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On the Picard Group of Polynomial Rings

FRIEDRICH ISCHEBECK

Mathematisches Institut der Universität Münster, Einsteinstraße 62, D-4400 Münster, West Germany

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IN MEMORIAM FRIEDRICH BACHMANN

Sei $A \subset B$ eine Erweiterung reduzierter Ringe, ${}^+_B A$ der seminormale Abschluß von A in B und p eine Primzahl oder $p = \infty$. Die Gruppe $\ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$ besitzt genau dann ein Element der Ordnung p, wenn dies für die (additive) Gruppe ${}^+_B A/A$ gilt.

For a commutative ring A—and all rings we consider here are supposed to be commutative—one has a natural splitting $\operatorname{Pic}(A[X_1,..., X_r]) \simeq \operatorname{Pic} A \oplus N_r \operatorname{Pic} A$.

Our results on $N_r \operatorname{Pic}$ depend heavily on Swan's paper [2], where he defines the notion of p-seminormality and proves that (for $r \geqslant 1$) $N_r \operatorname{Pic} A$ has no p-torsion iff A_{red} is p-seminormal. In the case $p = 0$ this means that $N_r \operatorname{Pic} A = 0$ iff A_{red} is seminormal. Here we give a relative version, showing that $G := \ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$ has no p-torsion iff $A \subset B$ is an extension of rings, such that $A_{\text{red}} \subset B_{\text{red}}$ is p-seminormal.

This solves a problem stated at the end of [2, 6].

Further we characterize the p-seminormality of an extension $A \subset B$ by the fact that the quotient of additive groups ${}^+_B A/A$ has no p-torsion. Here ${}^+_B A$ is the seminormal closure of A in B.

On the other hand we show that (for reduced A, B) G is a torsion group, iff ${}^+_B A/A$ is so.

Therefore we may state summarily: Let $A \subset B$ be an extension of reduced rings and p a prime number or $p = \infty$. Then G has an element of order p iff ${}^+_B A/A$ has one.

We show how to derive the absolute case from the relative one, which seems to be a bit delicate if the ring has infinitely many minimal prime ideals.

In the Appendix we say something on divisibility of G.

An idea of C. A. Weibel is used in parts (e) and (k) of the proof.

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1. Following [2] for a natural number p an extension $A \subset B$ of commutative rings is called p-seminormal if $x \in B$, $x^2, x^3, px \in A$ imply $x \in A$. Here the case $p = 0$ is included and "0-seminormal" means the same as "seminormal."

The smallest ring C between A and B, such that $C \subset B$ is p-seminormal (and such a ring exists), is called the p-seminormal closure of A in B and is denoted by ${}_B^{+p}A$. Instead of ${}_B^{+0}A$ one writes ${}_B^{+}A$.

Look in [2, 4] for these notions.

Before we give another characterization of p-seminormality, let us remark that one immediately verifies the following fact: If C/A is the S-torsion part of the additive group B/A, then C is a subring of B. Here S denotes any multiplicative subset of $\mathbb{Z} - \{0\}$.

2. Proposition. *An extension $A \subset B$ is p-seminormal iff the additive group ${}_B^{+}A/A$ has no p-torsion. The group ${}_B^{+p}A/A$ is the p-torsion part of ${}_B^{+}A/A$. (Here and further as in [2] by the 0-torsion part of an abelian group G we mean G itself. "G has no 0-torsion" means "$G = 0$.")*

Proof. The two sentences are clearly equivalent. Let us prove the first. If $x \in B$, $x^2, x^3 \in A$, then $x \in {}_B^{+}A$. So the implication from the right to the left is trivial.

Now assume that $A \subset B$ is p-seminormal. Let C be the ring between A and ${}_B^{+}A$ such that C/A is the p-torsion part of ${}_B^{+}A/A$. Since ${}_C^{+}A = C$, we must prove only that ${}_C^{+}A = A$, i.e., $A \subset C$ is seminormal. So let $x \in C$, x^2, $x^3 \in A$ and n be minimal with $p^n x \in A$. If $n = 0$, we are through. But otherwise we would have $(p^{n-1}x)^2$, $(p^{n-1}x)^3$, $p(p^{n-1}x) \in A$ and hence by p-seminormality $p^{n-1}x \in A$. This contradicts the minimality of n.

