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International Journal of Solids and Structures 45 (2008) 894–907

INTERNATIONAL JOURNAL OF SOLIDS AND STRUCTURES

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Three-dimensional stochastic analysis using a perturbation-based homogenization method for elastic properties of composite material considering microscopic uncertainty

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Received 4 June 2007; received in revised form 28 August 2007
Available online 19 September 2007

Abstract

This paper discusses evaluation of influence of microscopic uncertainty on a homogenized macroscopic elastic property of an inhomogeneous material. In order to analyze the influence, the perturbation-based homogenization method is used. A higher order perturbation-based analysis method for investigating stochastic characteristics of a homogenized elastic tensor and an equivalent elastic property of a composite material is formulated.

As a numerical example, macroscopic stochastic characteristics such as an expected value or variance, which is caused by microscopic uncertainty in material properties, of a homogenized elastic tensor and homogenized equivalent elastic property of unidirectional fiber reinforced plastic are investigated. The macroscopic stochastic variation caused by microscopic uncertainty in component materials such as Young's modulus or Poisson's ratio variation is evaluated using the perturbation-based homogenization method. The numerical results are compared with the results of the Monte-Carlo simulation, validity, effectiveness and a limitation of the perturbation-based homogenization method is investigated. With comparing the results using the first-order perturbation-based method, effectiveness of a higher order perturbation is also investigated.

Keywords: Microscopic uncertainty; Stochastic response; Perturbation method; Composite material; Homogenized elasticity; Homogenization method

1. Introduction

Inhomogeneous materials such as a composite material can be designed to materialize a highly functional material for a special use, and this property will be desired in industrial use. However, a composite material generally has a complex microstructure, dispersion of a microstructure or microscopic material properties

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0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2007.09.008

sometimes occur. This microscopic uncertainty may cause dispersion of a homogenized macroscopic material property. Uncertainty of a homogenized property caused by a microscopic uncertainty, therefore, should be taken into account in manufacturing.

In recent, several results of a multi-scale uncertainty analysis using the finite element method have been reported. Huyse and Maes (1999, 2000, 2001) discussed a homogenized material property with the random field modeling. Kami'nski and Kleiber (1996, 2000) reported a stochastic structural analysis considering uncertainty of interface defects in fiber composites and a perturbation-based homogenization analysis for material properties of composite materials considering Young's modulus variation under plane stress condition. Kami'nski (2001) also reported a perturbation-based stochastic homogenization analysis of heat conduction problem of composites. Koishi et al. (1996) reported a first order perturbation theory-based homogenization method, and validity of the first-order perturbation-based multi-scale stress analysis has been discussed with comparing the numerical result obtained using the stochastic finite element analysis. Ostoja-Starzewski (1994) reported a mechanics for continuum random fields, Zohdi and Wriggers (2001) reported macro–micro testing using computer simulation. Ostoja-Starzewski (2002) also discussed scale-dependent hierarchies for accomplishment of stochastic homogenization of material response in thermomechanics. Niekawa et al. (2004) reported a stochastic finite element analysis using Mori-Tanaka theory. Xu and Brady (2006) reported a computational method for stochastic homogenization of random media.

In this paper, for the stochastic analysis of homogenized material elastic property considering the microscopic uncertainty, the perturbation theory-based homogenization method is formulated. The homogenization method (Babuka, 1976; Guedes and Kikuchi, 1990) will be effective to estimate a homogenized material property of an inhomogeneous material, several results have been reported (for example, Terada and Kikuchi, 1996; Terada et al., 2000; Laschet, 2002; Wu and Ohno, 1999).

A stochastic response analysis method based on the homogenization method has been discussed in some literatures (citebib7; Koishi et al., 1996), however we cannot find a result, which discusses an equivalent elastic property. An effect of a higher order perturbation term on the estimation with considering both Young's modulus and Poisson's ratio variation in a microscopic material has been not discussed yet. Also, a detailed three-dimensional analysis has not been performed. Therefore, in this study, a perturbation-based homogenization method considering a higher perturbation term, which can analyze a stochastic characteristic of a homogenized elastic tensor of composite materials, is formulated at first. The stochastic response analysis method for an uivalent elastic property assuming an orthogonal material is also proposed.

In order to investigate validity, effectiveness and a limitation of the perturbation-based homogenization method, a comparison between the result of the proposed method and that of the Monte-Carlo simulation is performed. As a numerical example for a stochastic analysis, stochastic characteristics of homogenized elastic properties of a unidirectional fiber reinforced plastics (FRP) are investigated.

2. Influence of microscopic uncertainty on homogenized elastic property of composite media

In this section, influence of uncertainty in microstructure on homogenized macroscopic properties of an inhomogeneous material is discussed. As an example, uncertainty in microscopic elastic properties of a unidirectional fiber reinforced composite is taken into account. A stochastic response of the homogenized elastic properties caused by microscopic uncertainty is evaluated using the Monte-Carlo simulation with Box-Muller randomization technique (see, Press et al., 1993). A homogenized elastic property is computed using the homogenization method-based three-dimensional finite element analysis. Fig. 1 shows an example of a finite element model of a microstructure of the unidirectional FRP. Fig. 1(a) shows a scheme of a periodic microstructure. A finite element models of a unit cell with square or hexagonal fiber arrangements are shown in Fig. 1(b) or (c).

In this case, it is considered that Young's modulus and Poisson's ratio of fiber and matrix have a certain variance. The expected values of elastic properties for fiber and matrix are listed in Table 1. The properties of fiber and matrix are employed correspond to E-glass and Epoxy resin. Volume fraction of fiber (V_f) is 0.2513 in this example.

