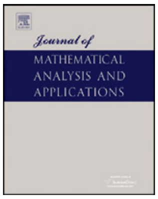
J. Math. Anal. Appl. 380 (2011) 163–176

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



Minimal tori with low nullity

David L. Johnson [a], Oscar Perdomo [b,*]

[a] *Department of Mathematics, Lehigh University, Bethlehem, PA 18015-3174, United States*
[b] *Department of Mathematics, Central Connecticut State University, New Britain, CT 06050, United States*

ARTICLE INFO

Article history:
Received 9 January 2010
Available online 22 February 2011
Submitted by Steven G. Krantz

Keywords:
Minimal tori on spheres
Stability operator
Nullity

ABSTRACT

The *nullity* of a minimal submanifold $M \subset S^n$ is the dimension of the nullspace of the second variation of the area functional. That space contains as a subspace the effect of the group of rigid motions $SO(n+1)$ of the ambient space, modulo those motions which preserve M, whose dimension is the *Killing nullity* $kn(M)$ of M. In the case of 2-dimensional tori M in S^3, there is an additional naturally-defined 2-dimensional subspace that contributes to the nullity; the dimension of the sum of the action of the rigid motions and this space is the *natural nullity* $nnt(M)$. In this paper we will study minimal tori in S^3 with natural nullity less than 8. We construct minimal immersions of the plane $\mathbb{R}^2$ in S^3 that contain all possible examples of tori with $nnt(M) < 8$. We prove that the examples of Lawson and Hsiang with $kn(M) = 5$ also have $nnt(M) = 5$, and we prove that if the $nnt(M) \leqslant 6$ then the group of isometries of M is not trivial.

1. Introduction

Let $\tilde{\rho}: M \to S^3$ be a minimal immersion of an oriented surface without boundary M in the unit three dimensional sphere $S^3 \subset \mathbb{R}^4$. Let $N: M \to S^3$ be the Gauss map, i.e. $N(m) \perp T_m M$ and $\langle N(m), m\rangle = 0$. For any $m \in M$, $a(m)$ will denote the nonnegative principal curvature of M at m and $W_1(m)$ and $W_2(m)$ will denote two unit tangent vectors such that $dN_m(W_1(m)) = -a(m)W_1(m)$ and $dN_m(W_2(m)) = a(m)W_2(m)$. When M is a torus, it is known that for every m, $a(m)$ is positive [1], therefore in this case we can choose $W_1(m)$ and $W_2(m)$ so that they define smooth vector fields on M. In the following, for M a torus, W_1 and W_2 denote such unit tangent vector fields and $a: M \to \mathbb{R}$ will be the smooth function given by the positive principal curvature. Since M is minimal, M is a critical point of the area functional. Since $M \hookrightarrow S^3$ has codimension 1, any variation of the surface M is given by a function $f \in C^\infty(M)$. The second variation of the area function at this critical point is given by the stability operator

$$J: C^\infty(M) \to C^\infty(M) \quad \text{given by } J(f) := -\Delta f - 2a^2 f - 2f.$$

The *nullity* of a minimal surface is defined as the dimension of the kernel of the operator J and will be denoted by $n(M)$. Elements of the nullity are infinitesimal variations of M which, up to order 2, do not change the area of M.

1.1. Killing nullity

Given a fixed matrix $B \in so(4)$, define $f_B: M \to \mathbb{R}$ by $f_B = \langle B\tilde{\rho}(m), N(m)\rangle$. It is clear that f_B satisfies the elliptic equation $J(f_B) = 0$ because, when we move the immersion M by the group of isometries $e^{Bt}: S^3 \to S^3$ we induce a family that leaves the area and second fundamental form constant; f_B is the function associated with this family.

---

[*] Corresponding author.
E-mail addresses: david.johnson@lehigh.edu (D.L. Johnson), perdomoosm@ccsu.edu (O. Perdomo).

0022-247X/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2011.02.057

In [2], Lawson and Hsiang classify all the minimal surfaces that are invariant under a 1-parametric group of isometries in S^3. One way to see this classification is the following: Define the Killing nullity by setting $KS := \{f_B\colon\ B \in so(4)\}$ to be the space of all variations arising from $SO(4)$, and the Killing nullity is defined as $kn(M) := \dim(KS)$. We have that $kn(M) \leqslant n(M)$ and in general the Killing nullity is expected to be 6 since the dimension of $so(4)$ is 6. Lawson and Hsiang classify all the examples of surfaces with $kn(M) < 6$, i.e. they classify all minimal surfaces with *not full* Killing nullity. More precisely, their classification can be described in the following way,

$$K_3 = \left\{M \subset S^3\colon\ kn(M) = 3\right\},$$

which is the set of totally geodesic spheres. Up to rigid motions there is only one example.

$$K_4 = \left\{M \subset S^3\colon\ kn(M) = 4\right\}$$

is the set of Clifford tori, and

$$K_5 = \left\{M \subset S^3\colon\ kn(M) = 5\right\}$$

is a collection of immersed minimal tori. There are infinitely many non-isometric examples in K_5. One of the goals of this paper will be to provide a better understanding of this set.

1.2. Natural nullity

Minimal tori in S^3 will have a potentially larger nullity than the Killing nullity. For a minimal torus, since W_1 and W_2 are globally defined, we can define $h_\theta : M \to \mathbb{R}$ as the directional derivative of $-2a^{-\frac{1}{2}}$ in the direction $\cos(\theta)W_1 + \sin(\theta)W_2$, that is, $h_\theta = \cos(\theta)a^{-\frac{3}{2}}W_1(a) + \sin(\theta)a^{-\frac{3}{2}}W_2(a)$. For tori, $HS = \{\lambda h_\theta\colon\ \lambda \in \mathbb{R}, \theta \in S^1\}$ form a subspace of $\ker(J)$, which follows by a direct computation. In general, $\dim(HS) := hn(M)$ is expected to be 2. Recall that the functions f_B defined above also satisfy $J(f_B) = 0$, that is, they represent infinitesimal variations of the M through minimal tori. The functions f_B are not only infinitesimal variations but actually generate a family of minimal tori, namely the family $t \to e^t M$. The functions h_θ are also infinitesimal variations of M through minimal immersions since $J(h_\theta) = 0$, however, the authors have not yet been able to determine whether or not the functions h_θ generate a family of minimal tori.

The principle focus of this paper is to study the space $KS + HS := NS$, the subspace of the nullity arising from these two natural sources. We call the *natural nullity* of the space M $nnt(M) := \dim(NS)$. In this paper we classify all minimal tori with $nnt(M) < 8$.

The Lawson–Hsiang examples, beyond the Clifford torus, will be shown by a Liouville argument to be the immersed minimal tori with $\dim(HS) = 1$, as well as those having Killing nullity 5. We also show, using a result by Ramanathan about isometries of a minimal surfaces of S^3, that if $\dim(HS) = 1$ then $M \in K_5$. In other words, we have that

$$NN_5 = \left\{M \subset S^3\colon\ nnt(M) = 5\right\} = K_5 = H_1,$$

where $H_1 = \{M \subset S^3\colon\ \dim(HS) = 1\}$.

We construct *every* possible torus with $nnt(M) < 8$ by building, for any angle $\theta \in S^1$ and any skew-symmetric matrix B, an integrable distribution $\mathcal{D}_{B,\theta}$ in $SO(4) \times \mathbb{R}^2$ with the property that the projection onto the first column in $SO(4)$ always defines a smooth minimal immersion of $\mathbb{R}^2$. If M is a torus with $nnt(M) < 8$ then there is a B and θ for which M is the image of such a leaf. In particular, by our previous result we have that if $M \in K_5$ then $\dim(HS) = 1$, so $h_\theta = 0$ for some θ. This observation tells us that K_5 can be obtained as coming from examples in the distribution $\mathcal{D}_{B,\theta}$. We prove that if $M \in K_5$ then $h_{\theta+\frac{\pi}{2}} \in KS$, i.e. we prove that not only is $kn(M) = 5$ but also $nnt(M) = 5$.

To describe our last result we point out that if $M \in K_5$ then $nnt(M) = 5$ and M is invariant under infinitely many isometries (a 1-parameter group to be precise). We prove that if $nnt(M) = 6$ then M has some nontrivial isometry.

The authors would like to thank the referee for numerous suggestions and comments.

2. Preliminaries

This section reviews some known results that will be used later on. The first result, due to Blaine Lawson, has already been used in the introduction in order to define the unit tangent smooth vector fields W_1 and W_2 in an immersed minimal torus of S^3.

Theorem 2.1. *(See Lawson [1].) If $M \subset S^3$ is a closed minimal surface and $a : M \to \mathbb{R}$ denotes the nonnegative principal curvature function, then a is positive everywhere if and only if $\chi(M) = 0$.*

The next theorem also was mentioned in the introduction in order to define the natural nullity for tori. Even though this is a known result, for completeness sake we will provide a proof at the end of this section.

Theorem 2.2. *If $M \subset S^3$ is a minimal immersed torus, and $W_1 : M \to S^3$ and $W_2 : M \to S^3$ are unit vector fields that define the principal directions, then the functions*

$$h_0, h_{\frac{\pi}{2}} : M \to \mathbb{R} \quad \text{given by } h_0 = a^{-\frac{3}{2}} W_1(a) \text{ and } h_{\frac{\pi}{2}} = a^{-\frac{3}{2}} W_2(a)$$

satisfy

$$J(h_0) = -\Delta h_0 - 2h_0 - 2a^2 h_0 = 0 = J(h_{\frac{\pi}{2}}).$$

There is a correspondence between constant mean curvature (CMC) surfaces in Euclidean space and minimal surfaces in S^3. The proof of Theorem 2.2 for the case of CMC surfaces in Euclidean space is established in Sections §2 and §3 of [3].

In Section 4 we construct a family of minimal immersions of the plane into S^3. The following theorem will be used to show that Lawson–Hsiang examples correspond to a subfamily of those immersions.