3. Corollary. *Let $A \subset B$ be p-seminormal and $x \in B$ be an element for which there is an $n \in \mathbb{N}$, such that $x^{n+i} \in A$ for all $i \in \mathbb{N}$. If further $p^r x \in A$ for some $r \in \mathbb{N}$, then $x \in A$.*

Proof. By induction on n one easily shows that an $x \in B$, which fulfills the first condition, belongs to ${}_B^{+}A$. The rest follows from the proposition above.

4. Definition. Let B be a ring and $\mathfrak{a}$ a subgroup of its additive group. Then $\mathcal{O}_B(\mathfrak{a}) := \{x \in B \mid x\mathfrak{a} \subset \mathfrak{a}\}$.

Remark. $\mathcal{O}_B(\mathfrak{a})$ is a subring of B.

If—as usual—$\mathfrak{a}$ is an ideal in a subring A of B, then $\mathcal{O}_B(\mathfrak{a}) \supset A$ and $\mathfrak{a}$ is a common ideal of A and $\mathcal{O}_B(\mathfrak{a})$.

5. Lemma. *Let $A \subset B$ be a nontrivial extension with A noetherian, and $\mathfrak{a}$*

an ideal of A with $\mathfrak{a}B \subset A$. Then there is a prime ideal $\mathfrak{p} \supset \mathfrak{a}$ of A, such that $\mathcal{O}_B(\mathfrak{p}) \neq A$. Consequently if $\mathfrak{c}$ is maximal in the set of all conductors of extensions $A \subset C$, with $C \subset B$ and $A \neq C$, then $\mathfrak{c}$ is prime in A.

Proof. Let $\mathfrak{k}$ be the conductor of $A \subset B$ and $\mathfrak{p} \in \mathrm{Ass}(A/\mathfrak{k})$. Then $\mathfrak{k} \supset \mathfrak{a}$, hence $\mathfrak{p} \supset \mathfrak{a}$. Now by the definition of "Ass" there is an $a \in A - \mathfrak{k}$ with $a\mathfrak{p} \subset \mathfrak{k}$. To that a we find a $b \in B$ with $ba \notin A$, since $a \notin \mathfrak{k}$. But then $ba\mathfrak{p} \subset b\mathfrak{k} \subset \mathfrak{k} \subset \mathfrak{p}$, so that $ba \in \mathcal{O}_B(\mathfrak{p}) - A$.

6. LEMMA. *Let $A \subset B \subset C$ be two extensions, where $A \subset B$ is p-seminormal, and $\mathfrak{a}$ a radical ideal of A (i.e., $\mathfrak{a} = \sqrt{\mathfrak{a}}$) such that $\mathfrak{a}C \subset B$. Then $\mathcal{O}_C(\mathfrak{a}) \subset C$ is also p-seminormal.*

(This is shown in the proof of Theorem 6.1 in [2] in the case $B = C$. We need here the more general formulation.)

Proof. Let $x \in C$ be an element with $x^2, x^3, px \in \mathcal{O}_C(\mathfrak{a})$. We have to show $x \in \mathcal{O}_C(\mathfrak{a})$, i.e., $xa \in \mathfrak{a}$ for every $a \in \mathfrak{a}$. Now $x^2a, x^3a, (px)a \in \mathfrak{a}$ implies $(xa)^2, (xa)^3, p(xa) \in \mathfrak{a}$. By hypothesis we also have $xa \in B$. So by p-seminormality we get $xa \in A$, hence $xa \in \mathfrak{a}$, since $(xa)^2 \in \mathfrak{a}$ and $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

7. LEMMA [3]. *Let $A \subset B$ be an extension of reduced rings, and A noetherian. Then ${}^{+p}_{B}A \subset \bar{A}$, where $\bar{A}$ is the integral closure of A in $Q(A)$, its full ring of fractions.*

Proof. ${}^{+p}_{B}A \subset {}^{+}_{B}A$ and according to the classical definition of seminormality in [4] ${}^{+}_{B}A$ is the biggest ring between A and B, which is integral over A and has the "same" spectrum and the "same" residue fields (in every point of the spectrum) as A. Apply this to the minimal prime ideals of A.

