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Approximating survivable networks with β-metric costs

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ABSTRACT

The Survivable Network Design (SND) problem seeks a minimum-cost subgraph that satisfies prescribed node-connectivity requirements. We consider SND on both directed and undirected complete graphs with *β-metric costs* when $c(xz) \leqslant \beta[c(xy) + c(yz)]$ for all $x, y, z \in V$, which varies from uniform costs ($\beta = 1/2$) to metric costs ($\beta = 1$).
For the k-Connected Subgraph (k-CS) problem our ratios are: $1 + \frac{2\beta}{k(1-\beta)} - \frac{1}{2k-1}$ for undirected graphs, and $1 + \frac{4\beta^3}{k(1-3\beta^2)} - \frac{1}{2k-1}$ for directed graphs and $\frac{1}{2} \leqslant \beta < \frac{1}{\sqrt{3}}$. For undirected graphs this improves the ratios $\frac{\beta}{1-\beta}$ of Böckenhauer et al. (2008) [3] and $2 + \beta\frac{k}{n}$ of Kortsarz and Nutov (2003) [11] for all $k \geqslant 4$ and $\frac{1}{2} + \frac{3k-2}{2(4k^2-7k+2)} \leqslant \beta \leqslant \frac{k^2}{(k+1)^2-2}$. We also show that SND admits the ratios $\frac{2\beta}{1-\beta}$ for undirected graphs, and $\frac{4\beta^3}{1-3\beta^2}$ for directed graphs with $1/2 \leqslant \beta < 1/\sqrt{3}$. For two important particular cases of SND, so-called Subset k-CS and Rooted SND, our ratios are $\frac{2\beta^3}{1-3\beta^2}$ for directed graphs and $\frac{\beta}{1-\beta}$ for subset k-CS on undirected graphs.

1. Introduction

1.1. Problems considered

For a graph H, let $\kappa_H(u, v)$ denote the *uv-connectivity in H*, that is, the maximum number of pairwise internally node-disjoint uv-paths in H. We consider variants of the following fundamental problem:

Survivable Network Design (SND)
Instance: A directed/undirected complete graph $G = (V, E)$ with non-negative edge-cost $\{c(e)\colon e \in E\}$, and integral connectivity requirements $\{r(u, v)\colon u, v \in V\}$.
Objective: Find a min-cost subgraph H of G satisfying $\kappa_H(u, v) \geqslant r(u, v)$ for all $u, v \in V$.

Important particular cases of SND are:

- k-Connected Subgraph (k-CS), when $r(u, v) = k$ for all $u, v \in V$.
- Subset k-CS, when $r(u, v) = k$ for all $u, v \in T \subseteq V$ and $r(u, v) = 0$ otherwise.
- Rooted SND, when there is a node $s \in V$ so that $r(u, v) > 0$ implies $u = s$.

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Table 1
Approximation ratios and hardness of approximation results for SND and k-CS (recall that in the case of β-metric costs and undirected graphs we assume $1/2 \leqslant \beta < 1$). Here $k = \max_{u,v\in V} r(u, v)$ denotes the maximum requirement of an SND instance.

Costs	Requirements	Approximability	
		Undirected	*Directed*
General	General	$O(\min\{k^3 \log n, n^2\})$ [8], $\Omega(k^\varepsilon)$ [7]	$\Omega(2^{\log^{1-\varepsilon} n})$ for $k = 1$ [9]
General	k-CS	$O(\log \frac{n}{n-k} \log k)$ [14]	$O(\log \frac{n}{n-k} \log k)$ [14]
General	Subset k-CS	$O(\min\{k^2 \log k, n^2\})$ [15], $\Omega(k^\varepsilon)$ [7]	$O(n^2)$
General	Rooted SND	$O(\min\{k^2, n\})$ [15]	$O(n)$
Metric	General	$O(\log k)$ [6]	$\Omega(2^{\log^{1-\varepsilon} n})$ for $k = 1$ [9]
Metric	k-CS	$2 + (k-1)/n$ [11]	$2 + k/n$ [11]
β-metric	General	–	–
β-metric	k-CS	$2 + \beta\frac{k}{n}$ [11], $\frac{\beta}{1-\beta}$ [3], APX-hard [2]	–