Theorem 2.3. *(See Ramanathan [4].) Let $\tilde{\rho} : M \to S^3$ be a minimal immersion of an oriented compact surface. Suppose that M admits a one parameter group of isometries $\phi_t : M \to M$ with respect to the induced metric. Then, there exists a one-parameter family of orientation preserving isometries Φ_t of S^3 such that $\tilde{\rho} \circ \phi_t = \Phi_t \circ \tilde{\rho}$ for all $t \in \mathbb{R}$.*

The next theorem is a consequence of the uniformization theorem applied to a minimal torus in S^3.

Theorem 2.4. *For every minimal immersion of a torus $\tilde{\rho} : M \to S^3$, there exists a covering map $\tau : \mathbb{R}^2 \to M$, a doubly periodic conformal immersion $\rho : \mathbb{R}^2 \to S^3$, a Gauss map $\nu : \mathbb{R}^2 \to S^3$, and a fixed angle α, so that*

$$\rho(u,v) = \tilde{\rho}\big(\tau(u,v)\big), \qquad \nu(u,v) \perp \rho_*\big(T_{(u,v)}\mathbb{R}^2\big), \qquad \nu(u,v) \perp \rho(u,v),$$

and

$$\begin{aligned}
\frac{\partial^2 \rho}{\partial u^2} &= -\frac{\partial r}{\partial u}\frac{\partial \rho}{\partial u} + \frac{\partial r}{\partial v}\frac{\partial \rho}{\partial v} + \cos(2\alpha)\nu - \mathrm{e}^{-2r}\rho,\\
\frac{\partial^2 \rho}{\partial v^2} &= \frac{\partial r}{\partial u}\frac{\partial \rho}{\partial u} - \frac{\partial r}{\partial v}\frac{\partial \rho}{\partial v} - \cos(2\alpha)\nu - \mathrm{e}^{-2r}\rho,\\
\frac{\partial^2 \rho}{\partial u \partial v} &= -\frac{\partial r}{\partial v}\frac{\partial \rho}{\partial u} - \frac{\partial r}{\partial u}\frac{\partial \rho}{\partial v} - \sin(2\alpha)\nu,\\
\frac{\partial \nu}{\partial u} &= \mathrm{e}^{2r}\left(-\cos(2\alpha)\frac{\partial \rho}{\partial u} + \sin(2\alpha)\frac{\partial \rho}{\partial v}\right),\\
\frac{\partial \nu}{\partial v} &= \mathrm{e}^{2r}\left(\sin(2\alpha)\frac{\partial \rho}{\partial u} + \cos(2\alpha)\frac{\partial \rho}{\partial v}\right),
\end{aligned}$$

where $\mathrm{e}^{-2r} = \langle \frac{\partial \rho}{\partial u}, \frac{\partial \rho}{\partial u} \rangle = \langle \frac{\partial \rho}{\partial v}, \frac{\partial \rho}{\partial v} \rangle$. Moreover, $\Delta r + 2\sinh(2r) = 0$.

Proof. The idea of the proof is the following: the existence of the conformal map ρ and the covering τ follows from the uniformization theorem, the existence of the constant α follows from the fact that

$$f(z) = f(u + iv) = \left\langle \frac{\partial^2 \rho}{\partial u^2}, \nu \right\rangle - i \left\langle \frac{\partial^2 \rho}{\partial u \partial v}, \nu \right\rangle$$

is an analytic, doubly periodic function in the whole plane, and therefore is constant. Clearly this constant function f is not identically zero otherwise M would be totally geodesic. By scaling the coordinates u and v by a constant, we can make $f(u + iv) = \cos(2\alpha) + i\sin(2\alpha)$ for some constant angle α.

To complete the proof, the equations for the second derivatives of ρ are just the standard computation of the Christoffel symbols, and the elliptic equation of r follows from computing the Gauss curvature using the Christoffel symbols and setting it to $1 - \mathrm{e}^{4r}$, i.e. this elliptic equation follows from the Gauss equation. □

Remark 2.5. We can change the angle α to any value by rotating the coordinates u and v.

Corollary 2.6. *Using the same notation as in Theorem 2.4, the principal directions of the minimal immersion are given by*

$$V_1 = \mathrm{e}^r\left(\cos(\alpha)\frac{\partial \rho}{\partial u} - \sin(\alpha)\frac{\partial \rho}{\partial v}\right) \quad \text{and} \quad V_2 = \mathrm{e}^r\left(\sin(\alpha)\frac{\partial \rho}{\partial u} + \cos(\alpha)\frac{\partial \rho}{\partial v}\right).$$

More precisely,

$$d\nu\big(\{d\rho_{(u,v)}\}^{-1}(W_1\circ\tau)\big)=-\mathrm{e}^{2r}V_1\quad\textit{and}\quad d\nu\big(\{d\rho_{(u,v)}\}^{-1}(W_2\circ\tau)\big)=\mathrm{e}^{2r}V_2.$$

Moreover, it follows from the last expression that the principal curvatures are $\pm a$ *where the function* $a:M\to\mathbb{R}$ *is defined by* $a(\tau(u,v))=\mathrm{e}^{2r(u,v)}$. *We also have that* $h_\alpha\circ\tau=2\frac{\partial r}{\partial u}$ *and* $h_{\alpha+\frac{\pi}{2}}\circ\tau=2\frac{\partial r}{\partial v}$.

Proof.

$$\begin{aligned}-d\nu\big(\{d\rho_{(u,v)}\}^{-1}(W_1\circ\tau)\big)&=\mathrm{e}^{r}d\nu\left(-\cos(\alpha)\frac{\partial}{\partial u}+\sin(\alpha)\frac{\partial}{\partial v}\right)\\&=\mathrm{e}^{3r}\left(-\cos(\alpha)\left(-\cos(2\alpha)\frac{\partial\rho}{\partial u}+\sin(2\alpha)\frac{\partial\rho}{\partial v}\right)+\sin(\alpha)\left(\sin(2\alpha)\frac{\partial\rho}{\partial u}+\cos(2\alpha)\frac{\partial\rho}{\partial v}\right)\right)\\&=\mathrm{e}^{2r}V_1.\end{aligned}$$

Similarly, $d\nu(\{d\rho_{(u,v)}\}^{-1}(W_2\circ\tau))=\mathrm{e}^{2r}V_2$. In the same fashion,

$$\begin{aligned}h_\alpha\circ\tau&=\big(\mathrm{e}^{2r}\big)^{-\frac{3}{2}}\big(\cos(\alpha)V_1\big(\mathrm{e}^{2r}\big)+\sin(\alpha)V_2\big(\mathrm{e}^{2r}\big)\big)\\&=\mathrm{e}^{-3r}\mathrm{e}^{r}\left(\cos(\alpha)\left(\cos(\alpha)\frac{\partial\rho}{\partial u}-\sin(\alpha)\frac{\partial\rho}{\partial v}\right)\big(\mathrm{e}^{2r}\big)+\sin(\alpha)\left(\sin(\alpha)\frac{\partial\rho}{\partial u}+\cos(\alpha)\frac{\partial\rho}{\partial v}\right)\big(\mathrm{e}^{2r}\big)\right)\\&=2\frac{\partial r}{\partial u},\end{aligned}$$

and $h_{\alpha+\frac{\pi}{2}}\circ\tau=2\frac{\partial r}{\partial v}$. □

Proof of Theorem 2.2. Take maps $\rho, V_1, V_2, \nu:\mathbb{R}^2\to S^3$, $\tau:\mathbb{R}^2\to M$ and $r:\mathbb{R}^2\to\mathbb{R}$ such that they satisfy the condition of Theorem 2.4 with $\alpha=0$, i.e. with $V_1(u,v)=W_1(\tau(u,v))=\mathrm{e}^{r(u,v)}\frac{\partial\rho}{\partial u}(u,v)$ and $V_2(u,v)=W_2(\tau(u,v))=\mathrm{e}^{r(u,v)}\frac{\partial\rho}{\partial v}(u,v)$. Since $\Delta_{\mathbb{R}^2}r+2\sinh(2r)=0$, we obtain that

$$\Delta_{\mathbb{R}^2}\frac{\partial r}{\partial u}+4\cosh(2r)\frac{\partial r}{\partial u}=0.$$

Since $\frac{\partial\rho}{\partial u}(u,v)=\mathrm{e}^{-r}V_1(u,v)=\mathrm{e}^{-r}W_1(\tau(u,v))$ and $a(\tau(u,v))=\mathrm{e}^{2r(u,v)}$, we have

$$\frac{\partial r}{\partial u}=a^{-\frac{1}{2}}W_1\left(\frac{1}{2}\ln(a)\right)=\frac{1}{2}a^{-\frac{3}{2}}W_1(a).$$

Denote by Δ_M the Laplacian in the surface. Since the metric induced by ρ in $\mathbb{R}^2$ is given by $ds^2=\mathrm{e}^{-2r}(du^2+dv^2)$, we obtain that,

$$\Delta_M\left(\frac{1}{2}a^{-\frac{3}{2}}W_1(a)\right)=a\Delta_{\mathbb{R}^2}\left(\frac{\partial r}{\partial u}\right)=-a\left(2\big(a+a^{-1}\big)\left(\frac{1}{2}a^{-\frac{3}{2}}W_1(a)\right)\right).$$

Therefore the function $h_0=a^{-\frac{3}{2}}W_1(a)$ satisfies $J(h_0)=0$. $J(h_{\frac{\pi}{2}})=0$ follows similarly. □

3. Minimal tori with $hn(M)<2$

The Lawson–Hsiang torus examples are characterized as those immersed minimal tori in S^3 that are preserved by a 1-parameter group of ambient isometries [2]. It is clear that if for some $B\in so(4)$, $M\subset S^3$ is invariant under the group of isometries $\{\mathrm{e}^{Bt}:S^3\to S^3\colon t\in\mathbf{R}\}$, then the function f_B vanishes. This is because the function f_B is the function associated with the variation $M_t=\mathrm{e}^{Bt}M$ and, under our assumption, $M_t=M$ for all t, therefore this variation is constant and f_B must be identically zero. We will start this section showing the converse of this observation.