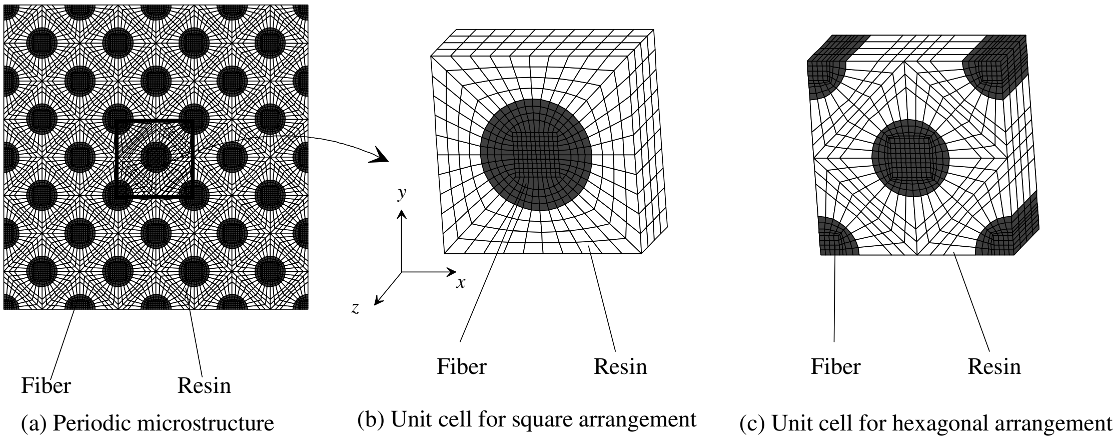


Fig. 1. Schematic view of composite material with periodic microstructure.

Table 1
Expected values of elastic properties for fiber and matrix

	Fiber (E-glass)	Matrix (Epoxy)
Young's modulus (GPa)	73.0	4.5
Poisson's ratio	0.2156	0.39

It is assumed that a microscopic elasticity is distributed according to the normal distribution. For example, an observed value of Young's modulus can be simply expressed using a random variable as:

$$E^* = E^0(1 + \varepsilon) \tag{1}$$

where E^0 is an expected value of Young's modulus, ε is a random variable. Stochastic characteristics of ε are assumed as:

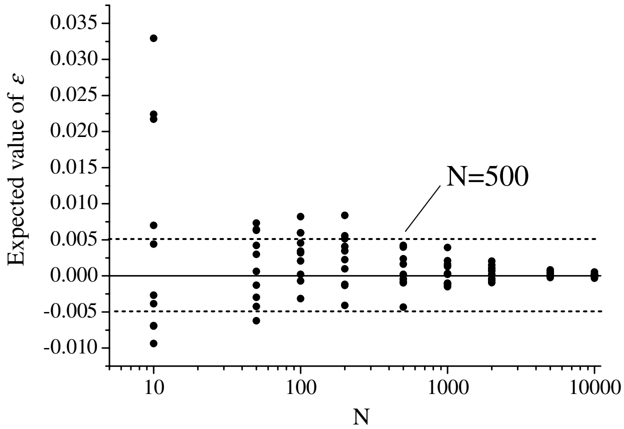
$$\left.\begin{array}{l} E[\varepsilon] = 0 \\ \mathrm{Var}[\varepsilon] = \sigma^2 \end{array}\right\} \tag{2}$$

Here $E[\varepsilon]$ is an expected value and $\mathrm{Var}[\varepsilon]$ is variance of a random variable ε. In this case, it is assumed that $\sigma = 0.055$.

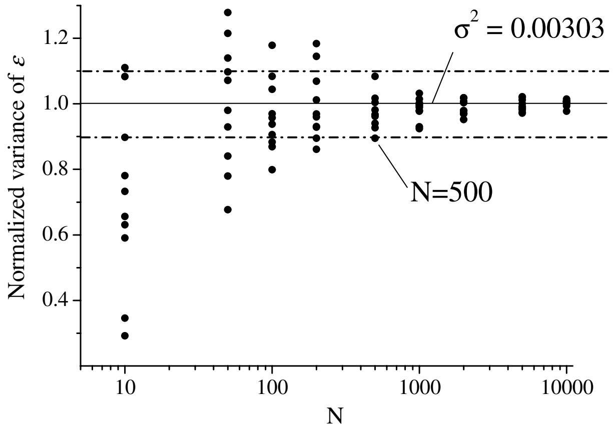
In order to determine a sampling size for the Monte-Carlo simulation, a relationship between dispersion of stochastic responses and the number of sampling data is investigated. Fig. 2(a) shows the relationship between the expected values of ε and the number of sampling data, and Fig. 2(b) shows the relationship between variance of ε and the number of sampling data. N in Fig. 2 shows the number of samples. Each result of the expected value and variance is obtained as a result of the Monte-Carlo simulation, Fig. 2(a) and (b) shows the results of 10th trials. From Fig. 2, it can be recognized that the dispersion of stochastic response is reduced according to increase of the number of sampling data. In this case, we adopt $N = 500$ (namely 500 sampling data are used) for the Monte-Carlo simulation, then the dispersion of the expected value will be within 0.5% and that of the variance will be within about 10.0%.

In this case, a microstructure of a unit cell for a unidirectional FRP with square fiber arrangement is assumed. A result for hexagonal fiber arrangement will be shown in a later section.

The computational results of the expectation and variance of homogenized elastic properties of the composite material, which are obtained using the Monte-Carlo simulation considering Young's modulus and Poisson's ratio variation of fiber and resin, are listed in Tables 2–5. CV in the tables means the coefficient of variation, which is defined as;



(a) relationship between expected values and the number of sampling data



(b) relationship between variance and the number of sampling data

Fig. 2. Relationship between a stochastic response and the number of sampling data for the Monte-Carlo simulation.