We consider instances of SND with *β-metric costs*, when *the input graph is complete* and for some $1/2 \leqslant \beta < 1$ the costs satisfy the *β-triangle inequality* $c(xz) \leqslant \beta[c(xy) + c(yz)]$ for all $x, y, z \in V$. When $\beta = \frac{1}{2}$ the costs are uniform, and we have the "cardinality version" of the problem, when we seek a minimum-size subgraph in a complete graph that satisfies the connectivity requirements. If we allow the case $\beta = 1$, then the costs satisfy the ordinary triangle inequality and we have the metric version of the problem. Many practical instances of the problem may have costs which are between metric and uniform. Let $k = \max_{u,v\in V} r(u, v)$ denote the maximum requirement of an SND instance. We will assume that $k \leqslant |V| - 1$, as otherwise the problem does not have a feasible solution.

1.2. Previous work and our results

The k-CS problem (and thus also Subset k-CS, SND, and by [1] also Rooted SND) with β-metric costs is APX-hard for $k = 2$ and any $\beta > 1/2$ [2]. Approximation ratios and hardness of approximation results for SND and k-CS are summarized in Table 1. In [3] is also given a $(1 + \frac{5(2\beta-1)}{9(1-\beta)})$-approximation algorithm for undirected 3-CS with β-metric costs. For a survey on various min-cost connectivity problems see [12]. For recent work on SND problems see [7,8,14,15]. We mention a recent result [13] that for $k = n/2 + k'$ the approximability of undirected SND is the same as that of directed SND with maximum requirement k'. This is so also for k-CS. However, the reduction in [13] does not preserve metric costs.

We analyze the algorithm of Cheriyan and Thurimella [5] originally suggested for k-CS with $\{1, \infty\}$-costs, and show that for β-metric costs it achieves the following ratios.

Theorem 1. *k-CS with β-metric costs admits the following approximation ratios*:

- $1 + \frac{2\beta}{k(1-\beta)} - \frac{1}{2k-1} \leqslant 1 + \frac{1}{k}(\frac{2\beta}{1-\beta} - \frac{1}{2})$ *for undirected graphs and* $1/2 \leqslant \beta < 1$.
- $1 + \frac{4\beta^3}{k(1-3\beta^2)} - \frac{1}{2k-1} \leqslant 1 + \frac{1}{k}(\frac{4\beta^3}{1-3\beta^2} - \frac{1}{2})$ *for directed graphs and* $1/2 \leqslant \beta < 1/\sqrt{3}$.

For undirected graphs, this improves the ratios $\frac{\beta}{1-\beta}$ of [3] and $2 + \beta\frac{k}{n}$ of [11][2] for all $k \geqslant 4$ and $\frac{1}{2} + f(k) \leqslant \beta \leqslant \frac{k^2}{(k+1)^2-2}$, where $f(k) = \frac{3k-2}{2(4k^2-7k+2)} \leqslant \frac{1}{2k}$ for $k \geqslant 5$ and $f(4) = \frac{5}{38}$.

For other versions of the problem our results are as follows (see Table 2 for a summary of our results).

Theorem 2. *SND with β-metric costs admits approximation ratios $\frac{2\beta}{1-\beta}$ for undirected graphs, and $\frac{4\beta^3}{1-3\beta^2}$ for directed graphs with $1/2 \leqslant \beta < 1/\sqrt{3}$. For Subset k-CS the ratios are $\frac{\beta}{1-\beta}$ for undirected graphs and $\frac{2\beta^3}{1-3\beta^2}$ for directed graphs with $1/2 \leqslant \beta < 1/\sqrt{3}$; for directed Rooted SND the ratio is $\frac{2\beta^3}{1-3\beta^2}$, $1/2 \leqslant \beta < 1/\sqrt{3}$.*

In our proofs, we will often use the following statement:

Lemma 3. *(See [2,4].) Let e, e' be a pair of edges in a complete graph G with β-metric costs.*

(i) *If G is undirected, and if e, e' are adjacent then $c(e) \leqslant \frac{\beta}{1-\beta} c(e')$; here $1/2 \leqslant \beta < 1$.*
(ii) *If G is directed, and if $\frac{1}{2} \leqslant \beta < \frac{1}{\sqrt{3}}$, then $c(e) \leqslant \frac{2\beta^3}{1-3\beta^2} c(e')$.*

[2] In [11] is given a $(2 + (k-1)/n)$-approximation algorithm for metric costs; a slight adjustment of the analysis of [11] shows that this algorithm has ratio $2 + \beta\frac{k}{n}$ for β-metric costs.