Proposition 3.1. *If* $\tilde\rho:M\to S^3$ *is an immersed closed minimal surface, such that* $f_B:M\to\mathbf{R}$ *vanishes for some* $B\neq\mathbf{0}$, *then* $\tilde\rho(M)$ *is invariant under the group* $\{\mathrm{e}^{tB}\colon t\in\mathbf{R}\}$, *so that M is one of the examples of Hsiang–Lawson.*

Proof. Let $X:S^3\to\mathbb{R}^4$ be the tangent vector field on S^3 given by $X(p)=Bp$. Since $0=f_B(m)=\langle B\tilde\rho(m),N(m)\rangle$, then X induces a unit tangent vector field on M. Therefore the integrals curves of the vector field X that start in $\tilde\rho(M)$ remains in $\tilde\rho(M)$, i.e. if $\tilde\rho(m)\in\tilde\rho(M)$ then $\mathrm{e}^{tB}\tilde\rho(m)\in\tilde\rho(M)$. □

We continue this section showing that if M is an example in K_5, then $hn(M)=1$.

Proposition 3.2. *If $\tilde{\rho}: M \to S^3$ is a minimal immersion of a torus in the set K_5, then, for some angle θ, $h_\theta : M \to \mathbb{R}$ vanishes, and therefore $hn(M) = 1$.*

Proof. Since $M \in K_5$, f_B vanishes for some $B \in so(4)$. As in the previous proposition, the vector field $X(m) = B\tilde{\rho}(m)$ defines a tangent vector field on M. Since the function a is invariant under isometries and X is a Killing vector field, then the function $X(a)$ is identically zero. We will prove the proposition by showing that for some fixed angle θ and some fixed real number λ, $X = \lambda a^{-\frac{1}{2}}(\cos(\theta)W_1 + \sin(\theta)W_2)$. Choose maps $\rho, \nu, V_1, V_2 : \mathbb{R}^2 \to S^3$, a covering $\tau : \mathbb{R}^2 \to M$ and a function $r : \mathbb{R}^2 \to \mathbb{R}$ using Theorem 2.4, and its corollaries, such that

$$W_1\big(\tau(u,v)\big) = V_1(u,v), \qquad W_2\big(\tau(u,v)\big) = V_2(u,v) \quad \text{and} \quad N\big(\tau(u,v)\big) = \nu(u,v).$$

With this special parametrization of this torus and having in mind that $a(\tau(u,v)) = \mathrm{e}^{2r(u,v)}$, we have that $\alpha = 0$ and

$$\begin{aligned}
V_1 &= \mathrm{e}^r \frac{\partial \rho}{\partial u},\\
V_2 &= \mathrm{e}^r \frac{\partial \rho}{\partial v},\\
W_1(a)\big(\tau(u,v)\big) &= 2\mathrm{e}^{3r(u,v)} \frac{\partial r}{\partial u}(u,v), \quad \text{and}\\
W_2(a)\big(\tau(u,v)\big) &= 2\mathrm{e}^{3r(u,v)} \frac{\partial r}{\partial v}(u,v).
\end{aligned}$$

Using the previous identities and Theorem 2.4 we can check that

$$\nabla_{W_1} W_2 = -\frac{W_2(a)}{2a} W_1 \quad \text{and} \quad \nabla_{W_2} W_1 = -\frac{W_1(a)}{2a} W_2. \tag{3.1}$$

Since X is a tangent vector field, $X(\tau(u,v)) = f(u,v)V_1(u,v) + g(u,v)V_2(u,v)$ for two doubly periodic smooth functions $f, g : \mathbb{R}^2 \to \mathbb{R}$. Since, moreover, X is a Killing vector field,

$$\begin{aligned}
\langle \nabla_{W_1} X, W_1\rangle\big(\tau(u,v)\big) &= V_1(f)(u,v) - \frac{W_2(a)}{2a}\big(\tau(u,v)\big)g(u,v) = \mathrm{e}^r\left(\frac{\partial f}{\partial u} - g\frac{\partial r}{\partial v}\right) = 0,\\
\langle \nabla_{W_2} X, W_2\rangle\big(\tau(u,v)\big) &= V_2(g)(u,v) - \frac{W_1(a)}{2a}\big(\tau(u,v)\big)f(u,v) = \mathrm{e}^r\left(\frac{\partial g}{\partial v} - f\frac{\partial r}{\partial u}\right) = 0, \quad \text{and}\\
\big(\langle \nabla_{W_1} X, W_2\rangle + \langle \nabla_{W_2} X, W_1\rangle\big)\big(\tau(u,v)\big) &= V_1(g)(u,v) + \frac{W_2(a)}{2a}\big(\tau(u,v)\big)f(u,v)\\
&\quad + V_2(f)(u,v) + \frac{W_1(a)}{2a}\big(\tau(u,v)\big)g(u,v)\\
&= \mathrm{e}^r\left(\frac{\partial g}{\partial u} + f\frac{\partial r}{\partial v} + \frac{\partial f}{\partial v} + g\frac{\partial r}{\partial u}\right)\\
&= 0.
\end{aligned}$$

A direct verification gives that the three equations above imply that the function $h(u+iv) = (\mathrm{e}^r f)(u,v) + i(\mathrm{e}^r g)(u,v)$ is an analytic function. Since h is doubly periodic in $\mathbb{R}^2$, and in particular is bounded, then we get that the function h is constant. We can write this constant as $\lambda\cos(\theta) + i\lambda\sin(\theta)$ with $\lambda \neq 0$. Since $f = \mathrm{e}^{-r}\lambda\cos(\theta)$, $g = \mathrm{e}^{-r}\lambda\sin(\theta)$ then $X = \lambda a^{-\frac{1}{2}}(\cos(\theta)W_1 + \sin(\theta)W_2)$. Since $X(a) = 0$ vanishes, then $h_\theta = \lambda^{-1}a^{-1}X(a)$ also vanishes. Notice that $hn(M)$ has to be 1, otherwise M would be a Clifford torus. □

The previous proposition shows that if H_1 is defined as in the introduction, then $K_5 \subset H_1$. The following proposition shows that H_1 is also a subset of K_5.

Proposition 3.3. *Let $\tilde{\rho}: M \to S^3$ be a minimal immersion of a torus. If for some θ, $h_\theta : M \to \mathbb{R}$ vanishes, then f_B vanishes for some nonzero skew-symmetric matrix B. Therefore, M is either a Clifford torus or a torus in K_5.*

Proof. Define the vector field X by $X = a^{-\frac{1}{2}}\cos(\theta)W_1 + a^{-\frac{1}{2}}\sin(\theta)W_2$. Using Eq. (3.1) we can prove the following identities which show that X is a Killing vector field on M.

$$\begin{aligned}
\langle \nabla_{W_1} X, W_1\rangle &= -\frac{1}{2}a^{-\frac{3}{2}}W_1(a)\cos(\theta) - a^{-\frac{1}{2}}\frac{1}{2a}W_2(a)\sin(\theta) = -\frac{1}{2a}h_\theta = 0,\\
\langle \nabla_{W_2} X, W_2\rangle &= -\frac{1}{2}a^{-\frac{3}{2}}W_2(a)\sin(\theta) - a^{-\frac{1}{2}}\frac{1}{2a}W_1(a)\cos(\theta) = -\frac{1}{2a}h_\theta = 0,\\
\langle \nabla_{W_1} X, W_2\rangle &= -\frac{1}{2}a^{-\frac{3}{2}}W_1(a)\sin(\theta) + a^{-\frac{1}{2}}\frac{1}{2a}W_2(a)\cos(\theta),\\
\langle \nabla_{W_2} X, W_1\rangle &= -\frac{1}{2}a^{-\frac{3}{2}}W_2(a)\cos(\theta) + a^{-\frac{1}{2}}\frac{1}{2a}W_1(a)\sin(\theta) = -\langle \nabla_{W_1} X, W_2\rangle.
\end{aligned}$$

Therefore the flow of the vector field X, $\Theta_X(t,\cdot): M \to M$ defines a 1-parameter group of isometries in M. By Theorem 2.3, M is invariant under a 1-parameter group of isometries of S^3, and therefore f_B vanishes for some nonzero $B \in so(4)$. □

The previous two propositions show that $H_1 = K_5$. For a minimal torus in K_5, we have that the space HS is one dimensional. What can we say about the function that spans this one dimensional space? We will prove, in Section 5, that this function is contained in KS, i.e. we will show that $HS \subset KS$.

4. Minimal tori with natural nullity less than 8

In this section we find an integrable distribution that produces *every* possible minimal torus with $nnt(M) < 8$. This distribution will be used to show that if $kn(M) = 5$, then $NS \subset KS$ and also that whenever $nnt(M) \leqslant 6$, then the group of isometries of M is not trivial.

Remark 4.1. The condition $nnt(M) < 8$ is equivalent that for some θ and some $B \in so(4)$, $h_\theta = 2f_B$.

Proof. Recall that $nnt(M) = \dim(NS)$, therefore, if $nnt(M) < 8$, then, there exist constants λ and θ and a matrix $B \in so(4)$ such that

$$\lambda h_\theta - 2f_B = 0.$$

If the space KS has dimension 6, then λ cannot be zero and then we can rescale so that $\lambda = 1$, which will give us the relation $h_\theta = 2f_B$. On the other hand, if the dimension of KS is less than 6 then M is one of the Lawson–Hsiang examples, i.e. M is either a Clifford torus or a torus in K_5. In either of these cases there exists an angle θ such that h_θ vanishes (3.2). Taking this θ and the zero matrix B, once again we obtain the relation $h_\theta = 2f_B$. □

4.1. Distributions that produce all examples of minimal tori with $nnt(M) < 8$

We define the integrable distributions $\mathcal{D}_{B,\theta}$, depending upon $B \in so(4)$ and $\theta \in S^1$, that generate all minimal immersions of the plane with $nnt(M) < 8$. As a bonus we will find a family of solutions the elliptic sinh-Gordon equation given by $\Delta r + 2\sinh(2r) = 0$, where $r: \mathbb{R}^2 \to \mathbb{R}$.