8. In order to achieve finer results, we introduce here the notion N^r Pic. Generally if F is a functor from the category of rings to that of abelian group, one has the natural splitting $F(A[X]) \simeq FA \oplus NFA$. Set $N^0F := F$, $N^rF := N(N^{r-1}F)$. Then one easily sees [1, XI, Section 7] that in a natural way $F(A[X_1,..., X_r]) = \oplus_{i \geqslant 0} N^iF(A)^{\binom{r}{i}}$. So

$$N_rF(A) = \ker[F(A[X_1,..., X_r]) \to FA] = \bigoplus_{i \geqslant 1} N^iF(A)^{\binom{r}{i}}.$$

9. THEOREM. *Let $r \geqslant 1$ be a natural number and $A \subset B$ be an extension of reduced rings. Then the following properties are equivalent:*

(i) *$A \subset B$ is p-seminormal;*

(ii) $\ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B]$ *has no p-torsion;*

(iii) $\ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$ *has no p-torsion.*

Remarks. (a) In the case $p=0$ this means: $A \subset B$ is seminormal iff $N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B$ (resp. $N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B$) is injective.

(b) We do not suppose that B is integral over A. So as a special case of the theorem we have: Let A be integrally closed in B, then $N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B$ is injective. I do not know, whether this has been proved before.

Proof of the Theorem. (a) As it is already remarked at the end of 6 in [2], the implication (ii) $\Rightarrow$ (i) can be shown in the same manner as in the proof of Theorem 6.1 [2].

Since $N^r \operatorname{Pic} A \subset N_r \operatorname{Pic} A$ canonically, (iii) $\Rightarrow$ (ii) is trivial.

(b) To prove that (i) $\Rightarrow$ (iii), we first reduce to the case that A, B are universally japanese. Remember that Pic and therefore N_r Pic commutes with filtered direct limits. Therefore, and since $A \subset C$ is p-seminormal for any C between A and B, we may assume $B = A[b_1,\dots,b_n]$. Now A is the filtered union of finitely generated $\mathbb{Z}$-algebras A_α. Set $A'_\alpha := {}^{+p}_{B_\alpha}A_\alpha$, where $B_\alpha = A_\alpha[b_1,\dots,b_n]$. Of course A is also the filtered union of the A'_α. So it remains to show that A'_α is finitely generated over $\mathbb{Z}$. Since A_α is noetherian and japanese, this follows from Lemma 7.

(c) Further we may assume that A is local (of depth 1). Namely, any nontrivial p-torsion element $\xi \in \ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B]$ survives in $N^r \operatorname{Pic} A_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$ (such that $\operatorname{depth} A_{\mathfrak{p}} = 1$); cf. Lemma 6.4 in [2] or Lemma 3.2 in [4]. One easily checks that $A_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ remains p-seminormal.

Also we use induction on $\dim A$.

(d) Let $\mathfrak{m}$ be the maximal ideal of A. (We may assume $\mathfrak{m} \neq 0$, hence $\operatorname{Ann}_A(\mathfrak{m}) = 0$.) Define $\mathfrak{b} = \operatorname{Ann}_B(\mathfrak{m})$, $C = \mathcal{O}_B(\mathfrak{m})$. Then $\mathfrak{b}$ is a common ideal of C and B. Further $C \subset B$ is p-seminormal by Lemma 6 and $\mathfrak{b} = \sqrt[B]{\mathfrak{b}}$. Namely, if $b^2 m = 0$ for all $m \in \mathfrak{m}$, then $(bm)^2 = 0$, so $bm = 0$, since B is reduced. Remark that C may not be noetherian; but $C/\mathfrak{b}$ is so, as we shall see later.