Table 2
Stochastic responses of homogenized elastic properties of square arranged unidirectional FRP for E_f variation

	E_x^H	E_z^H	G_{yz}^H	G_{xy}^H	v_{zx}^H	v_{xy}^H	E_f
Exp.	7.2697	21.636	2.5729	2.8381	0.3401	0.5496	72.956
Var.	0.0008	0.9903	0.0001	0.0002	0.0000	0.0000	17.231
CV	0.0039	0.0460	0.0029	0.0043	0.0002	0.0057	0.0569

Table 3
Stochastic responses of homogenized elastic properties of square arranged unidirectional FRP for v_f variation

	E_x^H	E_z^H	G_{yz}^H	G_{xy}^H	v_{zx}^H	v_{xy}^H	v_f
Exp.	7.2716	21.6479	2.5734	2.8389	0.3400	0.5498	0.2157
Var.	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
CV	0.0007	0.0002	0.0005	0.0008	0.0101	0.0010	0.0561

$$\mathrm{CV} = \frac{\sqrt{\mathrm{Var}[x]}}{E[x]} \tag{3}$$

where x is a stochastic variable. From Tables 2–5, it can be recognized that all CV of varying properties of the component materials are almost same, and correspond to about 0.055. On the other hand, CVs for homog-

Table 4
Stochastic responses of homogenized elastic properties of square arranged unidirectional FRP for E_m variation

	E_x^H	E_z^H	G_{yz}^H	G_{xy}^H	v_{zx}^H	v_{xy}^H	E_m
Exp.	7.2596	21.6422	2.5692	2.8342	0.3401	0.5499	4.4971
Var.	0.1348	0.0344	0.0176	0.0202	0.0000	0.0000	0.0659
CV	0.0506	0.0086	0.0516	0.0502	0.0002	0.0056	0.0571

Table 5
Stochastic responses of homogenized elastic properties of square arranged unidirectional FRP for v_m variation

	E_x^H	E_z^H	G_{yz}^H	G_{xy}^H	v_{zx}^H	v_{xy}^H	v_m
Exp.	7.3032	21.649	2.5710	2.8520	0.3416	0.5546	0.3906
Var.	0.0318	0.0001	0.0014	0.0021	0.0003	0.0019	0.0000
CV	0.0244	0.0005	0.0148	0.0159	0.0493	0.0790	0.0561

enized elastic properties are different from each other. For instance, CV of E_z^H for E_f variation is larger than the other values in Table 2. It can be also found that the most of CV of the homogenized elastic properties for v_f variation are very small in Table 3, CVs of E_x^H, G_{yz}^H and G_{xy}^H for E_m variation are larger in Table 4, and CVs of E_x^H, v_{zx}^H and v_{xy}^H for v_m variation are larger in Table 5. CV of v_{xy}^H for v_m variation is larger than CV of v_m itself, and the variance of E_x^H with v_m variation is not small, though it is assumed that Young's modulus of the component materials is independent of Poisson's ratio.

From these results, it can be recognized that different influence of microscopic uncertainty in material properties on macroscopic homogenized elastic properties can be found in each direction, and a kind of uncertainty, such as uncertainty of Young's modulus of a fiber or resin, has different influences from each other. Therefore, it can be considered that the stochastic responses in a homogenization problem for an inhomogeneous material are very complex, and it is important to investigate influence of microscopic uncertainty on a macroscopic homogenized property using a detailed three-dimensional analysis.

3. Perturbation-based stochastic response analysis for a homogenization problem

In order to evaluate influence of microscopic uncertainty to homogenized macroscopic elastic properties, the homogenization method with perturbation theory-based asymptotic expansion on a stochastic variation of microstructure may be effective, because the Monte-Carlo simulation will involve a higher computational cost especially in the case of using the large number of samples.

From a general formulation of the homogenization theory, a homogenized macroscopic elastic tensor $\boldsymbol{E}^H$ can be computed as:

$$\boldsymbol{E}^H = \frac{1}{|\boldsymbol{Y}|}\int_Y \boldsymbol{E}\left(\boldsymbol{I} - \frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{y}}\right)\mathrm{d}\boldsymbol{Y} \tag{4}$$

where $\boldsymbol{E}$ is an elastic tensor of microstructure, $|\boldsymbol{Y}|$ is the volume of a unit cell, $\boldsymbol{I}$ is a unit tensor. $\boldsymbol{\chi}$ is a characteristic displacement, which can be obtained as a solution of the following characteristic equation,

$$\int_Y \frac{\partial}{\partial \boldsymbol{y}}\boldsymbol{E}\frac{\partial \boldsymbol{\chi}}{\partial \boldsymbol{y}}\,\mathrm{d}\boldsymbol{Y} = \int_Y \frac{\partial}{\partial \boldsymbol{y}}\boldsymbol{E}\,\mathrm{d}\boldsymbol{Y} \tag{5}$$

Here, we can obtain a matrix form of Eq. (5) discretized using the finite element method as:

$$[K^Y][\boldsymbol{\chi}] = [\boldsymbol{F}^Y] \tag{6}$$

In this case, microscopic uncertainty, which arises in material properties or geometry of a microstructure, is taken into account. With an asymptotic expansion with respect to a microscopic stochastic variable based on the perturbation theory, an approximation form on the microscopic stochastic variables can be obtained as:

$$[E^*] = [E^0] + [E^1]\varepsilon + [E^2]\varepsilon^2 + \cdots \tag{7}$$

$$[B^*] = [B^0] + [B^1]\varepsilon + [B^2]\varepsilon^2 + \cdots \tag{8}$$

where ε is a small stochastic variation. $[E]$ is a stress-strain matrix and $[B]$ is a displacement-strain matrix. For example, $\boldsymbol{E}^i$ shows an ith-order differential for stochastic variation ε at $\varepsilon = 0$.

Substituting Eqs. (7) and (8) into Eq. (6), and using a stochastic expression of the characteristic displacement $\boldsymbol{\chi}$*, an approximated form of Eq. (6) can be written as:

$$[K^{Y*}][\boldsymbol{\chi}^*] = [\boldsymbol{F}^{Y*}] \tag{9}$$

where

$$\begin{aligned}[K^{Y*}] &= [K^{Y0}] + [K^{Y1}]\varepsilon + [K^{Y2}]\varepsilon^2 + \cdots \\ &= \int_Y [B^0]^T[E^0][B^0]\mathrm{d}V + \left(\int_Y [B^1]^T[E^0][B^0]\mathrm{d}V + \int_Y [B^0]^T[E^1][B^0]\mathrm{d}V + \int_Y [B^0]^T[E^0][B^1]\mathrm{d}V\right)\varepsilon \\ &\quad + \left(\int_Y [B^0]^T[E^2][B^0]\mathrm{d}V + \int_Y [B^1]^T[E^1][B^0]\mathrm{d}V + \int_Y [B^0]^T[E^1][B^1]\mathrm{d}V + \cdots\right)\varepsilon^2 + \cdots\end{aligned} \tag{10}$$