Table 2
Improvement ranges of our results.

Graph	Requirements	Approximability	Improvement range
Undirected	General	$\frac{2\beta}{1-\beta}$	$\frac{1}{2} \leqslant \beta < 1$
Undirected	k-CS	$1 + \frac{2\beta}{k(1-\beta)} - \frac{1}{2k-1}$	$k \geqslant 4$, $\frac{1}{2} + \frac{3k-2}{2(4k^2-7k+2)} < \beta < \frac{k^2}{(k+1)^2-2}$
Undirected	Subset k-CS	$\frac{\beta}{1-\beta}$	$\frac{1}{2} \leqslant \beta < 1$
Directed	General	$\frac{4\beta^3}{1-3\beta^2}$	$\frac{1}{2} \leqslant \beta < \frac{1}{\sqrt{3}}$
Directed	k-CS	$1 + \frac{4\beta^3}{k(1-3\beta^2)} - \frac{1}{2k-1}$	$\frac{1}{2} \leqslant \beta < \frac{1}{\sqrt{3}}$
Directed	Subset k-CS	$\frac{2\beta^3}{1-3\beta^2}$	$\frac{1}{2} \leqslant \beta < \frac{1}{\sqrt{3}}$
Directed	Rooted SND	$\frac{2\beta^3}{1-3\beta^2}$	$\frac{1}{2} \leqslant \beta < \frac{1}{\sqrt{3}}$

1.3. Notation

Given an instance $G = (V, E), c, r$ of SND we use the following notation. Let $V = \{v_1, \ldots, v_n\}$. For undirected graphs, the *requirement r_i of v_i* is the maximum requirement of a pair containing v_i. For directed graphs $r_i^{out} = \max_{v_j \in V} r(v_i, v_j)$ is the *out-requirement of* v_i, and $r_i^{in} = \max_{v_j \in V} r(v_j, v_i)$ is the *in-requirement of* v_i. For an edge set or a graph F and a node v, let $\deg_F(v)$ denote the degree of v in F. For directed graphs, let $\deg_F^{out}(v)$ and $\deg_F^{in}(v)$ denote the out-degree and the in-degree of v in F. Let opt denote the optimum solution value of an instance at hand.

2. Proof of Theorem 1

Definition 2.1. An edge set F on node set V is a *k-cover* (of V) if for all $v \in V$:

(i) $\deg_F(v) \geqslant k$ if F is undirected.
(ii) $\deg_F^{in}(v) \geqslant k$ and $\deg_F^{out}(v) \geqslant k$ if F is directed.

Lemma 4. *For both directed and undirected graphs, any k-cover J contains a $(k-1)$-cover F of cost $c(F) \leqslant (1 - \frac{1}{2k-1})c(J)$.*

Proof. The following procedure finds $M \subseteq J$ such that $F = J - M$ is a $(k-1)$-cover and $c(M) \geqslant c(J)/(2k-1)$. Start with $M = \emptyset$, $F = J$, and all edges in F unmarked, and iteratively do the following, until all edges that remain in F are marked. Among all unmarked edges in F, let $e = uv$ be one of the maximum cost. Remove e from F and add it to M. In the case of undirected graphs, if the degree in F of an endnode of e is exactly $k-1$, mark all edges incident to this endnode. In the case of directed graphs, if $\deg_F^{out}(u) = k-1$ mark all edges leaving u, and if $\deg_F^{in}(v) = k-1$ mark all edges entering v. It is easy to see that during, and thus also at the end of the procedure, $F = J - M$ is a $(k-1)$-cover. At every iteration, one edge e is moved from F to M, and at most $2k-2$ edges in F are marked; each of these edges is cheaper than e. Hence $c(M) \geqslant c(J)/(2k-1)$, as required. □

Let $F \subseteq E$ be a minimum-cost $(k-1)$-cover of V. Such F of minimum-costs can be computed in polynomial time, for both directed and undirected graphs, cf. [16]. As any feasible solution to k-CS is a k-cover, $c(F) \leqslant (1 - \frac{1}{2k-1})$opt, by Lemma 4. Now let $I \subseteq E - F$ be an inclusion-minimal augmenting edge set so that $H = (V, F + I)$ is k-connected. It is known that I is a forest in the case of undirected graphs, and $|I| \leqslant 2n-1$ in the case of directed graphs, cf. [5] and [12].