(e) Let $\xi \in \ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$ be a non-zero p-torsion element. Then its image $\xi' \in \ker[N_r \operatorname{Pic} C \to N_r \operatorname{Pic} B]$ is non-zero. Namely, since $\mathfrak{m}$ is a common ideal of A and C, we have an exact sequence

$$0 \to N_r U(C/\mathfrak{m}) \to N_r \operatorname{Pic} A \to N_r \operatorname{Pic} C.$$

(Notice that $N_r U(A/\mathfrak{m}) = N_r U(C) = N_r \operatorname{Pic}(A/\mathfrak{m}) = 0$.) Now, if $p=0$, A is seminormal in B and so in C. Therefore $C/\mathfrak{m}$ is easily seen to be reduced, i.e., $N_r U(C/\mathfrak{m}) = 0$. If $p > 0$, one derives from the p-seminormality of $A \subset C$ that $C/\mathfrak{m}$ is reduced or $\operatorname{char}(A/\mathfrak{m}) \nmid p$. But in the latter case $N_r U(C/\mathfrak{m})$ is a $\mathbb{Z}[1/p]$-module by [2, Corollary 8.2] and therefore has no p-torsion element.

(f) Define $C' := C/\mathfrak{b}$, $B' := B/\mathfrak{b}$. Then $C' \subset B'$ is p-seminormal and $N_r U(B') = 0$, since $\sqrt[B]{\mathfrak{b}} = \mathfrak{b}$. We get an exact sequence

$$0 \to N_r \operatorname{Pic} C \to N_r \operatorname{Pic} B \oplus N_r \operatorname{Pic} C' \to N_r \operatorname{Pic} B'.$$

The element $\xi' \in \ker[N_r \operatorname{Pic} C \to N_r \operatorname{Pic} B]$ goes to an element of the form $(0, \xi'')$ with a non-zero $\xi'' \in \ker[N_r \operatorname{Pic} C' \to N_r \operatorname{Pic} B']$. So it is enough to prove the theorem for $C' \subset B'$.

(g) We have $A \subset C' \subset B'$ (since $\operatorname{Ann}_A(\mathfrak{m}) = 0$) with $\operatorname{Ann}_{B'}(\mathfrak{m}) = 0$. Therefore $\mathfrak{m} \not\subset \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Ass} B'$. And so there is an $s \in \mathfrak{m}$, which is a non-zero divisor of B'. Since $C' \subset \mathcal{O}_{B'}(\mathfrak{m})$, we have $sC' \subset \mathfrak{m}$. Therefore $C' \subset A_s$ $(=A[1/s])$, and C' as an A-module, being isomorphic to sC', is finitely generated. Let $\bar{A}, \bar{B}'$ be the integral closures of A, B' in A_s, B'_s. They are finite modules over A, resp. B', since A and B' are japanese. Therefore—by double noetherian induction—we find a maximal p-seminormal extension $A'' \subset B''$ with $C' \subset A'' \subset \bar{A}$, $B' \subset B'' \subset \bar{B}'$, such that a non-zero p-torsion element $\eta \in \ker[N_r \operatorname{Pic} A'' \to N_r \operatorname{Pic} B'']$ exists. That means if there is a p-seminormal extension $A^* \to B^*$ with

$$\begin{array}{ccccc} A'' & \subset & A^* & \subset & \bar{A} \\ \cap & & \cap & & \cap \\ B'' & \subset & B^* & \subset & \bar{B}' \end{array}$$

and $A'' \neq A^*$ or $B'' \neq B^*$, then η goes to 0 in $N_r \operatorname{Pic} A^*$.