$$[\boldsymbol{\chi}^*] = [\boldsymbol{\chi}^0] + [\boldsymbol{\chi}^1]\varepsilon + [\boldsymbol{\chi}^2]\varepsilon^2 + \cdots \tag{11}$$

$$\begin{aligned}[\boldsymbol{F}^{Y*}] &= [\boldsymbol{F}^{Y0}] + [\boldsymbol{F}^{Y1}]\varepsilon + [\boldsymbol{F}^{Y2}]\varepsilon^2 + \cdots \\ &= \int_Y [B^0]^T[E^0]\mathrm{d}V + \left(\int_Y [B^0]^T[E^1]\mathrm{d}V + \int_Y [B^1]^T[E^0]\mathrm{d}V\right)\varepsilon \\ &\quad + \left(\int_Y [B^0]^T[E^2]\mathrm{d}V + \int_Y [B^1]^T[E^1]\mathrm{d}V + \int_Y [B^2]^T[E^0]\mathrm{d}V\right)\varepsilon^2 + \cdots\end{aligned} \tag{12}$$

By comparing coefficients for each order of ε, the following equations can be obtained:

$$\left.\begin{aligned}&[\boldsymbol{K}^{Y0}]\{\boldsymbol{\chi}^0\} = [\boldsymbol{F}^{Y0}] \\ &([\boldsymbol{K}^{Y1}]\{\boldsymbol{\chi}^0\} + [\boldsymbol{K}^{Y0}]\{\boldsymbol{\chi}^1\}) = [\boldsymbol{F}^{Y1}] \\ &([\boldsymbol{K}^{Y2}]\{\boldsymbol{\chi}^0\} + [\boldsymbol{K}^{Y1}]\{\boldsymbol{\chi}^1\} + [\boldsymbol{K}^{Y0}]\{\boldsymbol{\chi}^2\}) = [\boldsymbol{F}^{Y2}] \\ &\vdots\end{aligned}\right\} \tag{13}$$

By solving Eq. (13), each order perturbation term of a characteristic displacement vector for an optional order of ε can be obtained.

Similar to these formulations, an asymptotic expansion form of the homogenized elastic tensor can be also expressed as:

$$\begin{aligned}[E^{H^*}] &= [E^{H0}] + [E^{H1}]\varepsilon + [E^{H2}]\varepsilon^2 + \cdots \\ &= \frac{1}{|\boldsymbol{Y}|}\int_Y ([E^0] + [E^1]\varepsilon + [E^2]\varepsilon^2 + \cdots)\mathrm{d}\boldsymbol{Y} - \frac{1}{|\boldsymbol{Y}|}\int_Y ([E^0] + [E^1]\varepsilon + [E^2]\varepsilon^2 + \cdots) \\ &\quad \times ([B^0] + [B^1]\varepsilon + [B^2]\varepsilon^2 + \cdots) \times (\{\boldsymbol{\chi}^0\} + \{\boldsymbol{\chi}^1\}\varepsilon + \{\boldsymbol{\chi}^2\}\varepsilon^2 + \cdots)\mathrm{d}\boldsymbol{Y} + \cdots\end{aligned} \tag{14}$$

Therefore, each order stochastic variation of the homogenized elastic tensor can be obtained with comparing a coefficient for each order of ε. $[E^{Hi}]$ for stochastic variation of material properties of a microstructure can be computed as:

$$\left.\begin{aligned}&[E^{H0}] = \tfrac{1}{|\boldsymbol{Y}|}\textstyle\int_Y [E^0]\mathrm{d}\boldsymbol{Y} - \tfrac{1}{|\boldsymbol{Y}|}\int_Y [E^0][B][\chi^0]\mathrm{d}\boldsymbol{Y} \\ &[E^{H1}] = \tfrac{1}{|\boldsymbol{Y}|}\textstyle\int_Y [E^1]\mathrm{d}\boldsymbol{Y} - \tfrac{1}{|\boldsymbol{Y}|}\int_Y ([E^0][B][\chi^1] + [E^1][B][\chi^0])\mathrm{d}\boldsymbol{Y} \\ &[E^{H2}] = \tfrac{1}{|\boldsymbol{Y}|}\textstyle\int_Y [E^2]\mathrm{d}\boldsymbol{Y} - \tfrac{1}{|\boldsymbol{Y}|}\int_Y ([E^1][B][\chi^1] + [E^0][B][\chi^2] + [E^2][B][\chi^0])\mathrm{d}\boldsymbol{Y} \\ &\vdots\end{aligned}\right\} \tag{15}$$

Stochastic responses of a homogenized elasticity for microscopic stochastic variation can be estimated using Eq. (15). An expectation and variance of the homogenized elastic tensor can be computed by the second-order approximation (SA) (Nakagiri and Hisada (1985)) as;

$$\left.\begin{aligned} E[E^{H*}] &= [E^{H0}] + \tfrac{1}{2}\sum_i \sum_j [E^{H2}]_{ij}\mathrm{cov}[\varepsilon_i, \varepsilon_j] \\ \mathrm{Var}[E^{H*}] &= \sum_i \sum_j [E^{H1}]_i [E^{H1}]_j \mathrm{cov}[\varepsilon_i, \varepsilon_j] + \sum_i \sum_j \sum_k [E^{H1}]_i [E^{H2}]_{jk} E[\varepsilon_i \varepsilon_j \varepsilon_k] \\ &\quad + \tfrac{1}{4}\sum_i \sum_j \sum_k \sum_l \{[E^{H2}]_{ij}[E^{H2}]_{jk}(E[\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l] - \mathrm{cov}[\varepsilon_i, \varepsilon_j]\mathrm{cov}[\varepsilon_k, \varepsilon_l])\} \end{aligned}\right\} \tag{16}$$

where $\mathrm{cov}[\varepsilon_i,\varepsilon_j]$ is covariance of ε, $E[\varepsilon_i\varepsilon_j\varepsilon_k]$ and $E[\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l]$ are a third and fourth-order moment of ε. If the second order perturbation term of the homogenized elastic tensor equals zero, or only the first order perturbation term is taken into account, the first-order second moment method (FASM) can be also used for estimation of the expected value and variance. A stochastic analysis method using these formulations is called as the Perturbation-based Stochastic Homogenization Method (PSHM) in this paper.