In the case of undirected graphs, since I is a forest, there exists an orientation D of I (namely, D is a directed graph obtained by directing every edge of I) so that the out-degree of every node w.r.t. D is at most 1. Let D_i be the set of edges in D leaving v_i, so either $D_i = \emptyset$ or $|D_i| = 1$ for all i. Let J be an optimal solution, and let J_i be the set of edges in J incident to v_i. As $J_i \geqslant k$, we have $c(D_i) \leqslant c(J_i)\frac{\beta}{k(1-\beta)}$, by Lemma 3. Hence

$$c(I) = \sum_{i=1}^{n} c(D_i) \leqslant \frac{\beta}{k(1-\beta)} \sum_{i=1}^{n} c(J_i) \leqslant \frac{2\beta}{k(1-\beta)} c(J) = \frac{2\beta}{k(1-\beta)} \cdot \text{opt}.$$

Consequently,

$$c(H) = c(F) + c(I) \leqslant \left(1 - \frac{1}{2k-1}\right) \cdot \text{opt} + \frac{2\beta}{k(1-\beta)} \cdot \text{opt} = \left(1 + \frac{2\beta}{k(1-\beta)} - \frac{1}{2k-1}\right) \cdot \text{opt}.$$

In the case of directed graphs, $|I| \leqslant 2n-1$. As any feasible solution has at least kn edges, we have

$$c(I) \leqslant \frac{2n-1}{kn} \cdot \frac{2\beta^3}{1-3\beta^2} \cdot \text{opt} \leqslant \frac{4\beta^3}{k(1-3\beta^2)} \cdot \text{opt}.$$

Consequently,

$$c(H) = c(F) + c(I) \leqslant \left(1 - \frac{1}{2k-1}\right) \cdot \mathrm{opt} + \frac{4\beta^3}{k(1-3\beta^2)} \cdot \mathrm{opt} = \left(1 + \frac{4\beta^3}{k(1-3\beta^2)} - \frac{1}{2k-1}\right) \cdot \mathrm{opt}.$$

3. Proof of Theorem 2

To prove Theorem 2 we use the following statement.

Lemma 5. *Let H, J be subgraphs of a complete graph with β-metric costs and let $\alpha \geqslant 1$.*

(i) *In the case of undirected graphs and $\frac{1}{2} \leqslant \beta < 1$ the following holds:*
- *$c(H) \leqslant \alpha \cdot \frac{\beta}{1-\beta} c(J)$ if $\deg_H(v) \leqslant \alpha \cdot \deg_J(v)$ for every node v, or if there are orientations H' of H and J' of J so that $\deg^{out}_{H'}(v) \leqslant \alpha \cdot \deg^{out}_{J'}(v)$ for every node v.*
- *$c(H) \leqslant 2\alpha \cdot \frac{\beta}{1-\beta} c(J)$ if H has an orientation H' so that $\deg^{out}_{H'}(v) \leqslant \alpha \cdot \deg_J(v)$ for every node v.*

(ii) *In the case of directed graphs and $\frac{1}{2} \leqslant \beta < \frac{1}{\sqrt{3}}$ we have $c(H) \leqslant \alpha \cdot \frac{2\beta^3}{1-3\beta^2} c(J)$ if the number of edges in H is at most α times the number of edges in J.*

Proof. Let H_i and J_i be the set of edges incident to v_i in H and in J, and let H'_i and J'_i be the set of edges leaving v_i in the orientations H' and J'_i, respectively.

If $|H_i| \leqslant \alpha \cdot |J_i|$ then $c(H_i) \leqslant \alpha \cdot \frac{\beta}{1-\beta} \cdot c(J_i)$, by part (i) of Lemma 3. Thus

$$c(H) = \frac{1}{2}\sum_{i=1}^{n} c(H_i) \leqslant \alpha \cdot \frac{\beta}{1-\beta} \cdot \frac{1}{2}\sum_{i=1}^{n} c(J_i) = \alpha \cdot \frac{\beta}{1-\beta} \cdot c(J).$$