(h) Now again as in (c) there is a $\mathfrak{p} \in \operatorname{Spec} A''$ such that η survives in $N_r \operatorname{Pic} A''_{\mathfrak{p}}$ and $\operatorname{depth} A''_{\mathfrak{p}} = 1$. Since $\dim A'' = \dim A$, this is impossible—by induction on $\dim A$—if $\mathfrak{p}$ is not maximal. Hence we have $\mathfrak{p} \cap A = \mathfrak{m}$, so $s \in \mathfrak{p}$ and s is a non-zero divisor of $B''_{\mathfrak{p}}$. We prove now that $A''_{\mathfrak{p}} \subset B''_{\mathfrak{p}}$ fulfills, for the given η, an analogous maximality condition to that of $A'' \subset B''$. Assume there is a p-seminormal extension $R \subset S$ between $A''_{\mathfrak{p}} \subset B''_{\mathfrak{p}}$ and their integral closures in $(A''_{\mathfrak{p}})_s$, $(B''_{\mathfrak{p}})_s$, strictly bigger than $A''_{\mathfrak{p}} \subset B''_{\mathfrak{p}}$. Set $A^* := \{x \in \bar{A} \mid x/1 \in R\}$, $B^* = \{x \in \bar{B}' \mid x/1 \in S\}$. Then $R = A^*_{\mathfrak{p}}$, $S = B^*_{\mathfrak{p}}$, and $A^* \subset B^*$ is strictly bigger than $A'' \subset B''$. The p-seminormality of $A^* \subset B^*$ can easily be proved if one uses the fact that $\bar{A} \subset \bar{B}'$ is p-seminormal. But this is so, since $A_s \subset B'_s$ is p-seminormal because $A_s = C'_s$. So our η goes to zero already in $N_r \operatorname{Pic} A^*$, hence in $N_r \operatorname{Pic} R$.

(i) Now, writing $A \subset B$ instead of $A''_{\mathfrak{p}} \subset B''_{\mathfrak{p}}$, we have: (1) A is local of depth 1; (2) there is a non-zero divisor s of B in $\mathfrak{m}$, the maximal ideal of A; (3) there is a non-zero p-torsion element $\xi \in \ker(N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B)$, which goes to 0 in $N_r \operatorname{Pic} A^*$, if $A^* \subset B^*$ is a strictly bigger p-seminormal extension, contained in $\bar{A} \subset \bar{B}$, where $\bar{A}, \bar{B}$ are the integral closures of A, B in A_s, B_s. But we shall now construct such an extension $A^* \subset B^*$, where ξ will not be killed in $N_r \operatorname{Pic} A^*$. This contradiction will prove our theorem.

(j) Consider the ring $\mathscr{C}_{A_s}(\mathfrak{m}) = \mathscr{C}_{\bar{A}}(\mathfrak{m})$. First we see, exactly as in [2, p. 223], that $\mathscr{C}_{\bar{A}}(\mathfrak{m})$ is strictly bigger than A. (Namely, since depth $A = 1$, we have $\mathfrak{m} = (s) : a$ with some $a \in A - (s)$. Therefore $(a/s)\mathfrak{m} \subset A$. If $(a/s)\mathfrak{m} = A$, then $\mathfrak{m} = A(s/a)$, so A is a discrete valuation ring and $N_r \operatorname{Pic} A = 0$, which contradicts the existence of our ξ. Hence $(a/s)\mathfrak{m} \subset \mathfrak{m}$, so $a/s \in \mathscr{C}_{A_s}(\mathfrak{m})$, but $a/s \notin A$.) We distinguish two cases: 1. $B \cap \mathscr{C}_{\bar{A}}(\mathfrak{m}) \neq A$, 2. $B \cap \mathscr{C}_{\bar{A}}(\mathfrak{m}) = A$.

(k) In the first case define $A^* = B \cap \mathscr{C}_{\bar{A}}(\mathfrak{m})$ and $B^* = B$. If $b \in \mathscr{C}_B(\mathfrak{m})$, we have $bs \in \mathfrak{m}$, so $b \in A_s$. This shows $\mathscr{C}_B(\mathfrak{m}) = A^*$. Therefore by Lemma 6 (with $B = C$) the extension $A^* \subset B^*$ is p-seminormal.

But on the other hand, replacing $A \subset A^* \subset B$ by $A \subset C \subset B$ we are here in the same situation as in (e), where we have shown that ξ does not become 0 in $N_r \operatorname{Pic} C$.

(l) Concerning the second case we first remark that here $B \neq B$. Otherwise take an $x \in \mathscr{C}_{\bar{A}}(\mathfrak{m}) - A$. Since $x\mathfrak{m} \subset \mathfrak{m}$, we would have $x \in xB = x\mathfrak{m}B \subset \mathfrak{m}B = B$. So $x \in \mathscr{C}_{\bar{A}}(\mathfrak{m}) \cap B = A$, a contradiction.