$\mathcal{D}_{B,\theta}$ is a 2-dimensional distribution in the tangent bundle

$$T\big(SO(4) \times \mathbb{R}^2\big),$$

where, for a choice of $B \in so(4)$ and $\theta \in S^1$, at $(g,r,s) = ([p,\nu,V_1,V_2],r,s) \in SO(4) \times \mathbb{R}^2$, $Z, W \in \mathfrak{X}(SO(4) \times \mathbb{R}^2)$ spanning the distribution are defined by

$$Z_{(g,r,s)} := \left(g \begin{bmatrix} 0 & 0 & -e^{-r}\cos(\theta) & -e^{-r}\sin(\theta) \\ 0 & 0 & e^{r}\cos(\theta) & -e^{r}\sin(\theta) \\ e^{-r}\cos(\theta) & -e^{r}\cos(\theta) & 0 & -s \\ e^{-r}\sin(\theta) & e^{r}\sin(\theta) & s & 0 \end{bmatrix}, \right.$$
$$\left. \langle Bp,\nu\rangle, \cos(\theta)\big\langle BV_2, e^{-r}\nu - e^{r}p\big\rangle - \sin(\theta)\big\langle BV_1, e^{-r}\nu + e^{r}p\big\rangle \right),$$

$$W_{(g,r,s)} := \left(g \begin{bmatrix} 0 & 0 & e^{-r}\sin(\theta) & -e^{-r}\cos(\theta) \\ 0 & 0 & -e^{r}\sin(\theta) & -e^{r}\cos(\theta) \\ -e^{-r}\sin(\theta) & e^{r}\sin(\theta) & 0 & \langle Bp,\nu\rangle \\ e^{-r}\cos(\theta) & e^{r}\cos(\theta) & -\langle Bp,\nu\rangle & 0 \end{bmatrix}, \right.$$
$$\left. s, e^{-2r} - e^{2r} - \sin(\theta)\big\langle BV_2, e^{-r}\nu - e^{r}p\big\rangle - \cos(\theta)\big\langle BV_1, e^{-r}\nu + e^{r}p\big\rangle \right). \tag{4.1}$$

The following theorem will be used to generate the desired family of minimal immersions, and provides a family of solutions of the elliptic sinh-Gordon equation.

Theorem 4.2. *The vector fields Z and W commute, and if we define the map $\phi : \mathbb{R}^2 \to SO(4) \times \mathbb{R}^2$ to be the immersion of the plane so that $\phi_*(\partial/\partial u) = Z$, $\phi_*(\partial/\partial v) = W$,*

$$\phi(u,v) = \big(\phi_1(u,v), \phi_2(u,v), \phi_3(u,v)\big),$$

where $\phi_1 : \mathbb{R}^2 \to SO(4)$ and $\phi_2, \phi_3 : \mathbb{R}^2 \to \mathbb{R}$, we have:

(1) *The first column of $\phi_1(u,v)$, $\phi_1(u,v)(\mathbf{e}_1)$, gives a minimal immersion of $\mathbb{R}^2$ into S^3 with principal curvature function $a = \mathrm{e}^{2r}$.*
(2) *The function $r(u,v) = \phi_2(u,v)$ solves the equation $\Delta r + 2\sinh(2r) = 0$.*

Remark 4.3. Not only will the first column of ϕ_1 give a minimal immersion, but the Gauss map is the second column and the third and fourth columns are the principal directions V_1 and V_2. So, these immersions of the plane will also have the principal directions globally defined, and $a > 0$, whether or not they are compact.

Proof. Commutativity of Z and W is a direct computation. Using the definitions from (4.1)

$$\begin{aligned}
[Z,W] = \Bigg(& Z(g)\begin{bmatrix} 0 & 0 & \mathrm{e}^{-r}\sin(\theta) & -\mathrm{e}^{-r}\cos(\theta) \\ 0 & 0 & -\mathrm{e}^{r}\sin(\theta) & -\mathrm{e}^{r}\cos(\theta) \\ -\mathrm{e}^{-r}\sin(\theta) & \mathrm{e}^{r}\sin(\theta) & 0 & \langle Bp,\nu\rangle \\ \mathrm{e}^{-r}\cos(\theta) & \mathrm{e}^{r}\cos(\theta) & -\langle Bp,\nu\rangle & 0 \end{bmatrix} \\
& + gZ\left(\begin{bmatrix} 0 & 0 & \mathrm{e}^{-r}\sin(\theta) & -\mathrm{e}^{-r}\cos(\theta) \\ 0 & 0 & -\mathrm{e}^{r}\sin(\theta) & -\mathrm{e}^{r}\cos(\theta) \\ -\mathrm{e}^{-r}\sin(\theta) & \mathrm{e}^{r}\sin(\theta) & 0 & \langle Bp,\nu\rangle \\ \mathrm{e}^{-r}\cos(\theta) & \mathrm{e}^{r}\cos(\theta) & -\langle Bp,\nu\rangle & 0 \end{bmatrix}\right) \\
& - W(g)\begin{bmatrix} 0 & 0 & -\mathrm{e}^{-r}\cos(\theta) & -\mathrm{e}^{-r}\sin(\theta) \\ 0 & 0 & \mathrm{e}^{r}\cos(\theta) & -\mathrm{e}^{r}\sin(\theta) \\ \mathrm{e}^{-r}\cos(\theta) & -\mathrm{e}^{r}\cos(\theta) & 0 & -s \\ \mathrm{e}^{-r}\sin(\theta) & \mathrm{e}^{r}\sin(\theta) & s & 0 \end{bmatrix} \\
& - gW\left(\begin{bmatrix} 0 & 0 & -\mathrm{e}^{-r}\cos(\theta) & -\mathrm{e}^{-r}\sin(\theta) \\ 0 & 0 & \mathrm{e}^{r}\cos(\theta) & -\mathrm{e}^{r}\sin(\theta) \\ \mathrm{e}^{-r}\cos(\theta) & -\mathrm{e}^{r}\cos(\theta) & 0 & -s \\ \mathrm{e}^{-r}\sin(\theta) & \mathrm{e}^{r}\sin(\theta) & s & 0 \end{bmatrix}\right), \\
& Z(s) - W\big(\langle Bp,\nu\rangle\big), Z\big(\mathrm{e}^{-2r} - \mathrm{e}^{2r} - \sin(\theta)\langle BV_2, \mathrm{e}^{-r}\nu - \mathrm{e}^{r}p\rangle - \cos(\theta)\langle BV_1, \mathrm{e}^{-r}\nu + \mathrm{e}^{r}p\rangle\big) \\
& - W\big(\cos(\theta)\langle BV_2, \mathrm{e}^{-r}\nu - \mathrm{e}^{r}p\rangle - \sin(\theta)\langle BV_1, \mathrm{e}^{-r}\nu + \mathrm{e}^{r}p\rangle\big)\Bigg).
\end{aligned}$$

Continuing, noting that $Z(g) = Z$, $W(g) = W$, $Z(p) = \mathrm{e}^{-r}\cos(\theta)V_1 + \mathrm{e}^{-r}\sin(\theta)V_2$, etc., substituting for the various derivatives and canceling massively, $[Z,W] = 0$.

We now show that $r(u,v) = \phi_2(u,v)$ is a solution of the elliptic sinh-Gordon equation. We have that

$$\begin{aligned}
\Delta r &= \frac{\partial^2 r}{\partial u^2} + \frac{\partial^2 r}{\partial v^2} = \frac{\partial\langle Bp,\nu\rangle}{\partial u} + \frac{\partial s}{\partial v} \\
&= \big\langle B\big(\mathrm{e}^{-r}\big(\cos(\theta)V_1 + \sin(\theta)V_2\big)\big), \nu\big\rangle + \big\langle Bp, \mathrm{e}^{r}\big(-\cos(\theta)V_1 + \sin(\theta)V_2\big)\big\rangle \\
&\quad - 2\sinh(2r) - \sin(\theta)\langle BV_2, \mathrm{e}^{-r}\nu - \mathrm{e}^{r}p\rangle - \cos(\theta)\langle BV_1, \mathrm{e}^{-r}\nu + \mathrm{e}^{r}p\rangle \\
&= -2\sinh(2r).
\end{aligned}$$

That $\phi_1(u,v)(\mathbf{e}_1)$ is a minimal immersion of $\mathbb{R}^2$ into S^3 is straightforward. □

Theorem 4.4. *If $\tilde{\rho} : M \to S^3$ is a minimal immersed torus in S^3 such that, for some angle θ and some matrix $B \in so(4)$, $h_\theta = 2f_B$, then it is possible to choose a covering map $\tau : \mathbb{R}^2 \to M$, maps $\rho : \mathbb{R}^2 \to S^3$, $\nu : \mathbb{R}^2 \to S^3$, $V_1, V_2 : \mathbb{R}^2 \to S^3$, and a function $r : \mathbb{R}^2 \to \mathbb{R}$ using Theorem 2.4 and its corollaries, so that*

$$\phi(u,v) = \big(\phi_1(u,v), \phi_2(u,v), \phi_3(u,v)\big) = \left(\big(\rho(u,v), \nu(u,v), V_1(u,v), V_2(u,v)\big), r(u,v), \frac{\partial r}{\partial v}(u,v)\right)$$

is a solution of the system (4.1) with matrix B and angle θ.