If $|H'_i| \leqslant \alpha \cdot |J'_i|$ then $c(H'_i) \leqslant \alpha \cdot \frac{\beta}{1-\beta} c(J'_i)$, by part (i) of Lemma 3. Thus

$$c(H) = \sum_{i=1}^{n} c\bigl(H'_i\bigr) \leqslant \alpha \cdot \frac{\beta}{1-\beta} \sum_{i=1}^{n} c\bigl(J'_i\bigr) = \alpha \cdot \frac{\beta}{1-\beta} \cdot c(J).$$

If $|H'_i| \leqslant \alpha \cdot |J_i|$ then $c(H'_i) \leqslant \alpha \cdot \frac{\beta}{1-\beta} c(J_i)$, by part (i) of Lemma 3. Thus

$$c(H) = \sum_{i=1}^{n} c(D_i) \leqslant \alpha \cdot \frac{\beta}{1-\beta} \sum_{i=1}^{n} c(J_i) \leqslant 2\alpha \cdot \frac{\beta}{1-\beta} \cdot c(J).$$

Part (ii) of the lemma follows directly from part (ii) of Lemma 3. □

Now let J be an optimal solution for an SND instance. For both directed and undirected graphs, we will have $\alpha = 2$ for SND, and $\alpha = 1$ for Subset k-CS and Rooted SND.

3.1. General SND

For general SND we use the following simple construction.

Lemma 6. *Let $V = \{v_1, \ldots, v_n\}$ be a node set, and for $i = 1, \ldots, n$ let $r^{out}_i, r^{in}_i \leqslant n-1$ be non-negative integers. Let A^{out}_i be the set of edges from v_i to the first r^{out}_i nodes in $V - \{v_i\}$, and A^{in}_i be the set of edges from the first r^{in}_i nodes in $V - \{v_i\}$ to v_i. Namely:*

$$A^{out}_i = \begin{cases} \{v_i v_j\colon 1 \leqslant j \leqslant r^{out}_i\}, & \text{if } r^{out}_i < i, \\ \{v_i v_j\colon 1 \leqslant j \leqslant r^{out}_i + 1,\ j \neq i\}, & \text{otherwise}, \end{cases}$$

$$A^{in}_i = \begin{cases} \{v_j v_i\colon 1 \leqslant j \leqslant r^{in}_i\}, & \text{if } r^{in}_i < i, \\ \{v_j v_i\colon 1 \leqslant j \leqslant r^{in}_i + 1,\ j \neq i\}, & \text{otherwise}. \end{cases}$$

Then for any $i \neq j$, the graph $H_{ij} = (V, A^{out}_i \cup A^{in}_j)$ contains at least $\min\{r^{out}_i, r^{in}_j\}$ internally disjoint $v_i v_j$-paths.

Proof. Note that there is a set C of $\min\{r^{out}_i, r^{in}_j\} - 1$ nodes so that in H_{ij} there is an edge from v_i to every node in C and from every node in C to v_j; furthermore, either $v_i v_j \in H_{ij}$ or there is one more node that can be added to C. The statement follows. □

The algorithm is as follows. In the case of directed graphs, we compute the edge sets A_i^{out} and A_i^{in} as in Lemma 6, and output their union graph H. In the case of undirected graphs, we consider the directed problem on the bi-direction of G (the directed graph obtained from G by replacing every edge $e = uv$ of G by two opposite directed edges uv, vu of the same cost as e) with the requirements $r(v_i, v_j) \leftarrow \max\{r(v_i, v_j), r(v_j, v_i)\}$ for $i > j$ and $r(v_i, v_j) \leftarrow 0$ otherwise. Hence we will have $A_i^{in} = \emptyset$ for all i. Let H' be the union of the sets A_i^{out}. The output graph H is the underlying graph of the directed graph $H' = \bigcup_{i=1}^{n} A_i^{out}$. For both directed and undirected graphs we have $\kappa_H(v_i, v_j) \geqslant \min\{r(v_i), r(v_j)\} \geqslant r(v_i, v_j)$, hence H is a feasible solution. To establish the approximation ratio we will use Lemma 5, fixing J to be some optimal solution.