Now define $D := \mathscr{C}_{\bar{A}}(\mathfrak{m}) \cdot B$. Then $(\mathfrak{m}B)D = \mathfrak{m}D = \mathfrak{m}B \subset B$. So by Lemma 5 there is a prime ideal $\mathfrak{p}$ of B containing $\mathfrak{m}B$, such that $B^* := \mathscr{C}_D(\mathfrak{p}) \neq B$. We set $A^* := \mathscr{C}_{\bar{A}}(\mathfrak{m}) \cap B^*$, and consider the canonical morphism between the two "conductor squares":

$$\begin{array}{ccccccc} A & \longrightarrow & A^* & & B & \longrightarrow & B^* \\ \downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\ A/\mathfrak{m} & \longrightarrow & A^*/\mathfrak{m} & & B/\mathfrak{p} & \longrightarrow & B^*/\mathfrak{p} \end{array}$$

One derives a commutative diagram with exact lines:

$$\begin{array}{ccccccc} 0 \longrightarrow & N_r U(A^*/\mathfrak{m}) & \longrightarrow & N_r \operatorname{Pic} A & \longrightarrow & N_r \operatorname{Pic} A^* \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & N_r U(B^*/\mathfrak{p}) & \longrightarrow & N_r \operatorname{Pic} B & \longrightarrow & \operatorname{Pic} B^* \oplus N_r \operatorname{Pic}(B/\mathfrak{p}) \end{array}$$

Now by hypothesis we have $A^* \cap B = A$, hence $\mathfrak{p} \cap A^* = \mathfrak{p} \cap B \cap A^* = \mathfrak{p} \cap A = \mathfrak{m}$. Therefore the map $N_r U(A^*/\mathfrak{m}) \to N_r U(B^*/\mathfrak{p})$ is injective. Further the extension $A^* \subset B^*$ is strictly bigger than $A \subset B$ and contained in $\bar{A} \subset \bar{B}$. Also it is p-seminormal by Lemma 6. Namely, $A^* = \mathscr{C}_{B^*}(\mathfrak{m})$ and $\mathfrak{m}B^* \subset B$. So the nontrivial $\xi \in \ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$ goes to 0 in $N_r \operatorname{Pic} A^*$. But that contradicts the left-exactness of "ker."

10. **Proposition.** *Let $S \subset \mathbb{Z} - \{0\}$ be a mutiplicative set and $A \subset B$ any ring-extension. Then ${}^+_BA/A$ is an S-torsion group iff $x \in B$, $x^2, x^3 \in A$ implies that $mx \in A$ for some $m \in S$.*

Proof. The implication from the left to the right is trivial, since the $x \in B$ with $x^2, x^3 \in A$ belong to ${}_B^{+}A$. Concerning the other implication, it is enough to show that for any ring C between A and ${}_B^{+}A$, which is finitely generated (hence finite) over A, the group C/A is an S-torsion-group. By Theorem 2.8 of [2] for such C there is a chain of rings $A = A_0 \subset A_1 \subset \cdots \subset A_n = C$ with $A_i = A_{i-1}[x_i]$, and $x_i^2, x_i^3 \in A_{i-1}$. By induction we may assume that A_{n-1}/A is an S-torsion group. Especially for any $a \in A_{n-1}$ there is an $m \in S$, such that $m(ax_n)^2$, $m(ax_n)^3 \in A$. Therefore $(max_n)^2$, $(max_n)^3 \in A$ and by hypothesis we get an $m' \in S$ with $m'm(ax_n) \in A$. So $C/A = (A_{n-1} + x_n A_{n-1})/A$ is an S-torsion group.

11. PROPOSITION. *For any extension $A \subset B$ of reduced rings consider the groups ${}_B^{+}A/A$, $\ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B]$, $\ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$. If any one of these is S-torsion, so are the others.*

Proof. If ${}_B^{+}A/A$ is not S-torsion, there is an $x \in B - A$, with $x^2, x^3 \in A$, such that $mx \notin A$ for any $m \in S$. By the method of the first part of the proof of Theorem 6.1 in [2], we get an element $\xi \in \ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B]$ with $m\xi \neq 0$ for any $m \in S$. So if $\ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B]$ is S-torsion for some r, so is ${}_B^{+}A/A$. Further $N^r \operatorname{Pic} A \subset N_r \operatorname{Pic} A$ functorially, and so $\ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B] \subset \ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$.