4. Stochastic response analysis of homogenized equivalent elastic constants of orthogonal media

Several types of industrial materials can be regarded as an isotropic or orthotropic material, then the common material properties such as Young's modulus or Poisson's ratio for each direction will be used for evaluation of material characteristics. Since it is very important to investigate influence of microscopic uncertainty on such a macroscopic engineering constant, a stochastic variation of the engineering constants are also formulated.

The homogenized equivalent elastic properties of isotropic or orthotropic composite materials can be computed by the homogenized compliance. For instance, the homogenized material properties of orthotropic material can be computed as follows:

$$\left.\begin{aligned} &E_x^H = \tfrac{1}{S_{11}^H}, \quad E_y^H = \tfrac{1}{S_{22}^H}, \quad E_z^H = \tfrac{1}{S_{33}^H} \\ &\upsilon_{yz}^H = -E_y^H S_{23}^H, \quad \upsilon_{zx}^H = -E_z^H S_{31}^H, \quad \upsilon_{xy}^H = -E_x^H S_{12}^H \\ &G_{yz}^H = \tfrac{1}{S_{44}^H}, \quad G_{zx}^H = \tfrac{1}{S_{55}^H}, \quad G_{xy}^H = \tfrac{1}{S_{66}^H} \end{aligned}\right\} \tag{17}$$

where S_{ij}^H is a component of a homogenized compliance matrix, which is the inverse of the homogenized elastic matrix E_{ij}^H.

The perturbation-based asymptotic expansion form of the compliance matrix will be also necessary to compute each perturbation term of the compliance. In case of using the second-order approximation of the homogenized elastic tensor, for instance, each component of the compliance of an orthotropic material considering a microscopic stochastic variation can be expressed as:

$$S_{ij}^{H*} = \begin{cases} \dfrac{C_{ij}^0 + C_{ij}^1\varepsilon + C_{ij}^2\varepsilon^2 + \cdots}{D^0 + D^1\varepsilon + D^2\varepsilon^2 + \cdots} & : i,j = 1\text{–}3 \\ \dfrac{1}{E_{ij}^{H*}} & i,j = (4,4),(5,5),(6,6) \end{cases} \tag{18}$$

where C^k and D^k are coefficients computed from the kth order perturbation term of the homogenized elastic matrix. The other components of the compliance matrix are zero in case of an orthotropic material. For example, C_{11}^k can be expressed as:

$$\left.\begin{aligned} C_{11}^0 &= E_{22}^{H0}E_{33}^{H0} - E_{23}^{H0}E_{32}^{H0} \\ C_{11}^1 &= (E_{22}^{H0}E_{33}^{H1} + E_{22}^{H1}E_{33}^{H0}) - (E_{23}^{H0}E_{32}^{H1} + E_{23}^{H1}E_{32}^{H0}) \\ C_{11}^2 &= (E_{22}^{H0}E_{33}^{H2} + E_{22}^{H2}E_{33}^{H0} + E_{22}^{H1}E_{33}^{H1}) \\ &\quad - (E_{23}^{H0}E_{32}^{H2} + E_{23}^{H2}E_{32}^{H0} + E_{23}^{H1}E_{32}^{H1}) \\ &\vdots \end{aligned}\right\} \tag{19}$$

and D^0 can be expressed as;

$$\left.\begin{aligned} D^0 &= E_{11}^{H0}E_{22}^{H0}E_{33}^{H0} + E_{21}^{H0}E_{32}^{H0}E_{13}^{H0} + E_{31}^{H0}E_{12}^{H0}E_{23}^{H0} \\ &\quad -E_{11}^{H0}E_{32}^{H0}E_{23}^{H0} - E_{31}^{H0}E_{22}^{H0}E_{13}^{H0} - E_{21}^{H0}E_{12}^{H0}E_{33}^{H0} \\ D^1 &= (E_{11}^{H0}E_{22}^{H0}E_{33}^{H1} + E_{11}^{H0}E_{22}^{H1}E_{33}^{H0} + E_{11}^{H1}E_{22}^{H0}E_{33}^{H0}) \\ &\quad +(E_{21}^{H0}E_{32}^{H0}E_{13}^{H1} + E_{21}^{H0}E_{32}^{H1}E_{13}^{H0} + E_{21}^{H1}E_{32}^{H0}E_{13}^{H0}) \\ &\quad +\cdots \\ D^2 &= (E_{11}^{H0}E_{22}^{H0}E_{33}^{H2} + E_{11}^{H0}E_{22}^{H2}E_{33}^{H0} + E_{11}^{H2}E_{22}^{H0}E_{33}^{H0} \\ &\quad +E_{11}^{H0}E_{22}^{H1}E_{33}^{H1} + E_{11}^{H1}E_{22}^{H1}E_{33}^{H0} + E_{11}^{H1}E_{22}^{H0}E_{33}^{H1}) \\ &\quad +\cdots \end{aligned}\right\} \tag{20}$$

From Eqs. (18)–(20), each order perturbation term of the compliance matrix S_{ij}^{Hk} can be computed. Each order perturbation term of a homogenized equivalent elastic property can be computed by Eqs. (19) and (20) and differential of Eq. (18).

5. Numerical example of the perturbation-based stochastic response analysis of homogenized elastic properties

5.1. Stochastic analysis of unidirectional GFRP

As an example, stochastic characteristics such as expectation or variance of homogenized elastic tensor or homogenized elastic properties of unidirectional FRP caused by microscopic uncertainty are evaluated using the perturbation-based homogenization analysis. The expectation and variance of the homogenized elastic tensor and the homogenized elastic property of the unidirectional GFRP obtained from the PSHM analysis are evaluated using SA and FASM. In order to investigate validity of the proposed method, the numerical results using the methods are compared with those of the Monte-Carlo simulation. In this case, 1000 samples are used for the Monte-Carlo simulation. The finite element model of the microstructure, which is illustrated in Fig. 1(c), is used for the simulation. As shown in Fig. 1(c), it is assumed that the microstructure has the hexagonal fiber arrangement. The volume fraction of fiber (Vf) is 0.2513 in this case. The material properties of fiber and resin listed in Table 1 are also used in this section. Stochastic characteristics of a random variable ε in Eq. (1) are; $E[\varepsilon] = 0$ and $\sqrt{\mathrm{Var}[\varepsilon]} = 0.055$ are assumed.