Proof. We can rotate coordinates so that the maps ρ, ν, V_1, and V_2 in Theorem 2.4 and Corollary 2.6 satisfy

$$V_1(u,v)=W_1\big(\tau(u,v)\big),\qquad V_2(u,v)=W_2\big(\tau(u,v)\big),\qquad \nu(u,v)=N\big(\tau(u,v)\big)\quad\text{and}\quad \alpha=\theta,$$

with $a(\tau(u,v))=\mathrm{e}^{2r}$. Since $\alpha=\theta$,

$$V_1=\mathrm{e}^r\left(\cos(\theta)\frac{\partial\rho}{\partial u}-\sin(\theta)\frac{\partial\rho}{\partial v}\right)\quad\text{and}\quad V_2=\mathrm{e}^r\left(\sin(\theta)\frac{\partial\rho}{\partial u}+\cos(\theta)\frac{\partial\rho}{\partial v}\right),$$

if $2f_B=h_\theta$, then

$$\begin{aligned}2\langle B\rho,\nu\rangle&=\cos(\theta)\mathrm{e}^{-3r}\left(\mathrm{e}^r\left(\cos(\theta)\frac{\partial\rho}{\partial u}-\sin(\theta)\frac{\partial\rho}{\partial v}\right)\right)\left(\mathrm{e}^{2r}\right)+\sin(\theta)\mathrm{e}^{-3r}\left(\mathrm{e}^r\left(\sin(\theta)\frac{\partial\rho}{\partial u}+\cos(\theta)\frac{\partial\rho}{\partial v}\right)\right)\left(\mathrm{e}^{2r}\right)\\&=2\frac{\partial r}{\partial u}.\end{aligned}$$

So

$$2\langle B\rho,\nu\rangle=2\frac{\partial r}{\partial u}=h_\theta \tag{4.2}$$

and, similarly,

$$2\frac{\partial r}{\partial v}=2s=h_{\theta+\frac{\pi}{2}}. \tag{4.3}$$

From the formulas for V_1 and V_2 in Corollary (2.6), we have that

$$\frac{\partial\rho}{\partial u}=\mathrm{e}^{-r}\big(V_1\cos(\theta)+\sin(\theta)V_2\big)\quad\text{and}\quad\frac{\partial\rho}{\partial v}=\mathrm{e}^{-r}\big(-V_1\sin(\theta)+\sin(\theta)V_2\big).$$

Also, using the equation above and the formula for $\frac{\partial\nu}{\partial u}$ and $\frac{\partial\nu}{\partial v}$ in Theorem 2.4, we get that

$$\frac{\partial\nu}{\partial u}=\mathrm{e}^r\big(-\cos(\theta)V_1+\sin(\theta)V_2\big)\quad\text{and}\quad\frac{\partial\nu}{\partial v}=\mathrm{e}^r\big(\sin(\theta)V_1+\cos(\theta)V_2\big).$$

A direct computation shows that derivatives of $\frac{\partial V_i}{\partial u}$ combine with the above to satisfy the equations for ϕ_1 to be an integral submanifold of the distribution. In order to complete the proof of this theorem, let us check the equation for $\frac{\partial s}{\partial v}$. We have that

$$\begin{aligned}\frac{\partial s}{\partial v}=\frac{\partial^2 r}{\partial v^2}&=-2\sinh(2r)-\frac{\partial^2 r}{\partial u^2}\\&=-2\sinh(2r)-\frac{\partial}{\partial u}\langle B\rho,\nu\rangle\\&=-2\sinh(2r)-\left\langle B\frac{\partial\rho}{\partial u},\nu\right\rangle-\left\langle B\rho,\frac{\partial\nu}{\partial u}\right\rangle\\&=-\sin(\theta)\big\langle BV_2,\mathrm{e}^{-r}\nu-\mathrm{e}^r p\big\rangle-\cos(\theta)\big\langle BV_1,\mathrm{e}^{-r}\nu+\mathrm{e}^r p\big\rangle,\end{aligned}$$

which verifies the equation in the system (4.1). The equation for $\frac{\partial s}{\partial u}$ is similar. □

Remark 4.5. Arguing as in the proof of the previous theorem, if

$$\phi(u,v)=\big(\rho(u,v),\nu(u,v),V_1(u,v),V_2(u,v),r(u,v),s(u,v)\big)$$

is a doubly-periodic solution of the system (4.1) and M is the torus $\frac{\mathbb{R}^2}{\sim}$, then,

$$h_\theta(u,v)=2\frac{\partial r}{\partial u}(u,v)\quad\text{and}\quad h_{\theta+\frac{\pi}{2}}(u,v)=2\frac{\partial r}{\partial v}(u,v)=2s.$$

Moreover, for any 4×4 skew-symmetric matrix $\tilde{B}$, $f_{\tilde{B}}(u,v)=\langle\tilde{B}\rho(u,v),\nu(u,v)\rangle$. Finally, since ϕ satisfies the system (4.1), then $h_\theta=2f_B$.

Note 4.6. It follows that doubly-periodic solutions of the system (4.1) induce minimal immersions of tori with natural nullity less than 8, since for the B and θ defining the distribution, $2f_B=h_\theta$. So far, the authors have not been able to find a method to determine which solutions are doubly periodic.

The previous theorem shows that for any choice of $B \in so(4)$, $\theta \in S^1$ and $\mathbf{x}_0 \in SO(4) \times \mathbb{R}^2$ we have a solution of the sinh-Gordon equation. The following theorem shows that this solution and its derivatives are defined in the whole plane and are bounded. Recall that $\frac{\partial r}{\partial v} = s$ and that $\frac{\partial r}{\partial u}$ is an algebraic function of the component functions of $(\phi_1(u,v), \phi_2(u,v), \phi_3(u,v))$.

Theorem 4.7. *The functions $\phi_1(u,v), \phi_2(u,v) = r(u,v)$, and $\phi_3(u,v) = s(u,v)$ are defined in the whole plane and are bounded in $T_*(SO(4) \times \mathbb{R}^2)$.*

The proof of this result appears in Appendix A.

The main tool we use to study minimal tori with natural nullity less than 8 is that we have a representation of them in term of integral submanifolds of the distribution $\mathcal{D}_{B,\theta}$ (4.1). Recall that by Remark 4.1, for every torus $M \subset S^3$ with $nnt(M) < 8$ there exist θ and $B \in so(4)$ such that $h_\theta = 2 f_B$.

4.2. Auxiliary identities

In order to simplify the study of the system (4.1) we give additional relationships among components of the solutions.

Theorem 4.8. *Let $\phi_1 := (p, \nu, V_1, V_2) : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to SO(4)$ and $\phi_2, \phi_3 := r, s : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to \mathbb{R}$ be a solution of the system* (4.1)*, that is, an integral submanifold of $\mathcal{D}_{B,\theta}$. If $\tilde{B} \in so(4)$ is any skew symmetric matrix, and if we define the functions*

$$\xi_1 = \langle \tilde{B}p, \nu\rangle, \qquad \xi_2 = \langle \tilde{B}V_1, V_2\rangle, \qquad \xi_3 = \langle \tilde{B}V_1, p\rangle, \qquad \xi_4 = \langle \tilde{B}V_2, p\rangle, \qquad \xi_5 = \langle \tilde{B}V_1, \nu\rangle, \qquad \xi_6 = \langle \tilde{B}V_2, \nu\rangle$$

then, the following identities hold:

$$\frac{\partial}{\partial u}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & e^r\cos(\theta) & -e^r\sin(\theta) & e^{-r}\cos(\theta) & e^{-r}\sin(\theta) \\ 0 & 0 & -e^{-r}\sin(\theta) & e^{-r}\cos(\theta) & -e^r\sin(\theta) & -e^r\cos(\theta) \\ -e^r\cos(\theta) & e^{-r}\sin(\theta) & 0 & s & 0 & 0 \\ e^r\sin(\theta) & -e^{-r}\cos(\theta) & -s & 0 & 0 & 0 \\ -e^{-r}\cos(\theta) & e^r\sin(\theta) & 0 & 0 & 0 & s \\ -e^{-r}\sin(\theta) & e^r\cos(\theta) & 0 & 0 & -s & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix}$$

and

$$\frac{\partial}{\partial v}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -e^r\sin(\theta) & -e^r\cos(\theta) & -e^{-r}\sin(\theta) & e^{-r}\cos(\theta) \\ 0 & 0 & -e^{-r}\cos(\theta) & -e^{-r}\sin(\theta) & -e^r\cos(\theta) & e^r\sin(\theta) \\ e^r\sin(\theta) & e^{-r}\cos(\theta) & 0 & -\langle Bp, \nu\rangle & 0 & 0 \\ e^r\cos(\theta) & e^{-r}\sin(\theta) & \langle Bp, \nu\rangle & 0 & 0 & 0 \\ e^{-r}\sin(\theta) & e^r\cos(\theta) & 0 & 0 & 0 & -\langle Bp, \nu\rangle \\ -e^{-r}\cos(\theta) & -e^r\sin(\theta) & 0 & 0 & \langle Bp, \nu\rangle & 0 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix}.$$

Proof. This is a long direct computation. □

4.3. Solutions of the system with $hn(M) < 2$ and natural nullity of the Lawson–Hsiang examples

The following theorem characterizes the integral submanifolds of the system (4.1) that contain every torus M with $hn(M) < 2$ in terms of the matrix B. Recall from Eq. (4.3) in the proof of Theorem 4.4 that $s(u,v) = \frac{\partial r}{\partial v}$, so that $s = 0$ implies that $hn(M) < 2$.

Theorem 4.9. *Let $\phi : \mathbb{R}^2 \to SO(4) \times \mathbb{R}^2$, $\phi = (\phi_1, \phi_2, \phi_3)$, be an integral submanifold of $\mathcal{D}_{B,\theta}$, and let $r(u,v) = \phi_2(u,v)$ and $s(u,v) = \phi_3(u,v)$. Assume that $\phi(0,0) = x^0 = (I, r_0, 0)$ and $\frac{\partial r}{\partial u}(0,0) = 0$. If*

$$B = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ -b_1 & 0 & b_4 & b_5 \\ -b_2 & -b_4 & 0 & b_6 \\ -b_3 & -b_5 & -b_6 & 0 \end{pmatrix},$$

then, s vanishes, and so $hn(M) < 2$, if and only if $b_1 = b_6 = 0$ and

(1) $-e^{r_0}\cos(\theta)b_2 + e^{r_0}\sin(\theta)b_3 - e^{-r_0}\cos(\theta)b_4 - e^{-r_0}\sin(\theta)b_5 = 2\sinh(2r_0)$,

(2) $-e^{r_0}\sin(\theta)b_2 - e^{r_0}\cos(\theta)b_3 - e^{-r_0}\sin(\theta)b_4 + e^{-r_0}\cos(\theta)b_5 = 0$, *and*

(3) $-e^{-r_0}\cos(\theta)b_2 - e^{-r_0}\sin(\theta)b_3 - e^{r_0}\cos(\theta)b_4 + e^{r_0}\sin(\theta)b_5 = 0$.