Let us suppose now that ${}_B^{+}A/A$ is an S-torsion group. By Theorem 9 we know that $N_r \operatorname{Pic} {}_B^{+}A \to N_r \operatorname{Pic} B$ is injective. So we may assume ${}_B^{+}A = B$. Now we have a commutative square,

$$\begin{array}{ccc} S^{-1}N_r \operatorname{Pic} A & \longrightarrow & S^{-1}N_r \operatorname{Pic} B \\ \downarrow & & \downarrow \\ N_r \operatorname{Pic}(S^{-1}A) & \longrightarrow & N_r \operatorname{Pic}(S^{-1}B), \end{array}$$

where the vertical arrows are isomorphisms by [2, Theorem 8.1] and the bottom arrow is so by hypothesis. Therefore $S^{-1}(\ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]) = 0$. (This also holds for the cokernel.)

Remark. Of course it would have been enough to consider only the case $S = \mathbb{Z} - (0)$ in the last proposition. Then Theorem 9 gives us the rest.

As a summary of Theorem 9 and Proposition 11 we have:

12. THEOREM. *Let p be a prime number or $p = \infty$, and $A \subset B$ be an extension of reduced rings. Consider the groups ${}_B^{+}A/A$, $\ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B]$, $\ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$ for all $r \geqslant 1$. If in any one of these groups there is an element of order p, then such an element exists in every one of the others.*

13. I want to give another formulation: Let $A \subset B$ be a subintegral extension of reduced rings ("$A \subset B$ is subintegral" means "$B = {}^+_B A$"; cf. [2]). For any ring C between A and B define: $\varphi(C) = \ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} C]$, which is a subgroup of $G := \ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B]$. Then $\varphi(A) = \{0\}$ $\varphi(B) = G$, and φ maps the ring C, such that C/A is the S-torsion part of B/A to the S-torsion part of G. Further φ is injective on the set of these rings. (Question: Is it injective on the whole?)

14. Proposition. *Let $A \subset B$ be a subintegral extension of reduced rings, such that B/A is a torsion group. If S is a multiplicative subset of $\mathbb{Z} - \{0\}$ and C the ring between A and B such that C/A is the S-torsion part of B/A, then the canonical map*

$$\alpha\colon \ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B] \to \ker[N^r \operatorname{Pic} C \to N^r \operatorname{Pic} B]$$

is surjective.

Proof. By the method of the proof of Proposition 11 we see that $\operatorname{coker}[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} C]$ is an S-torsion group. By the lemma of the serpent or even an easier diagram lemma this also yields that $\operatorname{coker} \alpha$ is an S-torsion group.

On the other hand $\ker[N^r \operatorname{Pic} C \to N^r \operatorname{Pic} B]$ and its factor groups are S'-torsion groups by Proposition 11, where S' denotes the multiplicative subset of $\mathbb{Z}$, generated by those primes p, which do not divide any $m \in S$. Therefore $\operatorname{coker} \alpha = 0$.

Question: Is the above proposition true if B/A is not supposed to be torsion?

15. Finally, I want to show how the absolute case, which is handled in [2], can be reduced to the relative case I have treated here. We must only prove that $N_r \operatorname{Pic} A = 0$ for a seminormal ring A. Let P be the set of minimal prime ideals of A (which may be infinite) and $K_{\mathfrak{p}}$ be an algebraic closure of $Q(A/\mathfrak{p})$ for every $\mathfrak{p} \in P$. Then one easily sees that $A = {}^+_B A$, where $B = \prod_{\mathfrak{p} \in P} K_{\mathfrak{p}}$. So by Theorem 9 we only have to show that $N_r \operatorname{Pic} B = 0$. But since this is classically true for any noetherian B, which is normal, i.e., reduced and integrally closed in its full ring of fractions, we need only show the following.