In the case of directed graphs, let J_i^{out} and J_i^{in} be the sets of edges in J leaving and entering v_i, respectively. Note that $|A_i^{out}| = r_i^{out}$ and $|A_i^{in}| = r_i^{in}$ while $|J_i^{out}| \geqslant r_i^{out}$ and $|J_i^{in}| \geqslant r_i^{in}$. Hence the number of edges in the constructed solution is at most $\sum_{i=1}^{n}(r_i^{out} + r_i^{in})$, while any feasible solution has at least half this number of edges. The ratio $\frac{4\beta^3}{1-3\beta^2}$ follows from part (ii) of Lemma 5.

In the case of undirected graphs, let J_i be the set of edges incident to v_i in J. Note that $\deg_{H'}^{out}(v_i) = |A_i^{out}| = r_i$ and that $|J_i| \geqslant r_i$ for all i. The ratio $2\frac{\beta}{1-\beta}$ follows from part (i) of Lemma 5.

3.2. Subset k-CS

Recall that Subset k-CS is the case of SND when for some $T \subseteq V$ we have $r(u, v) = k$ for all $u, v \in T$. Let $t = |T|$. For the case $t > k$ we can apply our algorithm for k-CS while ignoring the nodes in $V - T$, thus obtaining ratios as in Theorem 1. We can also obtain the ratios as in Theorem 2. Such an algorithm is described in [3] for undirected graphs, and we extend it to directed graphs using the following statement.

Lemma 7. *For any integers k, n so that, $n \geqslant k + 1$ there exists a directed k-connected graph H on n nodes in which the out-degree of every node is exactly k (in particular, H has exactly kn edges), and such H can be constructed in polynomial time.*

Proof. Let $V = \{v_0, \ldots, v_{n-1}\}$. Let A_i be the set of k edges from v_i to $v_{i+1}, v_{i+2}, \ldots, v_{i+k}$, where the indices are modulo n. Let $A = \bigcup_{i=0}^{n-1} A_i$ and let $H = (V, A)$. Then the out-degree of every node in A is exactly k, by the construction. We will show that H is k-connected. A theorem of Whitney states that a directed/undirected graph $H = (V, A)$ is k-connected if, and only if, $uv \in A$ or $\kappa_H(u, v) \geqslant k$ for all $u, v \in V$. Since the construction is symmetric, it is sufficient to show a set of k pairwise internally disjoint paths from v_0 to any node u not adjacent to v_0. Consider the BFS layers L_i with root v_0. We have $L_0 = \{v_0\}$, and there are k nodes in every other layer except of possibly the last one. Namely, $L_1 = \{v_1, \ldots, v_k\}$, $L_2 = \{v_{k+1}, \ldots, v_{2k}\}$, and in general $L_j = \{v_{(j-1)k+1}, \ldots, v_{jk}\}$ (in the last layer the last index is $n - 1$). Let $j \geqslant 1$ and let $u \in L_{j+1}$ be arbitrary, say $u = v_{jk+i}$ for some $1 \leqslant i \leqslant k$. Let P'_q be the path $v_0 \to v_q \to v_{k+q} \to \cdots \to \cdots v_{(j-1)k+q}$. Let P_q be the v_0u path obtained by adding to P'_q: the edges $v_{(j-1)k+q} \to v_{jk+q} \to u$ if $q \leqslant i$, and the edge $v_{(j-1)k+q} \to u$ otherwise; note that the edges we add exist in A, by the definition of A. Now it is easy to see that $P_1, \ldots, P_k$ is a set of k pairwise internally-disjoint v_0u-paths, as required. $\square$

Assume $t = |T| \geqslant k + 1$. Let H_k be a k-connected graph on T with the following property. In the case of directed graphs, we require that H_k has kn edges. By Lemma 7 such directed graphs exist, and can be constructed in polynomial time. In the case of undirected graphs, we require that the degree of every node in H_k is exactly k, except that if $|T|$, k are both odd then one chosen node v has degree $k + 1$. Harary [10] showed that such graph exist and can be constructed in polynomial time. If $t \geqslant k + 1$ then our algorithm for k-CS returns any graph H_k as above, except the case of undirected graphs when k, $|T|$ are both odd. In the latter case, for every $v \in T$ let $H_k(v)$ be any graph H_k so that v has degree $k + 1$; we return the cheapest one among the subgraphs $H_k(v)$, $v \in T$.