Proof. We will use the identities of Theorem 4.8 with $\tilde{B} = B$. Notice that

$$b_1 = -\xi_1(0,0), \qquad b_6 = -\xi_2(0,0), \qquad b_2 = \xi_3(0,0), \qquad b_3 = \xi_4(0,0), \qquad b_4 = \xi_5(0,0), \qquad b_3 = \xi_6(0,0).$$

Assume that $s(u,v) = 0$ for every $(u,v) \in \mathbb{R}^2$. The equation $b_1 = 0$ follows because we are assuming that $\frac{\partial r}{\partial u}(0,0) = \xi_1(0,0) = 0$. Eq. (1) in the statement of the theorem follows from the equation $\frac{\partial s}{\partial v}(0,0) = 0$. Eq. (2) follows from the equation $\frac{\partial s}{\partial u}(0,0) = 0$. We now prove that $s \equiv 0$ also implies that $b_6 = 0$ and Eq. (3) in the statement of the theorem.

A direct computation shows the following two equations:

$$\begin{aligned}\frac{\partial^2 s}{\partial v \partial u} &= \xi_1\big(-2\cosh(2r) + \mathrm{e}^r\big(\sin(\theta)\xi_4 - \cos(\theta)\xi_3\big) + \mathrm{e}^{-r}\big(\sin(\theta)\xi_6 + \cos(\theta)\xi_5\big)\big) \\ &\quad + s\big(-\mathrm{e}^r\big(\sin(\theta)\xi_3 + \cos(\theta)\xi_4\big) + \mathrm{e}^{-r}\big(\sin(\theta)\xi_5 - \cos(\theta)\xi_6\big)\big) - 2\sin(2\theta)\xi_2\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 s}{\partial v^2} &= s\big(-4\cosh(2r) + \mathrm{e}^r\big(\sin(\theta)\xi_4 - \cos(\theta)\xi_3\big) + \mathrm{e}^{-r}\big(\sin(\theta)\xi_6 + \cos(\theta)\xi_5\big)\big) \\ &\quad + \xi_1\big(\mathrm{e}^r\big(\sin(\theta)\xi_3 + \cos(\theta)\xi_4\big) + \mathrm{e}^{-r}\big(\cos(\theta)\xi_6 - \sin(\theta)\xi_5\big)\big) - 2\cos(2\theta)\xi_2.\end{aligned}$$

From these equations we get that $\xi_2(0,0) = -b_6 = 0$ and that $\frac{\partial \xi_2}{\partial v}(0,0) = 0$ because $\xi_1(0,0) = 0$, and

$$\frac{\partial \xi_1}{\partial v}(0,0) = \frac{\partial s}{\partial u}(0,0) = 0.$$

A direct computation shows that Eq. (3) in the statement of the theorem is equivalent to the equation $\frac{\partial \xi_2}{\partial v}(0,0) = 0$. So we have shown one implication in the theorem.

We now show the other implication. Assume that Eqs. (1), (2) and (3) of the statement of the theorem hold, and also $b_1 = b_6 = 0$. These 5 conditions are equivalent to the conditions

$$\xi_1(0,0) = 0, \qquad \xi_2(0,0) = 0, \qquad \frac{\partial \xi_1}{\partial v}(0,0) = \frac{\partial s}{\partial u}(0,0) = 0, \qquad \frac{\partial s}{\partial v}(0,0) = 0, \quad \text{and} \quad \frac{\partial \xi_2}{\partial v}(0,0) = 0.$$

Notice also that by assumption $s(0,0) = 0$. Using the identities of Theorem 4.8, the initial conditions above imply that

$$\frac{\partial \xi_i}{\partial u}(0,0) = \frac{\partial \xi_i}{\partial v}(0,0) = 0, \quad \text{for } i = 2,3,5,6, \tag{4.4}$$

and, also, by induction, given $n \geqslant 1$, k and l nonnegative integers such that $k + l = n$, there exists a polynomial $P = P(t_1,\ldots,t_9)$ such that

$$\frac{\partial^n r}{\partial u^l \partial v^k} = P\big(\mathrm{e}^r, \mathrm{e}^{-r}, s, \xi_1, \ldots, \xi_6\big).$$

Along with the equations in (4.4), these equations imply that

$$\frac{\partial^n s}{\partial u^l \partial v^{k+1}}(0,0) = \frac{\partial\big(\frac{\partial^n r}{\partial u^l \partial v^k}\big)}{\partial v}(0,0) = \frac{\partial P(\mathrm{e}^r, \mathrm{e}^{-r}, s, \xi_1, \ldots, \xi_6)}{\partial v}(0,0) = 0.$$

In the last equation we also used the hypothesis that $\frac{\partial \xi_1}{\partial v}(0,0) = \frac{\partial \xi_2}{\partial v}(0,0) = 0$. We should point out that we have used the fact that the function r is real analytic, which follows from the fact that $\Delta r + 2\sinh(2r) = 0$. □

The next theorem shows that for the Lawson–Hsiang examples not only is $kn(M) = 5$ but also $nnt(M) = 5$ by showing that the space $NS \subset KS$.

Theorem 4.10. *If $M \subset S^3$ is an immersed minimal torus invariant under a one-parameter group of isometries of S^3, then $nnt(M) = kn(M)$ and therefore the natural nullity $nnt(M) \leqslant 5$.*

Proof. By Proposition 3.2 we know that for some angle θ, $(\cos(\theta)V_1 + \sin(\theta)V_2)(a) = 0$ where $a : M \to \mathbf{R}$ is a positive function such that the principal curvatures of M at p are $\pm a(p)$. Without loss of generality, we can assume that

$$e_1 \in M, \qquad \nu(e_1) = e_2, \qquad V_1(e_1) = e_3, \qquad V_2(e_1) = e_4, \qquad \ln a(e_1) = 2r_0, \quad \text{and} \quad \nabla a(e_1) = \mathbf{0}.$$

Therefore, M defines a solution of the system (4.1) associated with the matrix $B = \mathbf{0}$ and θ. Call this solution $\phi : \mathbb{R}^2 \to SO(4) \times \mathbb{R}^2$. Without loss of generality we can assume that $\phi(0,0) = (I, r_0, 0)$.

Define $\tilde{\phi}$ to be the solution of the system (4.1) associated with a matrix $B = \{b_{ij}\}$ that satisfies the conditions in the previous lemma and $\tilde{\theta} = \theta - \frac{\pi}{2}$. Moreover we will take the initial solution that satisfies

$$\tilde{\phi}(0,0) = (I, r_0, 0).$$

Now consider the map $\hat{\phi} : \mathbb{R}^2 \to SO(4) \times \mathbb{R}^2$ given by

$$\begin{aligned}\hat{\phi}(u,v) &= \big(\big(\hat{\rho}(u,v), \hat{\nu}(u,v), \hat{V_1}(u,v), \hat{V_2}(u,v)\big), \hat{r}(u,v), \hat{s}(u,v)\big)\\ &= \big(\big(\tilde{\rho}(-v,u), \tilde{\nu}(-v,u), \tilde{V_1}(-v,u), \tilde{V_2}(-v,u)\big), \tilde{r}(-v,u), -\langle B\tilde{\rho}, \tilde{\nu}\rangle\big),\end{aligned}$$

where

$$\tilde{\phi}(\tilde{u},\tilde{v}) = \big(\big(\tilde{\rho}(\tilde{u},\tilde{v}), \tilde{\nu}(\tilde{u},\tilde{v}), \tilde{V_1}(\tilde{u},\tilde{v}), \tilde{V_2}(\tilde{u},\tilde{v})\big), \tilde{r}(\tilde{u},\tilde{v}), \tilde{s}(\tilde{u},\tilde{v})\big).$$

It is clear that $\hat{\phi}(0,0) = (I, r_0, 0)$. Notice that, by the way B was chosen, we have that $\tilde{s} = 0$ for every $(\tilde{u},\tilde{v}) \in \mathbb{R}^2$. Also, a direct computation shows that $\hat{\phi}$ is a solution of the system (4.1) with $B = \mathbf{0}$ and the angle θ, therefore, $\hat{\phi}(u,v) = \phi(u,v)$, and so

$$\frac{\partial r}{\partial v} = -\frac{\partial \tilde{r}}{\partial \tilde{u}} = -\langle B\rho, \nu\rangle.$$

This equality is equivalent to the fact that $\sin(\theta)u_1 - \cos(\theta)u_2 = f_B$, where the functions $u_1 = h_0$, $u_2 = h_{\pi/2}$, and f_B are defined in the first section. This last equation implies that $h_{\theta+\frac{\pi}{2}} = -f_B$, therefore, h_θ, which is identically zero, and $h_{\theta+\frac{\pi}{2}}$ are functions in $\{f_C\colon\ C \in so(4)\}$. Then, both functions u_1 and u_2 are also generated by the functions in the set $\{f_C\colon\ C \in so(4)\}$, i.e. the natural nullity is 5. Recall that the space $\{u_C\colon\ C \in so(4)\}$ is 5-dimensional for any torus invariant under a 1-parameter group of isometries in S^3. □

The results in Section 3 show that for a torus, the condition $kn(M) < 6$ is equivalent to the condition $hn(M) < 2$. Therefore, M is invariant under a group of isometries $\{e^{tB}\colon\ t \in \mathbf{R}\}$, if and only if, the function $a : M \to \mathbf{R}$ is invariant under a constant direction with respect to the principal directions. The following corollary establishes this relationship.