Figs. 3–6 show relative estimation errors between the expectation of the homogenized elastic properties computed using the PSHM and that of the Monte-Carlo simulation. FASM in the figures shows the result of the first-order perturbation method, SA shows the result of the second-order perturbation method. Fig. 3 shows the result of Young's modulus variation of fiber, Fig. 4 shows the result of Young's modulus

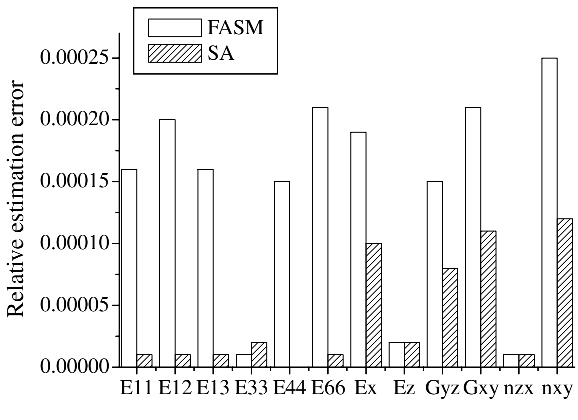


Fig. 3. Relative estimation errors in expectations of homogenized elastic properties by each method in case of Young's modulus variation of fiber.

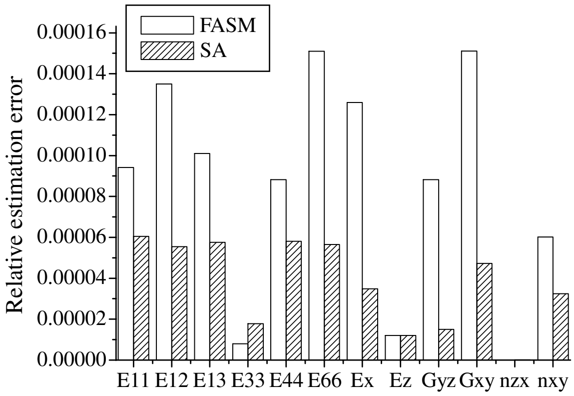


Fig. 4. Relative estimation errors in expectations of homogenized elastic properties by each method in case of Young's modulus variation of resin.

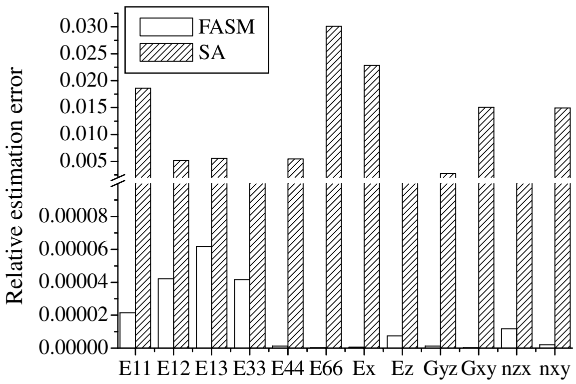


Fig. 5. Relative estimation errors in expectations of homogenized elastic properties by each method in case of Poisson's ratio variation of fiber.

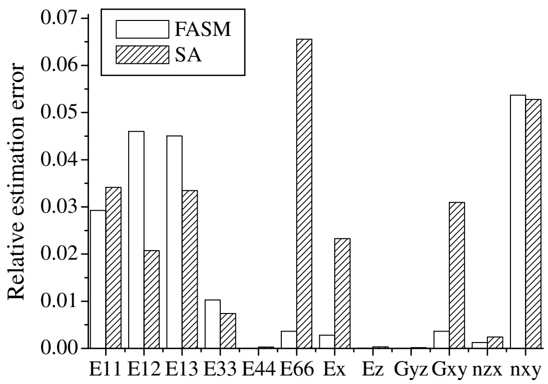


Fig. 6. Relative estimation errors in expectations of homogenized elastic properties by each method in case of Poisson's ratio variation of resin.

variation of resin. Figs. 5 and 6 show the results of Poisson's ratio variation of fiber and resin. The results for the homogenized elastic tensor E_{ij} and the equivalent elastic constants such as E_x or G_{xy} are illustrated.

From these results, it can be recognized that the expectation of each component of the homogenized elastic tensor is well-estimated using FASM and SA, especially in case of Young's modulus variation of fiber or resin.

In case of Young's modulus variation of fiber or resin, the estimation error in SA is smaller than that of FASM. Though the expectations for Poisson's ratio variation of fiber are also well estimated, the estimation errors in SA are larger than that of FASM. In case of Poisson's ratio variation of resin, the estimation errors in E11, E12, E13 and nxy are larger than the other cases. The estimation errors in E12, E13, E33 and nxy using SA are smaller than that of FASM, but others of SA are larger than that of FASM.

Next, an estimation error in the estimated variance is also investigated. Figs. 7–10 show relative estimation errors between the variance of the homogenized elastic properties computed using the PSHM and that of the Monte-Carlo simulation. Fig. 7 shows the result of Young's modulus variation of fiber, Fig. 8 shows the result of Young's modulus variation of resin. Figs. 9 and 10 show the results of Poisson's ratio variation of fiber and resin.

From these figure, it can be recognized that the PSHM will be effective for estimating variance of the homogenized elastic properties in case of Young's modulus variation. SA improves accuracy of the estimation in case of Young's modulus variation of fiber.