Lemma. *Let $B = \prod_{i \in I} K_i$ be a product of algebraically closed fields, then B is the filtered union of normal noetherian subrings.*

Proof. B is the filtered union of subrings B_α which are finitely generated $\mathbb{Z}$-algebras. Now the maps $B_\alpha \to K_i$ (induced by the projections) can be extended to homomorphisms $\bar{B}_\alpha \to K_i$, where $\bar{B}_\alpha$ is the normalization of B_α. Here we use the hypothesis that the K_i are algebraically closed. We get a map $\bar{B}_\alpha \to B$, whose kernel is a radical ideal $\mathfrak{a}$ with $\mathfrak{a} \cap B_\alpha = \{0\}$. If $\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, where the $\mathfrak{q}_i$ are prime ideals, then $(\mathfrak{q}_1 \cap B_\alpha) \cap \cdots \cap (\mathfrak{q}_r \cap B_\alpha) = \{0\}$, so all minimal prime ideals of B_α are among the $\mathfrak{q}_i \cap B_\alpha$. So all minimal prime ideals of $\bar{B}_\alpha$ are among the $\mathfrak{q}_i$, hence $\mathfrak{a} = \{0\}$. Therefore we have a subring B'_α of B, which is isomorphic to $\bar{B}_\alpha$. The B'_α form a cofinal subset of the set of all B_α. So the lemma is proved.

Remark. It is not to hard to show directly that $\operatorname{Pic} B[X_1, ..., X_r] = \{0\}$ for any direct product B of fields. This would suffice as well.

Appendix

1. We need a fact which certainly is well known.

Lemma. *Let A be a ring, M a finitely generated A-module, and $s \in A$ an element whose homothesy on M is surjective (i.e., M is s-divisible). Then the residue class of s in $A/\operatorname{Ann} M$ is a unit.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of A containing $\operatorname{Ann} M$. We have to show that $s \notin \mathfrak{m}$. But since M is finitely generated and $\mathfrak{m} \supset \operatorname{Ann} M$, we have $\mathfrak{m} \in \operatorname{Supp} M$, i.e., $M_\mathfrak{m} \neq 0$. Since $s/1 \colon M_\mathfrak{m} \to M_\mathfrak{m}$ is surjective, by Nakayama $s/1$ is a unit in $A_\mathfrak{m}$.

2. Proposition. *Let $s > 1$ be a natural number and $A \subset B$ be an extension of reduced rings, such that ${}^+_B A/A$ is a finitely generated s-divisible A-module. Then $\ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B]$ is a $\mathbb{Z}[1/s]$-module. (This is also the case for $\ker[N_r \operatorname{Pic} A \to N_r \operatorname{Pic} B]$, since $N_r \operatorname{Pic} A \simeq \oplus_{1 \leqslant s \leqslant r} N^s \operatorname{Pic} A^{\binom{r}{s}}$.)*

Proof. We may assume $B = {}^+_B A$, since $N^r \operatorname{Pic} {}^+_B A \to N^r \operatorname{Pic} B$ is injective. Let $\mathfrak{c}$ be the conductor of $A \subset B$, i.e., $\mathfrak{c} = \operatorname{Ann}(B/A)$. Then we have the conductor sequence

$$N^r U(A/\mathfrak{c}) \to N^r U(B/\mathfrak{c}) \to N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B \oplus N^r \operatorname{Pic}(A/\mathfrak{c}) \to N^r \operatorname{Pic}(B/\mathfrak{c}).$$

We derive the exact sequence

$$N^r U(A/\mathfrak{c}) \to N^r U(B/\mathfrak{c}) \to \ker[N^r \operatorname{Pic} A \to N^r \operatorname{Pic} B] \to N^r \operatorname{Pic}(A/\mathfrak{c}) \to N^r \operatorname{Pic}(B/\mathfrak{c}).$$

By the lemma s is a unit in $A/\mathfrak{c}$ and $B/\mathfrak{c}$. So by [2, Corollary 8.2] the first two and the last two groups in the sequence above are $\mathbb{Z}[1/s]$-modules. Therefore this holds for the third one.

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