Corollary 4.11. *If M is a minimal immersed torus in S^3, then $nnt(M) \leqslant 5$ if and only if M is one of the examples of Hsiang and Lawson.*

Proof. If M has $nnt(M) \leqslant 5$, then $kn(M) \leqslant 5$. Therefore, for some nonzero skew-symmetric matrix B, f_B vanishes. By Proposition 3.1, M will be invariant under a 1-parameter subgroup of the rigid motions of S^3, which, following [2], implies that M is one of Hsiang and Lawson's examples. On the other hand, since any of the Hsiang–Lawson examples are preserved by a one-parameter subgroup of $SO(4)$, there is a $B \in so(4)$ for which $f_B = 0$. Then Theorem 4.10 implies $nnt(M) \leqslant 5$. □

4.4. Symmetry of tori with natural nullity less than 7

In this subsection we will prove that if the natural nullity of a torus is less than 7, then the group of isometries is not trivial. Let us start with the following lemma.

Lemma 4.12. *If for any solution of the system* (4.1), *the functions $\xi_1 \ldots \xi_6$ defined in Theorem* 4.8 *satisfy $r(0,0) = r_0$, $\xi_1(0,0) = s(0,0) = \xi_4(0,0) = 0$, then $r(u,v) = r(-u,-v)$.*

Proof. A direct computation using the identities of Theorem 4.8 shows that the conditions $\xi_1(0,0) = s(0,0) = \xi_2(0,0) = 0$ give

$$\frac{\partial \xi_i}{\partial u}(0,0) = \frac{\partial \xi_i}{\partial v}(0,0) = 0 \quad \text{for } i = 3, 4, 5, 6.$$

Let $C^\omega(\mathbb{R}^2)$ be the set of analytic functions on $\mathbb{R}^2$ and let P_0 be the ideal of $C^\omega(\mathbb{R}^2)$ generated by the functions $\{e^r, e^{-r}, \xi_2, \xi_3, \xi_5, \xi_6\}$. Given a nonnegative integer k, define P_k as the set of functions in $C^\omega(\mathbb{R}^2)$ that can be written as a homogeneous polynomial of degree k in the variables s, ξ_1 and ξ_2 with coefficients in P_0. A direct computation using again the identities in Theorem 4.8 give us that if $f \in P_0$, then $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are in P_1. In the same way, if $f \in P_k$ then $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ are in $P_{k+1} + P_{k-1}$. Now with these observations in mind, we proceed to show that the function r satisfies $r(u,v) = r(-u,-v)$, by showing that all the partial derivatives of odd order of the function r vanish at $(0,0)$. To achieve this we first notice that the first derivatives of r, the functions ξ_1 and s vanish at $(0,0)$. Then, notice that the second derivatives of r, i.e. the first derivatives of s and ξ_1, are functions in P_0. The last statement implies that the third derivatives of r are in P_1 and therefore vanish at $(0,0)$. Once we know that the third derivatives of r are in P_1 we get that the fourth derivatives or r are in P_0+P_2. If we continue with this process we notice that if k is a positive even integer, then the k-th derivatives of r are functions in $P_0+P_2+\cdots+P_{k-2}$, and in the case that k is an odd integer greater that 1, then, the k-th derivatives of r are in $P_1+P_3+\cdots+P_{k-2}$. Now, since $\xi_1(0,0)=s(0,0)=\xi_2(0,0)=0$, the odd derivatives of the function r vanish at $(0,0)$. $\square$

Theorem 4.13. *Let M be a minimal torus immersed in S^3. If $nnt(M)\leqslant 6$, then the group of isometries of M is not trivial.*

Proof. Unless there is some nonzero $B\in so(4)$ for which $f_B=0$, in which case Proposition 3.1 implies the existence of a one-parameter group of isometries of S^3 which restrict to isometries of M, then $nnt(M)\leqslant 6$ implies that the span of $\{u_1,u_2\}$, $u_1:=a^{-\frac{3}{2}}W_1(a)=h_0$ and $u_2:=a^{-\frac{3}{2}}W_2(a)=h_{\frac{\pi}{2}}$, will be contained in the span of $\{f_B|B\in so(4)\}$. Since then $u_1=2f_B$ for some $B\in so(4)$, then M defines a solution ϕ of the system (4.1) associated with the matrix B and with $\theta=0$. The condition $u_2=2f_{\tilde{B}}$ implies by Remark 4.5 that $s=\tilde{\xi_1}$, for the identities of Theorem 4.8 associated with two distinct matrices B, $\tilde{B}$ and $\theta=0$. As before, we will assume that $\xi_1(0,0)=s(0,0)=0$ and $r(0,0)=r_0$. Define the function $f=s-\tilde{\xi}_1$. The hypothesis in the theorem is equivalent to the condition that f is identically zero, in particular, $\tilde{\xi}_1(0,0)=0$, since $f(0,0)=0$. The theorem is a consequence of the previous lemma and will follow by showing that $\xi_2(0,0)=0$. A direct computation shows that

$$\frac{\partial f}{\partial u}=\mathrm{e}^{-r}\xi_6-\mathrm{e}^{r}\xi_4-\mathrm{e}^{-r}\tilde{\xi_5}-\mathrm{e}^{r}\tilde{\xi_3}$$

and

$$\begin{aligned}\frac{\partial^2 f}{\partial u^2}&=\xi_1\big(-\mathrm{e}^{-r}\xi_6-\mathrm{e}^{r}\xi_4+\mathrm{e}^{-r}\tilde{\xi}_5-\mathrm{e}^{r}\tilde{\xi}_3\big)+\mathrm{e}^{-r}\big(-s\xi_5+\mathrm{e}^{r}\xi_2\big)-\mathrm{e}^{r}\big(-s\xi_3-\mathrm{e}^{-r}\xi_2\big)\\&\quad-\mathrm{e}^{-r}\big(-s\xi_5+\mathrm{e}^{r}\xi_2\big)-\mathrm{e}^{r}\big(-s\xi_3-\mathrm{e}^{-r}\xi_2\big)-\mathrm{e}^{r}\big(s\tilde{\xi_6}-\mathrm{e}^{-r}\xi_1\big)-\mathrm{e}^{r}\big(s\tilde{\xi}_4-\mathrm{e}^{r}\tilde{\xi}_1\big)\\&=\xi_1\big(-\mathrm{e}^{-r}\xi_6-\mathrm{e}^{r}\xi_4-\mathrm{e}^{-r}\tilde{\xi}_5-\mathrm{e}^{r}\tilde{\xi}_3\big)+s\big(-\mathrm{e}^{-r}\xi_5+\mathrm{e}^{r}\xi_3-\mathrm{e}^{-r}\tilde{\xi}_6-\mathrm{e}^{r}\tilde{\xi}_4\big)+2\xi_2+2\cosh(2r)\tilde{\xi_1}.\end{aligned}$$

From the last equation, using the fact that $s(0,0)=\xi_1(0,0)=\tilde{\xi_1}(0,0)$ and $\frac{\partial^2 f}{\partial u^2}=0$, we conclude that $\xi_2(0,0)=0$, which implies, by the previous lemma, that $r(u,v)=r(-u,-v)$. To finish the proof of the theorem, we notice that the function $A(u,v)=-(u,v)$ preserves the lattice in $\mathbb{R}^2$ given by the double-periodicity of the function ϕ and therefore induces a function in the torus $\tau(\mathbb{R}^2)=M$, since the first fundamental form of M in the coordinates u and v is $ce^{-2r}(du^2+dv^2)$ where c is a positive constant, then, this function from M to M induced by A is an isometry. $\square$

Appendix A. First integrals and existence of global solutions

In this subsection we prove Theorem 4.7, that the integral submanifolds of $\mathcal{D}_{B,\theta}$ are defined in the whole of $\mathbb{R}^2$. The theorem will follow from the following lemmas.

Lemma A.1. *For a given solution of the system* (4.1)*, the functions $\xi_1,\dots,\xi_6$ defined in Theorem* 4.8 *satisfy the condition that*

$$M=\frac{1}{2}\big\{\xi_1^2+\cdots+\xi_6^2\big\}$$

is a constant.

Proof. A direct computation using Theorem 4.8 gives us that

$$\begin{aligned}\frac{\partial M}{\partial u}&=\xi_1\frac{\partial\xi_1}{\partial u}+\cdots+\xi_6\frac{\partial\xi_6}{\partial u}\\&=\xi_1\big(\mathrm{e}^{r}\big(\cos(\theta)\xi_3-\sin(\theta)\xi_4\big)+\mathrm{e}^{-r}\big(\cos(\theta)\xi_5+\sin(\theta)\xi_6\big)\big)+\xi_3\big(s\xi_4-\mathrm{e}^{r}\cos(\theta)\xi_1+\mathrm{e}^{-r}\sin(\theta)\xi_2\big)\\&\quad+\xi_4\big(-s\xi_3+\mathrm{e}^{r}\sin(\theta)\xi_1-\mathrm{e}^{-r}\cos(\theta)\xi_2\big)+\xi_2\big(\mathrm{e}^{r}\big(-\sin(\theta)\xi_5-\cos(\theta)\xi_6\big)+\mathrm{e}^{-r}\big(\cos(\theta)\xi_4-\sin(\theta)\xi_3\big)\big)\\&\quad+\xi_5\big(s\xi_6+\mathrm{e}^{r}\sin(\theta)\xi_2-\mathrm{e}^{-r}\cos(\theta)\xi_1\big)+\xi_6\big(-s\xi_5+\mathrm{e}^{r}\cos(\theta)\xi_2-\mathrm{e}^{-r}\sin(\theta)\xi_1\big)\\&=0.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial M}{\partial v}&=\xi_1\frac{\partial\xi_1}{\partial v}+\cdots+\xi_6\frac{\partial\xi_6}{\partial v}\\&=\xi_1\big(-\mathrm{e}^{r}\big(\cos(\theta)\xi_4+\sin(\theta)\xi_3\big)+\mathrm{e}^{-r}\big(\cos(\theta)\xi_6-\sin(\theta)\xi_5\big)\big)+\xi_3\big(-\xi_1\xi_4+\mathrm{e}^{r}\sin(\theta)\xi_1+\mathrm{e}^{-r}\cos(\theta)\xi_2\big)\\&\quad+\xi_4\big(\xi_1\xi_3+\mathrm{e}^{r}\cos(\theta)\xi_1+\mathrm{e}^{-r}\sin(\theta)\xi_2\big)+\xi_2\big(\mathrm{e}^{-r}\big(-\cos(\theta)\xi_3-\sin(\theta)\xi_4\big)\big)\end{aligned}$$

$$
\begin{aligned}
&+ \xi_5\bigl(-\xi_1\xi_6 + \mathrm{e}^r\cos(\theta)\xi_2 + \mathrm{e}^{-r}\sin(\theta)\xi_1\bigr) + \xi_6\bigl(\xi_1\xi_5 - \mathrm{e}^r\sin(\theta)\xi_2 - \mathrm{e}^{-r}\cos(\theta)\xi_1\bigr)\\
&= 0,
\end{aligned}
$$

therefore, M is a constant. □

Lemma A.2. *For a given solution of the system* (4.1),

$$E = \frac{1}{2}\bigl\{\langle p, p\rangle + \langle V_1, V_1\rangle + \langle V_2, V_2\rangle + \langle \nu, \nu\rangle\bigr\}$$

is a constant.