FA is also effective for estimating the variance in case of Poisson's ratio variation of fiber, but the estimation errors in the result of SA is very large in this case. It can be noticed that the relative estimation errors in case of Poisson's ratio variation of resin are larger than the other cases, and SA improves accuracy of the estimation for the homogenized elastic tensor except to E66. On the other hand, SA is not effective for improving accuracy of the estimation for the homogenized equivalent elastic constants in this case. These results show difficulty in using the second order perturbation method for estimating the stochastic characteristics of the

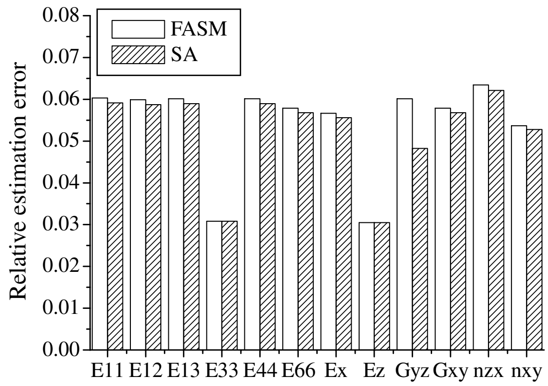


Fig. 7. Relative estimation errors in variances of homogenized elastic properties by each method in case of Young's modulus variation of fiber.

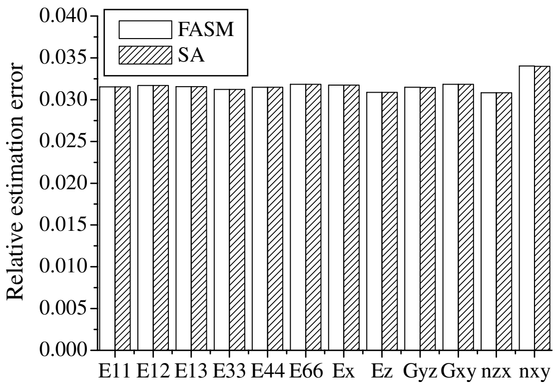


Fig. 8. Relative estimation errors in variances of homogenized elastic properties by each method in case of Young's modulus variation of resin.

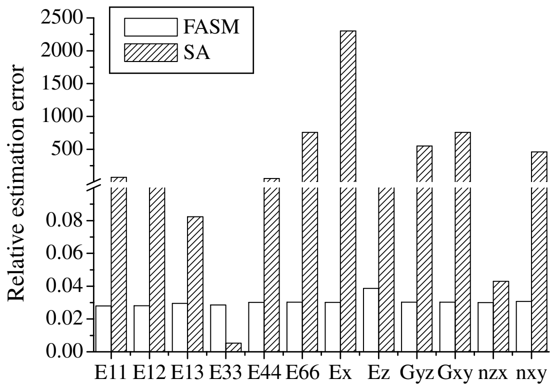


Fig. 9. Relative estimation errors in variances of homogenized elastic properties by each method in case of Poisson's ratio variation of fiber.

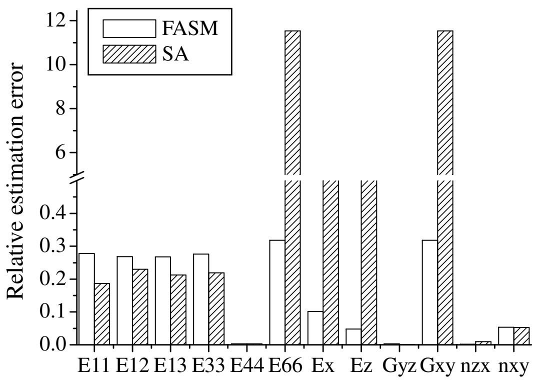


Fig. 10. Relative estimation errors in variances of homogenized elastic properties by each method in case of Poisson's ratio variation of resin.

homogenized elastic properties, especially the stochastic characteristics of the homogenized elastic constants for Poisson's ratio variation.

5.2. Influence of volume fraction of fiber

Geometry of a microstructure such as a volume fraction of fiber will have an influence of stochastic responses in homogenized elastic properties. In this paper, therefore, a relationship between accuracy of the perturbation-based homogenization method and volume fraction of fiber is also investigated.

As an example, relationships between the relative estimation error and volume fraction of fiber in case of Poisson's ratio variation of resin is illustrated. The number of samples used for the Monte-Carlo simulation is 1000, the stochastic characteristics of material properties of the component materials are assumed as the previous example.

Fig. 11 shows the relationship between a relative estimation error of expectation of the homogenized elastic tensor and volume fraction of fiber. Fig. 12 shows the relationship between that of variance and volume fraction of fiber. In these figures, as an example, estimation errors in E_{11}^H and E_{33}^H are illustrated.

From these figures, it is recognized that both of the estimation errors in expectations and variances in case of the second order perturbation method are less than that of the first order perturbation method. Estimation errors in expectations and variances of E_{11}^H and E_{33}^H decrease as the volume fraction of fiber increases, the second order perturbation method improves accuracy of the estimation within this range of the volume fraction.

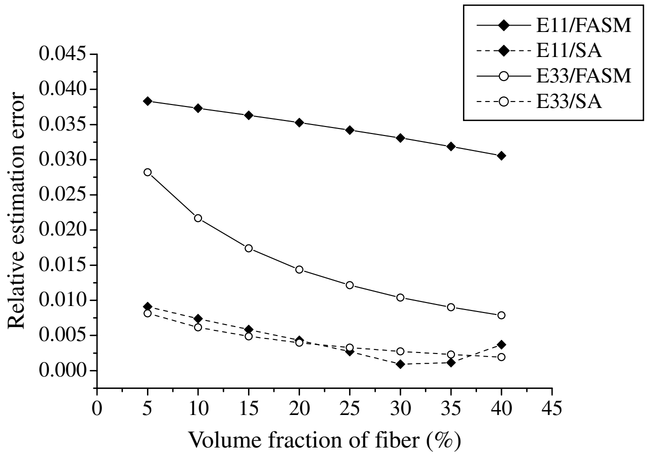


Fig. 11. Relationship between volume fraction of fiber and estimation error in expectation of homogenized elastic tensor in case of Poisson's ratio variation of resin.

