\perp}$. Moreover $\phi^1_0 \wedge \phi^2_0 \wedge \phi^2_1$ is a volume form on M.*
(ii) *The curvature 2-form $\Theta = \mathrm{d}\theta + \theta \wedge \theta$ satisfies*

$$\Theta = \begin{pmatrix} 0 & \mathcal{W}_1 \theta^1_0 \wedge \theta^2_0 & (\mathcal{W}_2 \theta^1_0 + \mathcal{F}_2 \theta^2_1) \wedge \theta^2_0 \\ 0 & 0 & \mathcal{F}_1 \theta^2_1 \wedge \theta^2_0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.1}$$

for some smooth functions $\mathcal{W}_1, \mathcal{W}_2, \mathcal{F}_1, \mathcal{F}_2 : B \to \mathbb{R}$.

Using this result and the Cartan–Kähler theorem we obtain local embeddability in the real analytic category:

Theorem 2. *Let (M, P_1, P_2) be a real analytic path geometry. Then for every point $p \in M$ there exists a p-neighborhood $U_p \subset M$ and a real analytic embedding $\varphi : U_p \to \mathbb{C}^2$ such that the path geometry induced by the splitting $(\{\omega_0\}, \{\phi_0\})$ is (P_1, P_2) on U_p.*

Proof. Let $(\pi : B \to M, \theta)$ denote the Cartan geometry of the path geometry (M, P_1, P_2). On $N = B \times \mathbb{R}^4$ consider the exterior differential system with independence condition $(\mathcal{I}, \zeta)$ where $\zeta = \zeta^1 \wedge \zeta^2 \wedge \zeta^3$ with $\zeta^1 = \theta^1_0$, $\zeta^2 = \theta^2_0$, $\zeta^3 = \theta^2_1$ and the differential ideal $\mathcal{I}$ is generated by the two 2-forms

$$\chi_1 = \theta^2_0 \wedge \theta^1_0 - \omega_0, \qquad \chi_2 = \theta^2_0 \wedge \theta^2_1 - \phi_0.$$

The dual vector fields to the coframing $(\theta^i_k, \mathrm{d}x^l)$ of N will be denoted by (T^i_k, ∂_{x^l}). Let $G_k(TN) \to N$ be the Grassmann bundle of k-planes on N and $G_3(TN, \zeta) = \{E \in G_3(TN) \mid \zeta_E \neq 0\}$ where ζ_E denotes the restriction of ζ to the 3-plane E. Let $V^k(\mathcal{I})$ denote the set of k-dimensional *integral elements* of $\mathcal{I}$, i.e. those $E \in G_k(TN)$ for which $\beta_E = 0$ for every form $\beta \in \mathcal{I}^k = \mathcal{I} \cap \mathcal{A}^k(N)$. The flag of integral elements $F = (E^0, E^1, E^2, E^3)$ of $\mathcal{I}$ given by $E^0 = \{0\}$, $E^1 = \{v_1\}$, $E^2 = \{v_1, v_2\}$, $E^3 = \{v_1, v_2, v_3\}$ where

$$v_1 = T^1_0 + T^2_0 + T^2_1 + \partial_{x^4},$$
$$v_2 = T^0_0 + T^1_0 - T^2_1 + \partial_{x^1} + \partial_{x^2},$$
$$v_3 = T^1_1 - T^2_1 + \partial_{x^1},$$

has Cartan characters $(s_0, s_1, s_2, s_3) = (0, 2, 4, 3)$. Therefore, by Cartan's test, $V^3(\mathcal{I})$ has codimension at least 8 at E^3. However the forms of $\mathcal{I}^3$ which impose independent conditions on the elements of $G_3(TN, \zeta)$ are the eight 3-forms $\mathrm{d}\chi_i$, $\chi_i \wedge \zeta^k$, $i = 1, 2$, $k = 1, 2, 3$. It follows that $V^3(\mathcal{I}) \cap G_3(TN, \zeta)$ has codimension 8 in $G_3(TN)$. Moreover computations show that $V^3(\mathcal{I}) \cap G_3(TN, \zeta)$ is a smooth submanifold near E^3, thus the flag F is Kähler regular and therefore the ideal $\mathcal{I}$ is involutive. Pick points $p \in M$ and $q = (b, 0) \in N$ with $\pi(b) = p$. By the Cartan–Kähler theorem there exists a 3-dimensional integral manifold $\bar{\psi} = (\bar{s}, \bar{\varphi}) : \Sigma \to B \times \mathbb{R}^4$ of $(\mathcal{I}, \zeta)$ passing through q and having tangent space E^3 at q. Every volume form on M pulls back under π to a nowhere vanishing multiple of ζ. Since $\bar{\phi}^*\zeta = \bar{s}^*\zeta \neq 0$, $\pi \circ \bar{s} : \Sigma \to M$ is a local diffeomorphism. Therefore $p \in M$ has a neighborhood U_p on which there exists a real analytic immersion $\psi = (s, \varphi) : U_p \to B \times \mathbb{R}^4$ such that the pair (ψ, U_p) is an integral manifold of the EDS $(N, \mathcal{I}, \zeta)$ and s a local section of $\pi : B \to M$. After possibly shrinking U_p we can assume that φ is an embedding. Since by construction $\varphi^*(\omega_0 + \mathrm{i}\phi_0) = s^*(\theta^2_0 \wedge (\theta^1_0 + i\theta^2_1))$, it follows that the path geometry induced by φ is (P_1, P_2) on U_p. □

Remark. Every nondegenerate hypersurface $M \subset \mathbb{C}^2$ also inherits a CR-structure (D, I) from the complex structure J on $\mathbb{C}^2$: For every $p \in M$ define D_p to be the largest J_p-invariant subspace of T_pM and $I_p : T_pM \to T_pM$ to be the restriction of J_p to D_p. Then (D, I) is easily seen to be compatible with the path geometry induced on M by $(\{\omega_0\}, \{\phi_0\})$.

Using this remark and Theorem 2 we get the well-known:

Corollary 1. *Let (D, I) be a nondegenerate real analytic CR-structure on a 3-manifold M. Then for every point $p \in M$ there exists a p-neighborhood U_p and a real analytic embedding $\varphi : U_p \to \mathbb{C}^2$, such that (D, I) is the CR-structure on U_p induced by the embedding φ.*

Proof. Pick a line bundle $P_2 \subset D$, define $P_1 = I(P_2)$ and apply Theorem 2. □

Remark. Corollary 1 also holds without the nondegeneracy assumption and in higher dimensions [1]. In [9], Nirenberg has constructed a smooth nondegenerate 3-dimensional CR-structure which is not induced by an embedding into $\mathbb{C}^2$. It follows that the real analyticity assumption in Theorem 2 is necessary.

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