Proof. As in the proof of the previous lemma, a direct computation shows that $\frac{\partial E}{\partial u} = \frac{\partial E}{\partial v} = 0$. □

Lemma A.3. *For a given solution of the system* (4.1), *the functions* $\xi_1, \dots, \xi_6$ *defined in Theorem* 4.8 *satisfy the identity that*

$$A = \mathrm{e}^r\bigl(\cos(\theta)\xi_3 - \sin(\theta)\xi_4\bigr) - \mathrm{e}^{-r}\bigl(\cos(\theta)\xi_5 + \sin(\theta)\xi_6\bigr) + \frac{1}{2}s^2 + \cosh(2r) - \frac{1}{2}(\xi_1)^2$$

is a constant.

Proof. Similarly to the previous two lemmas, we prove that $\frac{\partial A}{\partial u} = \frac{\partial A}{\partial v} = 0$.

Denote by

$$
\begin{aligned}
&B = \mathrm{e}^r\bigl(\cos(\theta)\xi_3 - \sin(\theta)\xi_4\bigr) - \mathrm{e}^{-r}\bigl(\cos(\theta)\xi_5 + \sin(\theta)\xi_6\bigr) \quad \text{and}\\
&C = \frac{\partial \xi_1}{\partial u} = \mathrm{e}^r\bigl(\cos(\theta)\xi_3 - \sin(\theta)\xi_4\bigr) + \mathrm{e}^{-r}\bigl(\cos(\theta)\xi_5 + \sin(\theta)\xi_6\bigr).
\end{aligned}
$$

Notice that $B + \frac{1}{2}s^2 - \frac{1}{2}\xi_1^2 + \cosh(2r) = A$. A direct computation shows that

$$
\begin{aligned}
\frac{\partial B}{\partial u} &= \xi_1 C + \mathrm{e}^r\bigl\{\cos(\theta)\bigl(s\xi_4 - \mathrm{e}^r\cos(\theta)\xi_1 + \mathrm{e}^{-r}\sin(\theta)\xi_2\bigr) - \sin(\theta)\bigl(-s\xi_3 + \mathrm{e}^r\sin(\theta)\xi_1 - \mathrm{e}^{-r}\cos(\theta)\xi_2\bigr)\bigr\}\\
&\quad - \mathrm{e}^{-r}\bigl\{\cos(\theta)\bigl(s\xi_6 + \mathrm{e}^r\sin(\theta)\xi_2 - \mathrm{e}^{-r}\cos(\theta)\xi_1\bigr) + \sin(\theta)\bigl(-s\xi_5 + \mathrm{e}^r\cos(\theta)\xi_2 - \mathrm{e}^{-r}\sin(\theta)\xi_1\bigr)\bigr\}\\
&= \xi_1\frac{\partial \xi_1}{\partial u} + s\bigl(\mathrm{e}^r\cos(\theta)\xi_4 + \mathrm{e}^r\sin(\theta)\xi_3 - \mathrm{e}^{-r}\cos(\theta)\xi_6 + \mathrm{e}^{-r}\sin(\theta)\xi_5\bigr)\\
&\quad + \xi_2\bigl(\cos(\theta)\sin(\theta) + \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta)\bigr)\\
&\quad + \xi_1\bigl(-\mathrm{e}^{2r}\cos^2(\theta)\xi_4 - \mathrm{e}^{2r}\sin^2(\theta)\xi_3 + \mathrm{e}^{-2r}\cos^2(\theta) + \mathrm{e}^{-2r}\sin^2(\theta)\bigr)\\
&= \xi_1\frac{\partial \xi_1}{\partial u} - s\frac{\partial s}{\partial u} - 2\xi_1\sinh(2r)\\
&= \frac{1}{2}\frac{\partial \xi_1^2}{\partial u} - \frac{1}{2}\frac{\partial s^2}{\partial u} - \frac{\partial\cosh(2r)}{\partial u}.
\end{aligned}
$$

Therefore $\frac{\partial A}{\partial u} = 0$. Similarly,

$$
\begin{aligned}
\frac{\partial B}{\partial v} &= sC + \mathrm{e}^r\bigl\{\cos(\theta)\bigl(-\xi_1\xi_4 + \mathrm{e}^r\sin(\theta)\xi_1 + \mathrm{e}^{-r}\cos(\theta)\xi_2\bigr) - \sin(\theta)\bigl(-\xi_1\xi_3 + \mathrm{e}^r\cos(\theta)\xi_1 + \mathrm{e}^{-r}\sin(\theta)\xi_2\bigr)\bigr\}\\
&\quad - \mathrm{e}^{-r}\bigl\{\cos(\theta)\bigl(-\xi_1\xi_6 + \mathrm{e}^r\cos(\theta)\xi_2 + \mathrm{e}^{-r}\sin(\theta)\xi_1\bigr) + \sin(\theta)\bigl(\xi_1\xi_5 - \mathrm{e}^r\sin(\theta)\xi_2 - \mathrm{e}^{-r}\cos(\theta)\xi_1\bigr)\bigr\}\\
&= s\left(-2\sinh(2r) - \frac{\partial s}{\partial v}\right) + \xi_1\bigl(-\mathrm{e}^r\cos(\theta)\xi_4 + \mathrm{e}^{2r}\cos(\theta)\sin(\theta) - \mathrm{e}^{2r}\sin(\theta)\cos(\theta) - \mathrm{e}^r\sin(\theta)\xi_3\bigr)\\
&\quad + \mathrm{e}^{-r}\cos(\theta)\xi_6 - \mathrm{e}^{-2r}\cos(\theta)\sin(\theta) - \mathrm{e}^{-r}\sin(\theta)\xi_5 + \mathrm{e}^{-2r}\sin(\theta)\cos(\theta)\\
&\quad + \xi_2\bigl(\cos^2(\theta) - \sin^2(\theta) + \cos^2(\theta) + \sin^2(\theta)\bigr)\\
&= -\frac{1}{2}\frac{\partial s^2}{\partial v} - \frac{\partial\cosh(2r)}{\partial v} + \frac{1}{2}\frac{\partial \xi_1^2}{\partial v}. \qquad \square
\end{aligned}
$$

Lemma A.4. *Given a solution of the system* (4.1). *If M and A are the constants given by Lemmas* A.1 *and* A.3, *respectively, if* (u_0, v_0) *is any point in the domain of the solution, and if R is a real number such that*

$$\cosh(2R) > A + 4M\cosh(R) + \frac{M^2}{2} \quad \textit{and} \quad R > \bigl|r(u_0, v_0)\bigr|,$$

then, $|r(u,v)| < R$ *and*

$$\frac{1}{2}s^2(u,v) + \cosh\bigl(2r(u,v)\bigr) \leqslant A + \frac{M^2}{2} + \cosh(2R)$$

for any (u,v) *in the domain of the solution.*

Proof. We have that

$$\begin{aligned}\frac{1}{2}s^2(u,v) + \cosh\bigl(2r(u,v)\bigr) &= A + \frac{1}{2}\xi_1^2 + \mathrm{e}^{-r}\bigl(\cos(\theta)\xi_5 + \sin(\theta)\xi_6\bigr) - \mathrm{e}^{r}\bigl(\cos(\theta)\xi_3 - \sin(\theta)\xi_4\bigr)\\ &\leqslant A + \frac{M^2}{2} + 4M\cosh(r).\end{aligned}$$

This inequality above shows that the result will follow once we prove that $|r(u,v)| \leqslant R$. We prove that $|r(u,v)| < R$ by contradiction. If, for some (u,v), $|r(u,v)| = R$, then, the inequality above implies that at that (u,v),

$$\cosh(2R) \leqslant A + \frac{M^2}{2} + 4M\cosh(R).$$

This is a contradiction with the choice of R given in the hypotheses. □

Theorem 4.7 is a corollary of the previous lemmas, since the solution of the system (4.1) remains bounded in $SO(4) \times \mathbb{R}^2$ for all (u,v), guaranteeing the existence of the solution for all (u,v).

References

[1] H.B. Lawson Jr., Complete minimal surfaces in S^3, Ann. of Math. (2) 92 (1970) 335–374.

[2] W.-Y. Hsiang, H.B. Lawson, Minimal submanifolds of low cohomogeneity, J. Differential Geom. 5 (1971) 1–38.

[3] U. Pinkall, I. Sterling, On the classification of constant mean curvature tori, Ann. of Math. 130 (1989) 407–451.

[4] J. Ramanathan, Rigidity of minimal surfaces in S^3, Manuscripta Math. 60 (1988) 417–422.