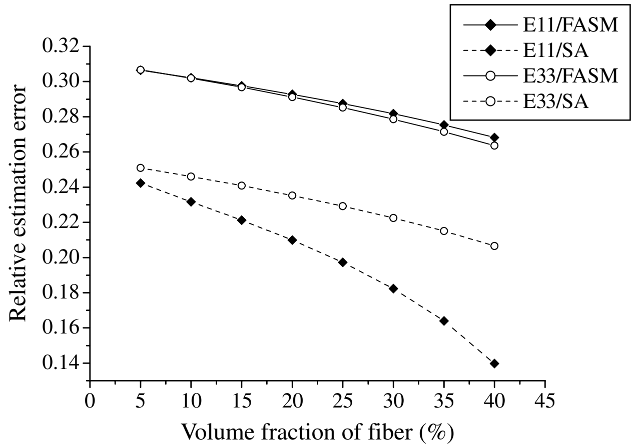


Fig. 12. Relationship between volume fraction of fiber and estimation error in variance of homogenized elastic tensor in case of Poisson's ratio variation of resin.

This result shows that the second order perturbation method is effective for improving accuracy of expectation and variance estimation of the homogenized elastic tensor within a wide range of the volume fraction of fiber in this case.

Next, a relationship between the relative estimation error in case of the equivalent elastic constants and volume fraction of fiber is investigated. Fig. 13 shows the relationship between a relative estimation error of expectation of the homogenized equivalent elastic constants and volume fraction of fiber. Fig. 14 shows the relationship between that of variance and volume fraction of fiber. In these figures, as an example, estimation errors in E_x^H and E_z^H are illustrated.

From these figures, it is recognized that the second order perturbation method is not effective for improving accuracy of the estimation. Especially, estimation errors in the expectation and variance of E_x^H rapidly increase as the volume fraction increases. On the other hand, the estimation error in the variance of E_z^H is very large for a small value of the volume fraction. This result shows that the second order perturbation method will not improve accuracy of the estimation in this case, and the second order perturbation method should not be used for stochastic analysis of homogenized equivalent elastic constants of a composite material, especially in case of considering Poisson's ratio variation.

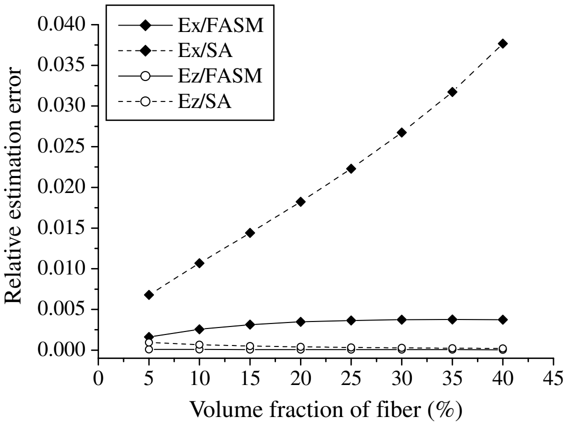


Fig. 13. Relationship between volume fraction of fiber and estimation error in expectation of equivalent elastic constants in case of Poisson's ratio variation of resin.

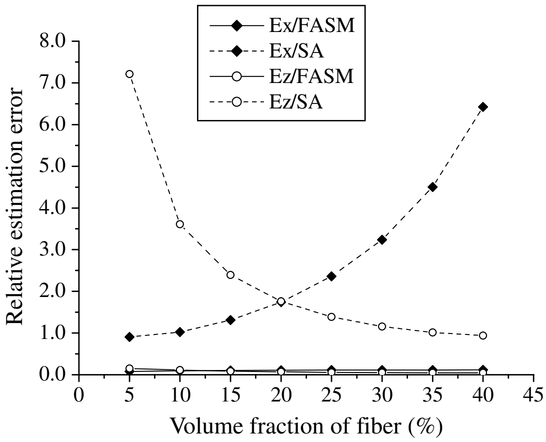


Fig. 14. Relationship between volume fraction of fiber and estimation error in variance of equivalent elastic constants in case of Poisson's ratio variation of resin.

6. Conclusion

In this paper, influence of uncertainty in microscopic material properties on homogenized elastic property using the perturbation-based three-dimensional homogenization analysis is discussed. The perturbation-based stochastic response analysis method using the homogenized method is formulated. Using this formulation, each order perturbation term of a homogenized elastic tensor or a homogenized equivalent elastic constant of an inhomogeneous material can be computed. The stochastic characteristics, such as the expectation or variance against a microscopic stochastic variation can be also computed using SA or FASM.

At first, a stochastic response analysis of homogenized elastic tensor for a unidirectional GFRP is performed using the Monte-Carlo simulation. This numerical result shows the necessary of a detailed thee-dimensional stochastic response analysis for a homogenization problem.

Next, the numerical analysis using the proposed formulation is performed. The numerical results obtained from the analysis using the proposed method are compared with the results of the Monte-Carlo simulation in detail. From the numerical results, it can be recognized that SA does not always improve accuracy of the stochastic estimation, and FASM will be useful in some cases of microscopic variation such as Young's modulus of fiber. However, the stochastic responses against some kinds of microscopic variation such as Poisson's ratio variation of resin cannot be well estimated using the PSHM.

Additionally, a relationship between volume fraction of fiber and accuracy of the perturbation-based stochastic response analysis is also investigated. From this result, it is recognized that the second order perturbation-based procedure may be effective for improving accuracy of the estimation for the homogenized elastic tensor, but it will not give an accurate estimation of expectation and variance of the homogenized equivalent elastic constants for Poisson's ratio variation.

From the numerical results, therefore, the perturbation-based analysis will be used in several limited cases of stochastic analysis for a three-dimensional homogenization problem. In concrete, the use of the perturbation-based method should be limited to Young's modulus variation at most, and the PSHM should be avoided to use a stochastic analysis of a homogenization problem including a nonlinear stochastic response such as Poisson's ratio variation considering a large stochastic variation. The second order perturbation method may improve accuracy of estimation, especially in case of the homogenized elastic tensor, but it should not be used for estimation of stochastic characteristics of a homogenized equivalent elastic constants. For more general use, it can be considered that an improved analysis procedure will be needed.

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