The approximation ratio is shown as follows. In the case of directed graphs, any feasible solution has at least kt edges. The ratio of $\frac{2\beta^3}{1-3\beta^2}$ now immediately follows from part (ii) of Lemma 3. In the case of undirected graphs, the degree in an optimal solution J of every node in T is at least k, and if $|T|$, k are both odd then one node v has degree at least $k + 1$. The ratio of $\frac{\beta}{1-\beta}$ now immediately follows from part (i) of Lemma 3.

Our construction for the case $t \leqslant k$ is a slight extension of this construction. Note that $|V| \geqslant k + 1$, as otherwise the problem has no feasible solution. We choose a set $U \subseteq V - T$ of arbitrary $k - t + 1$ nodes, and obtain a graph H by adding to a clique on T all possible edges between T and U. For any $u, v \in T$ there are k internally-disjoint uv-paths in H: the edge uv and $(t - 2) + |U| = k - 1$ internally disjoint paths of length 2 each through the nodes in $T \setminus \{u, v\}$ and U. Thus H is a feasible solution. For the analysis of the approximation ratio, we use the following simple observation.

Lemma 8. *Let J be a feasible solution to a Subset k-CS instance with $r(u, v) = k$ for all $u, v \in T$ and $r(u, v) = 0$ otherwise, and let $t = |T|$. Then:*

(i) *For undirected graphs, every node in T has in J at least $k - t + 1$ neighbors in $V - T$.*
(ii) *For directed graphs, J has at least $t(t - 1) + 2t(k - t + 1)$ edges.*

Proof. In undirected J, every node in T has at least k neighbors. At most $t-1$ of these neighbors can lie in T, hence all the other at least $k-t+1$ neighbors are in $V-T$. In directed J, every node has out-degree and in-degree at least k. At most $t-1$ edges can enter a node from nodes in T, or leave a node to a node in T. Hence for every $v \in T$, at least $k-t+1$ edges go from v to $V-T$, and at least $k-t+1$ edges go from $V-T$ to v. Thus the number of edges in J is at least $t(t-1)+2t(k-t+1)$, as claimed. □

For undirected graphs, we obtain orientations H' of H and J' of some optimal solution J as follows. Assume $T=\{v_1,\ldots,v_t\}$. Inside T edges go from v_i to nodes v_j with $j>i$, and all the other edges go from T to $V-T$. Using Lemma 8, it is easy to see that $\deg^{out}_{H'}(v) \leqslant \alpha \cdot \deg^{out}_{J'}(v)$ for every node v. For directed graphs our solution H has exactly $t(t-1)+2t(k-t+1)$ edges, while by Lemma 8 any feasible solution has at least this number of edges. Thus the ratios $\frac{\beta}{1-\beta}$ for undirected graphs and $\frac{2\beta^3}{1-\beta^2}$ for directed graphs follow from Lemma 3.

3.3. Directed Rooted SND

Let $G=(V,E)$, c, r, s be an instance of directed Rooted SND where s is the root. Let $T=\{u \in V\colon\ r(s,u)>0\}$ be the set of nodes with positive in-requirements.

Lemma 9. *Any feasible solution J for directed Rooted SND has at least $\sum_{v\in T} r^{in}(v)$ edges if $k \leqslant |T|$, and at least $\sum_{v\in T} r^{in}(v)+k-|T|$ edges if $k>|T|$.*

Proof. Clearly, $\deg^{in}_J(v) \geqslant r^{in}(v)$ and $\deg^{out}_J(r) \geqslant k$. Now consider the edges in J leaving r. At most $|T|$ of these edges can go to nodes in T, hence if $k>|T|$ then there are at least $k-|T|$ edges that go to nodes in $V-T$. The statement follows. □

Now we show that the lower bound in Lemma 9 is achievable. Construct a graph H as follows. Let $U \subseteq V \setminus \{s\}$ be an arbitrary set of $\max\{k,|T|\}$ nodes containing T, so $U=T$ if $|T| \geqslant k$. Take an edge from s to every node in U, and for every $v \in T$ take arbitrary $r^{in}(v)-1$ edges entering v from any $r^{in}(v)-1$ nodes in $U \setminus \{v\}$. It is easy to see that H is a feasible solution for the directed Rooted SND instance, and the number of edges in H coincides with the lower bound in Lemma 9. Applying Lemma 3(ii) we obtain $c(H) \leqslant \frac{2\beta^3}{1-3\beta^2} \cdot \mathrm{opt}$.

The proof of Theorem 2 is complete.

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